Miskolc Mathematical Notes
Vol. 21 (2020), No. 1, pp. 143–160

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2020.3118

FIXED POINT AND EXTENDED COUPLED FIXED POINT THEOREMS FOR MULTI-VALUED CONTRACTIONS WITH RESPECT TO THE POMPEIU FUNCTIONAL

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Received 07 November, 2019

Abstract. In this paper, we will prove some fixed point results using a Pompeiu type metric on $P_{cl}(X)$, for multi-valued operators satisfying two conditions: contractivity and monotonicity. The approach is based on a fixed point theorem for a multi-valued operator in the setting of a b-metric space. Several qualitative properties(well-posedness, Ulam-Hyers stability) are also obtained. In the second part of this paper, we will consider the extended coupled fixed point problem for a multi-valued operator. We will consider the problem of the existence of the solutions. Data dependence and well-posedness of the extended coupled fixed point problem are also discussed.

2010 *Mathematics Subject Classification:* 47H10; 54H25

Keywords: multi-valued operator, fixed point, ordered metric space, extended coupled fixed point, data dependence, well-posedness, Ulam-Hyers stability, limit shadowing

1. INTRODUCTION

Nadler's contraction principle [14] is an extension to the multi-valued case of the classical Banach's contraction principle. There are many applications of these results, mainly in the theory of operator equations and inclusions, see [2], [4], [6], [9] and [12]. On the other hand, several theorems were given for the so-called extended coupled fixed point problem([3], [17],[18]).

We will define first the notion of b-metric.

Definition 1 (Bakhtin [1], Czerwik [7])**.** Let X be a nonempty set and let $s \geq 1$ be a given real number. A functional $d : X \times X \to \mathbb{R}_+$ is said to be a *b*-metric with constant s if all the axioms of the metric take place with the following modification of the triangle inequality property:

$$d(x,z) \leq s[d(x,y) + d(y,z)],\ for\ all\ x,y,z \in X.$$

Under the above hypotheses, the pair (X,d) is called a *b*-metric space with constant s.

In the framework of a b-metric space, a set $Y \subset X$ is said to be closed if for any sequence $(x_n)_{n\in\mathbb{N}}$ in Y which converges (with respect to d) to an element x, we have $x \in Y$. The notions of bounded or compact sets are defined in a similar way to the case of usual metric spaces. For example, the set $Y \subset X$ is said to be bounded if its diameter $\delta(Y) := \sup_{x,y\in Y} d(x,y)$ is finite, while Y is called compact if every sequence sequence $(x_n)_{n\in\mathbb{N}}$ in Y has a convergent subsequence in Y.

Let (X,d) be a b-metric space and $\mathcal{P}$(X) be the set of all subsets of X. We denote

$$P(X) := \{Y \in \mathcal{P}(X) \mid Y \neq \varnothing\},$$
$$P_{cl}(X) := \{Y \in \mathcal{P}(X) \mid Y \text{ is closed}\},$$
$$P_{cp}(X) := \{Y \in \mathcal{P}(X) \mid Y \text{ is compact}\}.$$

Let (X,d) be a b-metric space. If $T : X \to P(X)$ is a multi-valued operator, then $x \in X$ is called fixed point for T if and only if

$$x \in T(x). \tag{P}$$

We denote by $Fix(T)$ the fixed point set of T and by $SFix(T)$ the set of all strict fixed points of T, i.e., the elements $x \in X$ such that $T(x) = \{x\}$.

If X,Y are two nonempty sets and $T : X \to P(Y)$, then we will denote by

$$Graph(T) := \{(x,y) \in X \times Y : y \in T(x)\}$$

the graph of T.

In the context of a b-metric space (X,d) the following generalized functionals are used in the main sections of the paper.

(1) The gap functional generated by d:

$$D_d : P(X) \times P(X) \to \mathbb{R}_+, D_d(A,B) = inf\{d(a,b) \mid a \in A, b \in B\}.$$

(2) The excess generalized functional:

$$e_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{\infty\}, e_d(A,B) = sup\{D_d(a,B) | a \in A\}.$$

(3) The Hausdorff-Pompeiu generalized functional :

$$H_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{\infty\}, H_d(A,B) = \max\{e_d(A,B), e_d(B,A)\}.$$

(4) The Pompeiu generalized functional:

$$H_d^+ : P(X) \times P(X) \to \mathbb{R}_+ \cup \{\infty\},\ H_d^+(A,B) := \frac{1}{2}\{e_d(A,B) + e_d(B,A)\}.$$

Definition 2 ([17])**.** Let X be a nonempty set, let "$\leq$" be a partial order on X and d be a b-metric on X with constant $s \geq 1$. Then the triple $(X, \leq, d)$ is called an ordered b-metric space if:

i. "$\leq$" is a partial order on X;
ii. d is a b-metric on X with constant $s \geq 1$;

iii. if $(x_n)_{n\in\mathbb{N}}$ is a monotone increasing sequence in X and $x_n \to x^* \in X$, then $x_n \leq x^*$, for all $n \in \mathbb{N}$;

iv. if $(y_n)_{n\in\mathbb{N}}$ is a monotone decreasing sequence in X, and $y_n \to y^* \in X$, then $y_n \geq y^*$, for all $n \in \mathbb{N}$.

Definition 3 ([17])**.** Let $(X,\leq)$ be a partially ordered set. Then, the partial order "$\leq$" induces on the product space $X \times X$ the following partial order:

$$\text{for } (x,y),(u,v) \in X \times X,\ (x,y) \leq_p (u,v) \Leftrightarrow x \leq u, y \geq v.$$

Definition 4 ([18])**.** Let (X,d), (Y,d) be two complete b-metric spaces, $T_1 : X \times Y \to P(X)$ and $T_2 : X \times Y \to P(Y)$ be two multi-valued operators. Then, by definition, a extended coupled fixed point for T_1 and T_2 is a pair $(x^*,y^*) \in X \times Y$ satisfying

$$\begin{cases} x^* \in T_1(x^*,y^*) \\ y^* \in T_2(x^*,y^*). \end{cases} \tag{P1}$$

We will denote by $CFix(T_1,T_2) = \{(x,y) \in X \times Y | x \in T_1(x,y), y \in T_2(x,y)\}$ the solution set for problem (P1).

In this paper, using a Pompeiu type metric on $P_{cl}(X)$, we will prove some fixed point results for multi-valued operators satisfying a combined condition (contractivity and monotonicity). Then, as applications, we will obtain some existence results for a general system of operator inclusions. Several qualitative properties(data dependence, well-posedness) are also obtained.

2. Preliminaries

We will start this section by recalling some useful notion and results.

Lemma 1 ([18])**.** *Let (X,d) be a b-metric space with constant $s \geq 1$, $A,B \in P(X)$ and $q > 1$. Then, for every $a \in A$ there exists $b \in B$ such that $e_d(A,B) \leq qd(a,b)$.*

Definition 5. Let $(X,\preceq)$ be a partially ordered set and $A,B \in P(X)$. We will denote:

a) $A \leq_{st} B \Leftrightarrow \ \forall a \in A,\ \forall b \in B$ we have $a \preceq b$;

b) $A \leq_{wk} B \Leftrightarrow \ \forall a \in A,\ \exists b \in B$ such that $a \preceq b$.

Remark 1. Notice that if $A,B,C \in P(X)$, then $A \leq_{st} B$ and $B \leq_{st} C$ implies $A \leq_{st} C$. The same property also holds for $\leq_{wk}$.

Definition 6. Let $(X,\preceq)$ a partially ordered set and $G : X \times X \to P(X)$. We say that G has the strict mixed monotone property with respect to the partial order "$\preceq$", if the following conditions hold:

(1) $x_0 \preceq x_1 \Rightarrow G(x_0,y) \leq_{st} G(x_1,y), \forall y \in X$;
(2) $y_0 \succeq y_1 \Rightarrow G(x,y_0) \leq_{st} G(x,y_1), \forall x \in X$.

Remark 2. Let (X,d) be a b-metric space with constant $s \geq 1$ and $Z := X \times X$. Then the functional $\tilde{d} : Z \times Z \to \mathbb{R}_+$ defined by

$$\tilde{d}((x,y),(u,v)) = d(x,u) + d(y,v), \text{ for all } (x,y),(u,v) \in Z$$

is a b-metric on Z with the same constant $s \geq 1$ and if (X,d) is a complete b-metric space, then $(Z,\tilde{d})$ is a complete b-metric space, too.
Moreover, for $x,y \in X$, $A,B,U,V \in P(X)$ we have:

$$D_{\tilde{d}}((x,y),A \times B) = D_d(x,A) + D_d(y,B),$$

$$e_{\tilde{d}}(U \times V, A \times B) = e_d(U,A) + e_d(V,B),$$

$$H_{\tilde{d}}(U \times V, A \times B) \leq H_d(U,A) + H_d(V,B),$$

$$H^+_{\tilde{d}}(U \times V, A \times B) = H^+_d(U,A) + H^+_d(V,B).$$

Additionally, by the properties of the gap functional D_d, if $(x,y) \in X \times X$ and $A,B \in P_{cl}(X)$, then

$$D_{\tilde{d}}((x,y),A \times B) = 0 \text{ if and only if } (x,y) \in A \times B.$$

Definition 7. Let (X,d) be a b-metric space with constant $s \geq 1$. A multi-valued operator $T : X \to P_{b,cl}(X)$ is called H^+-contraction with constant k if:

(1) there exists a fixed real number $0 < k < 1$ such that, for every $x,y \in X$, we have

$$H^+_d(T(x),T(y)) \leq kd(x,y),$$

(2) for every $x \in X$ and $y \in T(x)$ we have

$$D_d(y,T(y)) \leq H^+_d(T(x),T(y)).$$

In particular, a multivalued mapping $T : X \to P_{cl}(X)$ is called (H^+,α)-Lipschitz if $\alpha > 0$ and

$$H^+_d(T(x),T(y)) \leq \alpha d(x,y), \quad \forall x,y \in X.$$

Definition 8. Let $(X,\preceq)$ be a partially ordered set and $T : X \to P(X)$ be a multi-valued operator. We say that T is strong increasing (respectively strong decreasing) on X if, for every $x,y \in X$ with $x \preceq y$ we have that $T(x) \leq_{st} T(y)$ (respectively $T(x) \geq_{st} T(y)$).

Lemma 2 (R. Miculescu, A. Mihail [13])**.** *Every sequence* $(x_n)_{n\in\mathbb{N}}$ *of elements from b-metric space* (X,d) *of constant s, having the property that there exists* $\gamma \in [0,1)$ *such that*

$$d(x_{n+1},x_n) \leq \gamma d(x_n,x_{n-1}), \forall n \in \mathbb{N},$$

is Cauchy. Moreover

$$d(x_{n+1},x_{n+k}) \leq \gamma^n \frac{S}{1-\gamma} d(x_0,x_1),\ \forall n,k \in \mathbb{N}^*,\ \textit{where } S = \sum_{n\geq 1} \gamma^{2n\log_\gamma s + 2^{n-1}}.$$

Lemma 3. *Let (X,d) be a b-metric space with constant $s \geq 1$ and $T : X \to P_{b,cl}(X)$ be a $(H_d^+,k)-Lipschitz$. Then:*

1. *T has closed graph in $X \times X$;*
2. *T is H_d^+-l.s.c on X;*
3. *T is H_d^+-u.s.c on X;*
4. *If, additionally T has compact values, then T is l.s.c..*

Proof. (1) Let $(x_n,y_n) \subset X \times X$ such that $(x_n,y_n) \underset{\rightarrow}{d} (x,y)$, when $n \to \infty$ and $y_n \in T(x_n)$, for all $n \in \mathbb{N}$.

It follows that:

$$\begin{aligned} D(y,T(x)) &\leq s[d(y,y_n) + D(y_n,T(x))] \leq \\ &\leq s[d(y,y_n) + H_d(T(x_n),T(x))] \leq \\ &\leq s[d(y,y_n) + 2H_d^+(T(x_n),T(x))] \leq \\ &\leq s[d(y,y_n) + 2\alpha d(x_n,x)],\ \text{for all } n \in \mathbb{N}. \end{aligned}$$

Let us consider $n \to \infty$ and we obtain $D(y,T(x)) \leq 0$. Thus $y \in T(x)$ which means that T has a closed graph.

(2) Let $x \in X$ such that $x_n \to x$. We have:

$$\begin{aligned} e_d(T(x),T(x_n)) &\leq H_d(T(x),T(x_n)) \leq 2H_d^+(T(x),T(x_n)) \leq \\ &\leq 2\alpha d(x,x_n) \to 0. \end{aligned}$$

Then, T is H_d^+-l.s.c. on X.

(3) Using the relation:

$$\begin{aligned} e_d(T(x_n),T(x)) &\leq H(T(x_n),T(x)) \leq 2H_d^+(T(x_n),T(x)) \leq \\ &\leq 2\alpha d(x,x_n) \to 0, \end{aligned}$$

we obtain that T is H_d^+-u.s.c. on X.

(4) The conclusion follows by the fact that any H_d^+-l.s.c multivalued operator with compact values is l.s.c..

□

Lemma 4 (Cauchy's Lemma)**.** *Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two sequences of non-negative real numbers, such that $\sum_{p=0}^{\infty} a_p < +\infty$ and $\lim_{n\to\infty} b_n = 0$. Then,* $\lim_{n\to\infty} \left(\sum_{p=0}^{n} a_{n-p} b_p \right) = 0$.

3. Fixed point theorems for multi-valued contractions with respect to a Pompeiu type metric

In this paper, we will present some fixed point theorems for a multi-valued operators in b-metric spaces. See also [5], [11], [10] and [15] for related results.

Theorem 1. *Let $(X,\preceq,d)$ be an ordered b-metric space with constant $s\geq 1$ such that, the b-metric d is complete. Let $T:X\to P_{cl}(X)$ be a multi-valued strong increasing operator with closed graph. Assume that:*

(1) *there exists $k<1$ such that*

$$H_d^+(T(x),T(y))\leq kd(x,y),\forall x,y\in X,x\preceq y, \tag{3.1}$$

(2) *for every $x\in X$ and $y\in T(x)$ we have,*

$$D_d(y,T(y))\leq H_d^+(T(x),T(y)), \tag{3.2}$$

(3) *there exists an element $x_0\in X$ such that*

$$x_0\leq_{wk} T(x_0). \tag{3.3}$$

Then,

a) $Fix(T)\neq\varnothing$.

b) there exists a sequence of successive approximation for T starting from x_0 such that

$$d(x_{n+1},x^*)\leq\frac{\beta^n Ss}{1-\beta}d(x_0,x_1),\ \forall n\in\mathbb{N},\ \textit{where}\ \beta>0,\ \textit{with}\ k<\beta<1.$$

Proof. a) Let $x_0\in X$ such that $x_0\leq_{wk} T(x_0)$. By (3.3) we get that there exists $x_1\in T(x_0)$ such that $x_0\preceq x_1$.

Using (3.2) we obtain that,

$$D_d(x_1,T(x_1))\leq H_d^+(T(x_0),T(x_1))\leq kd(x_0,x_1). \tag{3.4}$$

For $k<\beta<1$ the relation (3.4) becames

$$D_d(x_1,T(x_1))<\beta d(x_0,x_1). \tag{3.5}$$

Thus, there exists $x_2\in T(x_1)$ such that

$$d(x_1,x_2)<\beta d(x_0,x_1).$$

Since $x_0\preceq x_1$ and T is a multi-valued strong increasing operator, we get that

$$T(x_0)\leq_{st} T(x_1).\ Thus\ x_1\preceq x_2.$$

By (3.2) we get that,

$$D_d(x_2,T(x_2))\leq H_d^+(T(x_1),T(x_2))\leq kd(x_1,x_2)<\beta d(x_1,x_2)<\beta^2 d(x_0,x_1). \tag{3.6}$$

By induction, we obtain a sequence $(x_n)_{n\in\mathbb{N}}\in X$ with the following properties:

(1) $x_{n+1}\in T(x_n),\forall n\in\mathbb{N}$,
(2) $d(x_n,x_{n+1})<\beta d(x_{n-1},x_n)$,

(3) $d(x_n,x_{n+1}) < \beta^n d(x_0,x_1), \forall n \in \mathbb{N}$.

By Lemma 2 we get that $(x_n)_{n\in\mathbb{N}}$ is Cauchy in (X,d). Then, there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. Since T has closed graph in (X,d) we obtain that $x^* \in$ Fix(T).

b) By Lemma 2 we have

$$d(x_{n+1},x_{n+k}) \leq \beta^n \frac{S}{1-\beta} d(x_0,x_1),\ \forall n,k \in \mathbb{N}^*,\ where\ S = \sum_{n\geq 1} \gamma^{2n\log_\gamma s + 2^{n-1}}.$$

By (1) we have that $x_n \to x^*$, $n \to \infty$. Then,

$$\frac{1}{s} d(x_{n+1},x^*) \leq d(x_{n+1},x_{n+k}) + d(x_{n+k},x^*) \leq \frac{\beta^n S}{1-\beta} d(x_0,x_1) + d(x_{n+k},x^*).$$

Letting $k \to \infty$, we get $\frac{1}{s} d(x_{n+1},x^*) \leq \beta^n \frac{S}{1-\beta} d(x_0,x_1)$. Hence $d(x_{n+1},x^*) \leq \beta^n \frac{Ss}{1-\beta} d(x_0,x_1)$.

□

The next result is a corollary of the previous theorem for the global case, where it isn't necessary to suppose that the operator T has to be with closed graph.

Corollary 1. *Let (X,d) be a complete b-metric space with constant $s \geq 1$. Let $T : X \to P_{cl}(X)$ be a multi-valued operator. Assume that:*

(1) *there exists $k < 1$ such that,*

$$H_d^+(T(x),T(y)) \leq kd(x,y), \forall x,y \in X, \tag{3.7}$$

(2) *for every $x \in X$ and $y \in T(x)$ we have,*

$$D_d(y,T(y)) \leq H_d^+(T(x),T(y)), \tag{3.8}$$

Then $Fix(T) \neq \varnothing$.

The following result is about the uniqueness of the solution for the fixed point problem (P).

Theorem 2. *Suppose that all the hypotheses of Theorem 1 are satisfied. Additionally, we suppose there exists $x^* \in X$ such that $T(x^*) = \{x^*\}$ and for every $\bar{x} \in Fix(T)$ we have $\bar{x} \preceq x^*$ or $x^* \preceq \bar{x}$. Then $Fix(T) = SFix(T) = \{x^*\}$.*

Proof. Let $x^* \in X$ such that $T(x^*) = \{x^*\}$. Then, for every $\bar{x} \in Fix(T)$ we have $\bar{x} \preceq x^*$ or $x^* \preceq \bar{x}$. Then,

$$d(\bar{x},x^*) = D_d(\bar{x},T(x^*)) \leq H_d^+(T(\bar{x}),T(x^*)) \leq kd(\bar{x},x^*).$$

Thus, $(1-k)d(\bar{x},x^*) \leq 0$ which means that $d(\bar{x},x^*) = 0$ and $\bar{x} = x^*$. Thus, $Fix(T) = SFix(T) = \{x^*\}$. □

We have the following global version of the Theorem 2.

Corollary 2. *Let* $(X, \preceq, d)$ *be a complete b-metric space with constant* $s \geq 1$. *Let* $T : X \to P_{cl}(X)$ *be a multi-valued operator. Assume that:*

(1) *there exists* $k < 1$ *such that,*

$$H_d^+(T(x), T(y)) \leq kd(x,y), \forall x, y \in X, \tag{3.9}$$

(2) *for every* $x \in X$ *and* $y \in T(x)$ *we have,*

$$D_d(y, T(y)) \leq H_d^+(T(x), T(y)), \tag{3.10}$$

(3) *there exists* $x^* \in X$ *such that* $T(x^*) = \{x^*\}$.

Then $Fix(T) = SFix(T) = \{x^*\}$.

In the next part of this section, we will present some properties of that fixed point problem (P). The first one is the data dependence problem for the fixed point problem (P).

Theorem 3. *Let (X,d) be a complete b-metric space with constant* $s \geq 1$ *and let* $T : X \to P_{cl}(X)$, $S : X \to P(X)$ *be two multi-valued operators. We suppose that:*

(1) *there exists* $k < \frac{1}{2s}$ *such that*

$$H_d^+(T(x), T(y)) \leq kd(x,y), \ \forall x, y \in X,$$

(2) *there exists* $x^* \in X$ *such that* $T(x^*) = \{x^*\}$,

(3) *there exists* $\eta > 0$ *such that*

$$e_d(T(x), S(x)) \leq \eta, \ \forall x \in X.$$

Then, $e_d(Fix(T), Fix(S)) \leq \frac{s\eta}{1-2sk}$.

Proof. By (1) and (2) we get that $Fix(T) = SFix(T) = \{x^*\}$. Let $u^* \in X$ such that $u^* \in S(u^*)$ and $x^* \in X$ such that $T(x^*) = \{x^*\}$. We have,

$$\begin{aligned} d(u^*, x^*) = D_d(u^*, T(x^*)) \leq \\ &\leq s[D_d(u^*, T(u^*)) + e_d(T(u^*), T(x^*))] \\ &\leq s[e_d(S(u^*), T(u^*)) + H_d(T(u^*), T(x^*))] \leq \\ &\leq s[\eta + 2H_d^+(T(u^*), T(x^*))] \leq \\ &\leq s[\eta + 2kd(u^*, x^*)]. \end{aligned}$$

Then, $d(u^*, x^*) \leq \frac{s\eta}{1-2sk}$.

Since $u^* \in Fix(S)$ are arbitrary chosen, we obtain that, $e_d(Fix(T), Fix(S)) \leq \frac{s\eta}{1-2sk}$.

□

The next problem we intend to study is the well-posedness of the fixed point problem (P).

Definition 9. We consider the fixed point problem (P). By definition, (P) is well-posed for T with respect to D_d if:

(1) $Fix(T) = \{x^*\}, x^* \in X$.
(2) if there exists a sequence $u_n \in X$ with $D_d(u_n, T(u_n)) \to 0$, then $d(u_n, x^*) \to 0$ as $n \to \infty$.

Theorem 4. *We suppose that all the hypotheses of Corollary 2 take place. Then the fixed point problem (P) is well-posed for T with respect to D_d.*

Proof. Since the multi-valued operator T satisfies the hypotheses of Corollary 1, we get that T has a unique fixed point, i.e., $Fix(T) = \{x^*\}$ where $x^* \in X$. Then we have, $T(x^*) = \{x^*\}$.

Let $u_n \subset X$, with $D_d(u_n, T(u_n)) \to 0$ as $n \to \infty$.

Then we have,

$$\begin{aligned} d(u_n, x^*) = D_d(u_n, T(x^*)) \leq \\ &\leq s[D_d(u_n, T(u_n)) + e_d(T(u_n), T(x^*))] \leq \\ &\leq s[D_d(u_n, T(x_n)) + H_d(T(u_n), T(x^*))] \leq \\ &\leq sD_d(u_n, T(u_n)) + 2sH_d^+(T(u_n), T(x^*)) \leq \\ &\leq sD_d(u_n, T(x_n)) + 2skd(u_n, x^*). \end{aligned}$$

Hence,

$$d(u_n, x^*) \leq \frac{s}{1-2sk} D_d(u_n, T(u_n)) \to 0, \ as \ n \to \infty.$$

□

Now we will present the Ulam-Hyers property of the fixed point problem (P).

Definition 10. Let (X,d) be a b-metric space with constant $s \geq 1$ and let $T : X \to P(X)$ be a multi-valued operator. Let us to consider the inclusion $x \in T(x)$ and the inequality

$$D_d(x, T(x)) \leq \varepsilon \ where \ \varepsilon > 0 \ and \ x \in X. \tag{3.11}$$

By definition, the inclusion (P) is called generalized Ulam-Hyers stable if and only if, there exists a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous in 0 with $\psi(0) = 0$, such that for each $\varepsilon > 0$ and for each solution $u^* \in X$ of the inequality (3.11), there exists a solution $x^* \in X$ of (P) such that,

$$d(u^*, x^*) \leq \psi(\varepsilon).$$

Theorem 5. *Let $T : X \to P_{cl}(X)$ be a multi-valued operator which verifies (1) and (3) from Corollary 2 with $k < \frac{1}{2s}$. Then the fixed point inclusion (P) is Ulam-Hyers stable.*

Proof. From Corollary 2 we have that T has a unique fixed point $x^* \in X$ such that $T(x^*) = \{x^*\}$.

Let $\varepsilon > 0$ and $u^* \in X$ be a ε-solution of the fixed point problem (P), i.e., a solution of the inequality 3.11.

Then, we have $D_d(u^*, T(u^*)) \leq \varepsilon$. Hence, we get

$$\begin{aligned} d(u^*,x^*) = D_d(u^*,T(x^*)) \leq \\ &\leq s[D_d(u^*,T(u^*)) + e_d(T(u^*),T(x^*))] \leq \\ &\leq s[D_d(u^*,T(u^*)) + H_d(T(u^*),T(x^*))] \leq \\ &\leq sD_d(u^*,T(u^*)) + 2sH_d^+(T(u^*),T(x^*)) \leq \\ &\leq sD_d(u^*,T(u^*)) + 2skd(u^*,x^*)). \end{aligned}$$

Thus,

$$d(u^*,x^*) \leq \frac{s}{1-2sk}\varepsilon.$$

Therefore, if we consider $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, given by $\psi(t) = ct$(with $c = \frac{s}{1-2sk} > 0$), we can conclude that the inclusion (P) is Ulam-Hyers stable. □

Using the global existence result for the fixed point problem(see Corollary 2), we can prove the limit shadowing property of the fixed point problem (P).

Definition 11. Let (X,d) be a metric space with constant $s \geq 1$ and $T : X \to P_{cl}(X)$ be a multivalued operator. By definition, the fixed problem (P) has the limit shadowing property if, for any sequence $(x_n)_{n\in\mathbb{N}}$ in X for which,

$$D_d(x_n, T(x_n)) \to 0 \text{ as } n \to \infty,$$

there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in X such that,

$$d(x_n, u_n) \to 0 \text{ as } n \to \infty.$$

Theorem 6. *Let (X,d) be a metric space with constant $s \geq 1$ and $T : X \to P_{cl}(X)$ be a multivalued operator which verifies (1) and (3) from Corollary 2. Then, the fixed point problem (P) has the limit shadowing property.*

Proof. By Corollary 2 we get that $Fix(T) = \{x^*\}$ and for any initial point $u_0 \in X$, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in X with $u_{n+1} \in T(u_n)$ for all $n \in \mathbb{N}$, such that $(u_n)_{n\in\mathbb{N}} \to x^*$ as $n \to \infty$.

Let $(x_n)_{n\in\mathbb{N}}$ be a sequences in X such that,

$$D_d(x_{n+1}, T(x_n)) \to 0 \; as \; n \to \infty.$$

Thus, $d(x_{n+1}, u_{n+1}) \leq s[d(x_{n+1}, x^*) + d(x^*, u_{n+1})]$.
Next, we can write,

$$\begin{aligned} d(x_{n+1}, x^*) = D_d(x_{n+1}, T(x^*)) \leq \\ &\leq s[D_d(x_{n+1}, T(x_n)) + e_d(T(x_n), T(x^*)] \leq \\ &\leq s[D_d(x_{n+1}, T(x_n)) + H_d(T(x_n), T(x^*)] \leq \\ &\leq s[D_d(x_{n+1}, T(x_n)) + 2H_d^+(T(x_n), T(x^*)] \leq \\ &\leq sD_d(x_{n+1}, T(x_n)) + 2ksd(x_n, x^*). \end{aligned}$$

We denote $2ks = \alpha < 1$, thus

$$\begin{aligned} d(x_{n+1}, x^*) &\leq sD_d(x_{n+1}, T(x_n)) + \alpha d(x_n, x^*) \leq \\ &\leq sD_d(x_{n+1}, T(x_n)) + \alpha[sD_d(x_n, T(x_{n-1})) + \alpha d(x_{n-1}, x^*)] \\ &= s[D_d(x_{n+1}, T(x_n)) + D_d(x_n, T(x_{n-1}))] + \alpha^2 d(x_{n-1}, x^*) \\ &\leq \cdots \leq \\ &\leq s[D_d(x_{n+1}, T(x_n)) + \ldots + \alpha^n D_d(x_1, T(x_0))] + \alpha^n d(x_0, x^*). \end{aligned}$$

By Cauchy's Lemma 4 we obtain that,

$$D_d(x_{n+1}, T(x_n)) + \ldots + \alpha^n D_d(x_1, T(x_0)) \to 0 \; as \; n \to \infty.$$

Thus, we obtain that $d(x_n, x^*) \to 0 \; as \; n \to \infty$.
Then, $d(x_{n+1}, u_{n+1}) \to 0 \; as \; n \to \infty$.

□

4. Extended coupled fixed point theorems for multi-valued contractions with respect to a Pompeiu type metric

The purpose of this section is to give some extended coupled fixed point theorems for multi-value operators satisfying a contraction type condition with respect to the excess functional. We will consider here the context of a b-metric space. Our results are new even for the case of metric spaces and they extend some theorems given in [8], [16], [5] and [19] and other papers in the literature.

Theorem 7. *Let $(X, \preceq, d)$, $(Y, \preceq, d)$ be two ordered complete b-metric spaces with constant $s \geq 1$. Let $T_1 : X \times Y \to P_{cl}(X)$ and $T_2 : X \times Y \to P_{cl}(Y)$ be two multi-valued operators with closed graph and having the strict mixed monotone property with respect to " $\preceq$ ". Assume that:*

(1) *there exist $k_1, k_2 > 0$ with $k_1 + k_2 \in (0,1)$ such that:*

$$e_d(T_1(x,y), T_1(u,v)) + e_d(T_1(u,v), T_1(x,y)) \leq k_1[d(x,u) + \rho(y,v)],$$

$\forall x \preceq u$ *and* $y \succeq v$,

$e_\rho(T_2(x,y),T_2(u,v)) + e_\rho(T_2(u,v),T_2(x,y)) \leq k_2[d(x,u)+\rho(y,v)]$,
$\forall x \preceq u$ *and* $y \succeq v$,

(2) *there exists* $(x_0,y_0) \in X \times Y$ *such that*

$$x_0 \preceq_{wk} T_1(x_0,y_0),\ y_0 \succeq_{wk} T_2(x_0,y_0)$$

or

$$x_0 \succeq_{wk} T_1(x_0,y_0),\ y_0 \preceq_{wk} T_2(x_0,y_0).$$

Then, the following conclusions hold:

(i) there exist $(x^*,y^*) \in X \times Y$ *and the sequences* $(x_n)_{n\in\mathbf{N}}$ *in* X *and* $(y_n)_{n\in\mathbf{N}}$ *in* Y *defined by*

$$\begin{cases} x_{n+1} \in T_1(x_n,y_n) \\ y_{n+1} \in T_2(x_n,y_n) \end{cases} \quad for\ all\ n \in \mathbf{N}, \tag{4.1}$$

such that $x_n \to x^*$ *and* $y_n \to y^*$ *as* $n \to \infty$ *and*

$$\begin{cases} x^* \in T_1(x^*,y^*) \\ y^* \in T_2(x^*,y^*) \end{cases} . \tag{4.2}$$

(ii) If we denote $\tilde{d}(z_n,z^*) = d(x_n,x^*) + \rho(y_n,y^*)$ *and* $\tilde{d}(z_0,z_1) = d(x_0,x_1) + \rho(y_0,y_1)$ *then we have the following estimation holds,*
$\tilde{d}(z_{n+1},z^*) \leq \dfrac{\beta^n Ss}{1-\beta}\tilde{d}(z_0,z_1)$, *where* $\beta > 0$ *is any number with* $\beta < \dfrac{k_1+k_2}{2}$.

Proof. Let $Z = X \times Y$, $T : Z \to P_{cl}(Z)$, $T(x,y) = T_1(x,y) \times T_2(x,y)$ and consider the functional $\tilde{d} : Z \times Z \to \mathbf{R}_+$, $\tilde{d}((x,y),(u,v)) = d(x,u) + \rho(y,v)$.

Then, for $z_0 = (x_0,y_0) \in Z$, there exist $x_1 \in T_1(x_0,y_0)$ and $y_1 \in T_2(x_0,y_0)$ such that

$$x_0 \preceq x_1 \ and\ y_0 \succeq y_1 \Rightarrow (x_0,y_0) \preceq_p (x_1,y_1) \Rightarrow z_0 \preceq_p z_1.$$

Thus, $z_0 \preceq_p T(z_0)$.

On the other hand, for $z = (x,y) \in Z$ and $w = (u,v) \in T(Z)$ with $z \preceq_p w$ we get

$$\begin{aligned} D_{\tilde{d}}(w,T(w)) &= D_{\tilde{d}}((u,v),T_1(u,v) \times T_2(u,v)) = \\ &= D_d(u,T_1(u,v)) + D_\rho(v,T_2(u,v)) \leq \\ &\leq e_d(T_1(x,y),T_1(u,v)) + e_\rho(T_2(x,y),T_2(u,v)) = \\ &= 2H_d^+(T_1(x,y) \times T_2(x,y),T_1(u,v) \times T_2(u,v)) = 2H_{\tilde{d}}^+(T(z),T(w)). \end{aligned}$$

Hence, we conclude that

$$D_{\tilde{d}}(w,T(w)) \leq 2H_{\tilde{d}}^+(T(z),T(w)),$$

for all $z \in Z$ and $w \in T(z)$ with $z \leq_p w$.

Now, we proof that:

$$H^+_{\tilde{d}}(T(z),T(w)) \leq k\tilde{d}(z,w)\ \forall z,w \in Z\ z \leq_p w.$$

$$\begin{aligned}
H^+_{\tilde{d}}(T(z),T(w)) &= H^+_{\tilde{d}}(T_1(z)\times T_2(z), T_1(w)\times T_2(w)) = \\
&= H^+_d(T_1(z),T_1(w)) + H^+_d(T_2(z),T_2(w)) = \\
&= \frac{1}{2}(e_d(T_1(x,y),T_1(u,v)) + e_d(T_1(u,v),T_1(x,y))) + \\
&\quad + \frac{1}{2}(e_\rho(T_2(x,y),T_2(u,v)) + e_\rho(T_2(u,v),T_2(x,y))) \leq \\
&\leq \frac{1}{2}k_1[d(x,u)+\rho(y,v)] + \frac{1}{2}k_2[d(x,u)+\rho(y,v)] = \\
&= \frac{1}{2}(k_1+k_2)[d(x,u)+\rho(y,v)] = \frac{1}{2}(k_1+k_2)\tilde{d}(z,w).
\end{aligned}$$

Hence, there exists $k := \frac{1}{2}(k_1+k_2) \in (0,1)$ such that

$$H^+_{\tilde{d}}(T(z),T(w)) \leq k\tilde{d}(z,w)\ \forall z,w \in Z\ z \leq_p w.$$

We will show now that T is an increasing operator, i.e, if $z \leq_p w$ we have $T(z) \leq_s tT(w)$. This is equivalent to the following relations

$$T_1(x,y)\times T_2(x,y) \leq_{st} T_1(u,v)\times T_2(u,v)$$

or

$$T_1(x,y) \leq_{st} T_1(u,v)\ and\ T_2(x,y) \leq_{st} T_2(u,v).$$

We have

$$x \preceq u \Rightarrow \begin{cases} T_1(x,y) \leq_{st} T_1(u,y), \forall y \in Y \\ T_2(x,y) \geq_{st} T_2(u,y), \forall y \in Y \end{cases}, \tag{4.3}$$

$$y \succeq v \Rightarrow \begin{cases} T_1(x,y) \leq_{st} T_1(x,v), \forall x \in X \\ T_2(x,y) \geq_{st} T_2(x,v), \forall x \in X \end{cases}, \tag{4.4}$$

If in the relation (4.4) we put $x := u$, then we have $T_1(u,y) \leq_{st} T_1(u,v)$. Using the relation (4.3) we get $T_1(x,y) \leq_{st} T_1(u,v)$.

On the other hand, for $x := u$ in the relation (4.4) we get $T_2(u,y) \geq_{st} T_2(u,v)$. Using the relation (4.3) we obtain $T_2(x,y) \geq_{st} T_2(u,v)$. Thus $T : Z \to Z$, $T(x,y) = T_1(x,y)\times T_2(x,y)$ is an increasing operator.

Thus, the operator $T : Z \to Z$ has the following properties:

(1) T is increasing on Z.
(2) There exists $z_0 = (x_0,y_0) \in Z$ such that $z_0 \leq_p T(z_0)$.

(3) There exists $k \in (0, \frac{1}{s})$ such that

$$H^{+}_{\tilde{d}}(T(z), T(w)) \leq k\tilde{d}(z,w) \ \forall z, w \in Z \ z \leq_p w.$$

(4) For every $z = (x,y) \in Z$ and $w = (u,v) \in T(z)$ with $z \leq_p w$, we have

$$D_{\tilde{d}}(w, T(w)) \leq 2H^{+}_{\tilde{d}}(T(z), T(w)).$$

Thus, by Theorem 1 we get that $Fix(T) \neq \varnothing$ and there exists a sequence $(z_n)_{n\in\mathbb{N}} \subset Z$, $z_n = (x_n, y_n) \in X \times Y$ of successive approximation of T starting from $z_0 \in Z$ which converges to a fixed point $z^* = (x^*, y^*)$ of T.

Thus,

$$z^* \in T(z^*) \Rightarrow (x^*, y^*) \in T(x^*, y^*) \Rightarrow (x^*, y^*) \in T_1(x^*, y^*) \times T_2(x^*, y^*)$$

which leads to

$$\begin{cases} x^* \in T_1(x^*, y^*) \\ y^* \in T_2(x^*, y^*). \end{cases} \tag{4.5}$$

By Theorem 1 (ii) we get that

$$\tilde{d}(z_{n+1}, z^*) \leq \frac{\beta^n Ss}{1-\beta}\tilde{d}(z_0, z_1), \ \textit{where} \ \beta < \frac{k_1 + k_2}{2}.$$

□

Corollary 3. *Let $(X, \preceq, d)$, $(Y, \preceq, d)$ be two orderd complete b-metric spaces with constant $s \geq 1$. Let $T_1 : X \times Y \to P_{cl}(X)$ and $T_2 : X \times Y \to P_{cl}(Y)$ be two multi-valued operators having the strict mixed monotone property with respect to "$\preceq$". Assume that, there exist $k_1, k_2 > 0$ with $k_1 + k_2 \in (0,1)$ such that:*

$$e_d(T_1(x,y), T_1(u,v)) + e_d(T_1(u,v), T_1(x,y)) \leq k_1[d(x,u) + \rho(y,v)],$$
$$\forall (x,y) \in (X \times Y),$$

$$e_\rho(T_2(x,y), T_2(u,v)) + e_\rho(T_2(u,v), T_2(x,y)) \leq k_2[d(x,u) + \rho(y,v)],$$
$$\forall (x,y) \in (X \times Y).$$

Then, the following conclusions hold:

a) there exists $(x^*, y^*) \in X \times Y$ such that the sequence $(x_n)_{n\in\mathbf{N}}$ in X and $(y_n)_{n\in\mathbf{N}}$ in Y with,

$$\begin{cases} x_{n+1} \in T_1(x_n, y_n) \\ y_{n+1} \in T_2(x_n, y_n) \end{cases} \quad for\ all\ n \in \mathbf{N} \tag{4.6}$$

such that $x_n \to x^*$ and $y_n \to y^*$ as $n \to \infty$ and

$$\begin{cases} x^* \in T_1(x^*, y^*) \\ y^* \in T_2(x^*, y^*) \end{cases} . \tag{4.7}$$

b) If we denote $\tilde{d}(z_n,z^*) = d(x_n,x^*) + \rho(y_n,y^*)$ and $\tilde{d}(z_0,z_1) = d(x_0,x_1) + \rho(y_0,y_1)$, then we have the following estimation holds,
$\tilde{d}(z_{n+1},z^*) \leq \dfrac{\beta^n Ss}{1-\beta}\tilde{d}(z_0,z_1)$*, where* $\beta > 0$ *is any number with* $\beta < \dfrac{k_1+k_2}{2}$.

The following result is about the uniqueness of the extended coupled fixed point.

Theorem 8. *In addition to the hypotheses of the Corollary 3 we suppose that:*

(1) *there exists* $(x^*,y^*) \in X \times Y$ *such that*

$$\begin{cases} T_1(x^*,y^*) = \{x^*\} \\ T_2(x^*,y^*) = \{y^*\} \end{cases}. \tag{4.8}$$

(2) *for every* $(\overline{x},\overline{y}) \in CFix(T_1,T_2)$ *we have* $\overline{x} \leq x^*$ *and* $\overline{y} \geq y^*$ *or* $x^* \leq \overline{x}$ *and* $y^* \geq \overline{y}$.

Then, we obtain that the extended coupled fixed point problem (P1) has an unique solution.

Proof. Let $Z := X \times Y$, $T : Z \to P_{cl}(Z)$, $T(x,y) = T_1(x,y) \times T_2(x,y)$ and consider the functional $\tilde{d} : Z \times Z \to \mathbb{R}_+$, $\tilde{d}((x,y),(u,v)) = d(x,u) + \rho(y,v)$.
Let $z^* \in Z$, $z^* = (x^*,y^*)$ such that $T(z^*) = \{z^*\}$ and $\overline{z} = (\overline{x},\overline{y}) \in Z$ such that $\overline{z} \in Fix(T)$ with $\overline{z} \leq z^*$ or $z^* \leq \overline{z}$.
By Theorem 2 we get that $Fix(T) = SFix(T) = \{z^*\}$. Hence $CFix(T_1,T_2) = \{z^*\}$. □

In the next part of this section, we will present some properties of the extended coupled fixed point problem (P1).
The first result is a data dependence problem for the extended coupled fixed point problem (P1).

Theorem 9. *Let (X,d), (Y,d) be two coupled b-metric spaces with constant* $s \geq 1$. *Let* $T_1 : X \times Y \to P_{cl}(X)$, $T_2 : X \times Y \to P_{cl}(Y)$, $S_1 : X \times Y \to P(X)$ *and* $S_2 : X \times Y \to P(Y)$ *be four multi-valued operators. We suppose that:*

(1) *there exist* $k_1,k_2 > 0$ *with* $k_1 + k_2 \in (0,1)$ *such that:*

$$e_d(T_1(x,y),T_1(u,v)) + e_d(T_1(u,v),T_1(x,y)) \leq k_1[d(x,u) + \rho(y,v)],\ \forall (x,y) \in X \times Y,$$

$$e_\rho(T_2(x,y),T_2(u,v)) + e_\rho(T_2(u,v),T_2(x,y)) \leq k_2[d(x,u) + \rho(y,v)],\ \forall (x,y) \in X \times Y,$$

(2) *there exists* $(x^*,y^*) \in X \times Y$ *such that*

$$\begin{cases} T_1(x^*,y^*) = \{x^*\} \\ T_2(x^*,y^*) = \{y^*\} \end{cases}.$$

(3) *there exists* $(u^*,v^*) \in X \times Y$ *such that*

$$\begin{cases} u^* \in S_1(u^*,v^*) \\ v^* \in S_2(u^*,v^*) \end{cases}.$$

(4) *there exists* $\eta > 0$ *such that*

$$e_{\tilde{d}}(T_1(x,y) \times T_2(x,y), S_1(x,y) \times S_2(x,y)) \leq \eta, \; \forall (x,y) \in X \times Y.$$

Then, $e_{\tilde{d}}(CFix(S_1,S_2), CFix(T_1,T_2)) \leq \dfrac{s\eta}{1-2sk}$.

Proof. Let $Z := X \times Y$, $T : Z \to P_{cl}(Z)$, $T(x,y) = T_1(x,y) \times T_2(x,y)$ and consider the functional $\tilde{d} : Z \times Z \to \mathbb{R}_+$, $\tilde{d}((x,y),(u,v)) = d(x,u) + \rho(y,v)$.

Let $z^* \in Z$, $z^* = (x^*,y^*)$ such that $T(z^*) = \{z^*\}$.

We consider $S : Z \to P(Z)$, $S(x,y) = S_1(x,y) \times S_2(x,y)$. Let $w^* \in Z$, $w^* = (u^*,v^*)$ such that $w^* \in S(w^*)$.

By Theorem 3 for the multi-valued operators T and S we obtain that

$$d(w^*,z^*) \leq \frac{s\eta}{1-2sk} \text{ where } k := \frac{k_1+k_2}{2}.$$

Since $w^* \in CFix(S)$ are arbitrary chosen, we obtain that

$$e_{\tilde{d}}(CFix(S_1,S_2), CFix(T_1,T_2)) \leq \frac{s\eta}{1-2sk}.$$

□

The second problem we intend to study is the well-posedness of the extended coupled fixed point problem.

Definition 12. We consider the extended coupled fixed point problem (P1). By definition, (P1) is well-posed for T_1 and T_2 with respect to D_d if:

(1) $CFix(T_1,T_2) = \{(u^*,v^*)\}$, $(u^*,v^*) \in X \times Y$,
(2) if there exists a sequences $(u_n,v_n) \in X \times Y$ with

$$\begin{cases} D_d(u_n, T_1(u_n,v_n)) \to 0 \\ D_d(v_n, T_2(u_n,v_n)) \to 0 \end{cases}.$$

then $d(u_n,u^*) + d(v_n,v^*) \to 0$ as $n \to \infty$.

Theorem 10. *We suppose that all the hypotheses of Theorem 9 take place. Then the extended coupled fixed problem (P1) is well-posed for* T_1 *and* T_2 *with respect to* D_d.

Proof. Let $Z := X \times Y$, $T : Z \to P_{cl}(Z)$, $T(x,y) = T_1(x,y) \times T_2(x,y)$ and consider the functional $\tilde{d} : Z \times Z \to \mathbb{R}_+$, $\tilde{d}((x,y),(u,v)) = d(x,u) + \rho(y,v)$.

Since the multi-valued operator T satisfies the hypotheses of Corollary 9, we get that T has a unique extended coupled fixed point. Let $z^* \in Z$, $z^* = (x^*, y^*)$ such that $T(z^*) = \{z^*\}$.

Let $w_n = (u_n, v_n) \in Z$, with

$$\begin{cases} D_d(u_n, T_1(u_n, v_n)) \to 0 \\ D_d(v_n, T_2(u_n, v_n)) \to 0 \end{cases}.$$

By Theorem 4 for the multi-valued operator T we obtained that $d(w_n, z^*) \to 0$, as $n \to \infty$.

□

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