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WEIGHTED SIMPSON'S TYPE INTEGRAL INEQUALITIES FOR HARMONICALLY-PREINVEX FUNCTIONS

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Abstract. In this paper, some new weighted Simpson's type integral inequalities are presented for the class of harmonically-preinvex functions by using power-mean integral inequality and Hölder's integral inequality.

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1. Introduction

It is known that a function $\vartheta : I \subset \mathbb{R} \to \mathbb{R}$ is convex if the inequality

$$\vartheta(vx + (1-v)y) \leq v\vartheta(x) + (1-v)\vartheta(y)$$

holds for all $x, y \in I$ and $v \in [0,1]$.

Convex functions can also be characterized by an important inequality known as Hermite-Hadamard inequality. This inequality is stated as follows.

Let $\vartheta : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function defined on a convex interval I of real numbers and $\lambda, \mu \in I$ with $\lambda < \mu$. Then

$$\vartheta\left(\frac{\lambda+\mu}{2}\right) \leq \frac{1}{\mu-\lambda}\int_{\lambda}^{\mu} \vartheta(x)\,dx \leq \frac{\vartheta(\lambda)+\vartheta(\mu)}{2}.$$

Convex functions play a key role in the development of the filed of mathematical inequalities. The filed of mathematical inequalities using generalizations of the convex functions is growing rapidly.

Harmonically-preinvex functions is one of the generalizations of the convex functions stated as follows.

Definition 1. [19] A function $\vartheta : I_{h\varphi} = [\lambda, \lambda + \varphi(\mu,\lambda)] \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is said to be harmonic preinvex function with respect to the bifunction $\varphi(.,.)$ if

$$\vartheta\left(\frac{x(x+\varphi(y,x))}{x+(1-v)\varphi(y,x)}\right) \leq v\vartheta(y) + (1-v)\vartheta(x)$$

for all $x,\ y \in I_{h\varphi}$ and $v \in [0,1]$. The function ϑ is said to be harmonic preconcave function with respect to $\varphi(.,.)$, if and only if, $-\vartheta$ is harmonic preinvex function.

Let us recall the notion of harmoncally symmetric functions that will also be useful to establish our resutls.

Definition 2. [19] A function $\vartheta : I_{h\varphi} = [\lambda, \lambda + \varphi(\mu,\lambda)] \subseteq \mathbb{R} \backslash \{0\} \to \mathbb{R}$ is said to be harmonically symmetric with respect to $\frac{2\lambda(\lambda+\varphi(\mu,\lambda))}{2\lambda+\varphi(\mu,\lambda)}$ if

$$\vartheta(x) = \vartheta\left(\frac{\lambda(\lambda+\varphi(\mu,\lambda))x}{(2\lambda+\varphi(\mu,\lambda))x - \lambda(\lambda+\varphi(\mu,\lambda))}\right)$$

holds for all $x \in [\lambda, \lambda+\varphi(\mu,\lambda)]$.

One of the important quadrature formulae for functions of one real variable is the Simpson's rule.

The Simpson's inequality states that if $\vartheta : [\lambda,\mu] \to \mathbb{R}$ is a four times continuously differentiable mapping on (λ,μ) and $\left\|\vartheta^{(4)}\right\|_\infty = \sup_{v\in(\lambda,\mu)} \left|\vartheta^{(4)}(v)\right| < \infty$, then

$$\left|\int_\lambda^\mu \vartheta(x)\,dx - \frac{\mu-\lambda}{3}\left[\frac{\vartheta(\lambda)+\vartheta(\mu)}{2} + 2\vartheta\left(\frac{\lambda+\mu}{2}\right)\right]\right| \leq \frac{1}{2880}\left\|\vartheta^{(4)}\right\|_\infty \cdot (\mu-\lambda)^5. \quad (1.1)$$

The inequality (1.1) has attracted a considerable attention of a number of researchers since it is very important and remarkable in the filed of mathematical inequalities and numerical analysis. Plenty of new Simpson's type inequalities using different types of convex functions have been established. There is a substantial literature on the extensions, generalizations and refinements of Simpson's inequality and on Simpson's type integral inequalities using a variety of convexity conditions, see for example [1–11], [12–18], [20–31] and the references cited in these papers.

In Section 2, we prove new Simpson's type integral inequalities using harmonically-preinvex and harmonically symmetric weight functions. The results of our manuscript provide extensions of previously established Simpson's type integral inequalities.

2. Weighted Simpson's type inequalities for harmonically-preinvex functions

Throughout in this section, we will use the notations $U_1(v)$ and $U_2(v)$ respectively for $\frac{2\lambda(\lambda+\varphi(\mu,\lambda))}{2\lambda+(1+v)\varphi(\mu,\lambda)}$ and $\frac{2\lambda(\lambda+\varphi(\mu,\lambda))}{2\lambda+(1-v)\varphi(\mu,\lambda)}$.

In order to prove the results of this paper, we need the following lemma.

Lemma 1. *Let $\vartheta : I_{h\varphi} \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a differentiable function on $I^{\circ}_{h\varphi}$ and λ, $\lambda+\varphi(\mu,\lambda) \in I^{\circ}_{h\varphi}$ with $\varphi(\mu,\lambda) > 0$ and let $w : [\lambda, \lambda+\varphi(\mu,\lambda)] \to [0,\infty)$ be a continuous positive mapping and harmonically symmetric to $\frac{2\lambda(\lambda+\varphi(\mu,\lambda))}{2\lambda+\varphi(\mu,\lambda)}$.*

If $\vartheta' \in L([\lambda, \lambda+\varphi(\mu,\lambda)])$, the following equality holds

$$\frac{1}{8}\left(\frac{\lambda(\lambda+\varphi(\mu,\lambda))}{\varphi(\mu,\lambda)}\right)\left[\vartheta(\lambda)+6\vartheta\left(\frac{2\lambda(\lambda+\varphi(\mu,\lambda))}{2\lambda+\varphi(\mu,\lambda)}\right)+\vartheta(\lambda+\varphi(\mu,\lambda))\right] \tag{2.1}$$

$$\times \int_{\lambda}^{\lambda+\varphi(\mu,\lambda)} \frac{w(z)}{z^2}dz - \frac{\lambda(\lambda+\varphi(\mu,\lambda))}{\varphi(\mu,\lambda)} \int_{\lambda}^{\lambda+\varphi(\mu,\lambda)} \frac{w(z)\vartheta(z)}{z^2}dz = \frac{\varphi(\mu,\lambda)}{4(\lambda(\lambda+\varphi(\mu,\lambda)))}$$

$$\times \left\{\int_0^1 p_1(v)(U_1(v))^2\vartheta'(U_1(v))\,dv + \int_0^1 p_2(v)(U_2(v))^2\vartheta'(U_2(v))\,dv\right\},$$

where

$$p_1(v) = \frac{3}{4}\int_0^1 w(U_1(s))\,ds - \int_0^v w(U_1(s))\,ds$$

and

$$p_2(v) = -\frac{3}{4}\int_0^1 w(U_2(s))\,ds + \int_0^v w(U_2(s))\,ds.$$

Proof. By integration by parts, we have

$$I_1 = \int_0^1 p_1(v)(U_1(v))^2\vartheta'(U_1(v))\,dv$$

$$= -\left(\frac{2\lambda(\lambda+\varphi(\mu,\lambda))}{\varphi(\mu,\lambda)}\right)\int_0^1 p_1(v)\left[-\left(\frac{\varphi(\mu,\lambda)}{2\lambda(\lambda+\varphi(\mu,\lambda))}\right)\right](U_1(v))^2\vartheta'(U_1(v))\,dv$$

$$= -\left(\frac{2\lambda(\lambda+\varphi(\mu,\lambda))}{\varphi(\mu,\lambda)}\right)\int_0^1\left[\frac{3}{4}\int_0^1 w(U_1(s))\,ds - \int_0^v w(U_1(s))\,ds\right]d\left[\vartheta(U_1(v))\right]dv$$

$$= -\left(\frac{2\lambda(\lambda+\varphi(\mu,\lambda))}{\varphi(\mu,\lambda)}\right)\left[\frac{3}{4}\int_0^1 w(U_1(s))\,ds - \int_0^v w(U_1(s))\,ds\right]\vartheta(U_1(v))\Bigg|_0^1$$

$$-\left(\frac{2\lambda(\lambda+\varphi(\mu,\lambda))}{\varphi(\mu,\lambda)}\right)\int_0^1 w(U_1(v))\vartheta(U_1(v))\,dv = \left(\frac{2\lambda(\lambda+\varphi(\mu,\lambda))}{\varphi(\mu,\lambda)}\right)$$

$$\times\left[\frac{1}{4}\vartheta(\lambda)\int_0^1 w(U_1(v))\,dv+\frac{3}{4}\vartheta\left(\frac{2\lambda(\lambda+\varphi(\mu,\lambda))}{2\lambda+\varphi(\mu,\lambda)}\right)\int_0^1 w(U_1(v))\,dv\right]$$
$$-\left(\frac{2\lambda(\lambda+\varphi(\mu,\lambda))}{\varphi(\mu,\lambda)}\right)\int_0^1 w(U_1(v))\,\vartheta(U_1(v))\,dv.$$

By making the substitution $z=U_1(v)$, we get

$$I_1=\left(\frac{\lambda(\lambda+\varphi(\mu,\lambda))}{\varphi(\mu,\lambda)}\right)^2\left[\vartheta(\lambda)+3\vartheta\left(\frac{2\lambda(\lambda+\varphi(\mu,\lambda))}{2\lambda+\varphi(\mu,\lambda)}\right)\right]\int_{\lambda}^{\frac{2\lambda(\lambda+\varphi(\mu,\lambda))}{2\lambda+\varphi(\mu,\lambda)}}\frac{w(z)}{z^2}dz$$
$$-\left(\frac{2\lambda(\lambda+\varphi(\mu,\lambda))}{\varphi(\mu,\lambda)}\right)^2\int_{\lambda}^{\frac{2\lambda(\lambda+\varphi(\mu,\lambda))}{2\lambda+\varphi(\mu,\lambda)}}\frac{w(z)\,\vartheta(z)}{z^2}dz.$$

Similarly, we can have

$$I_2=\int_0^1 p_2(v)\,(U_2(v))^2\,\vartheta'(U_2(v))\,dv$$
$$=\left(\frac{\lambda(\lambda+\varphi(\mu,\lambda))}{\varphi(\mu,\lambda)}\right)^2\left[\vartheta(\lambda+\varphi(\mu,\lambda))+3\vartheta\left(\frac{2\lambda(\lambda+\varphi(\mu,\lambda))}{2\lambda+\varphi(\mu,\lambda)}\right)\right]$$
$$\times\int_{\frac{2\lambda(\lambda+\varphi(\mu,\lambda))}{2\lambda+\varphi(\mu,\lambda)}}^{\lambda+\varphi(\mu,\lambda)}\frac{w(z)}{z^2}dz-\left(\frac{2\lambda(\lambda+\varphi(\mu,\lambda))}{\varphi(\mu,\lambda)}\right)^2\int_{\frac{2\lambda(\lambda+\varphi(\mu,\lambda))}{2\lambda+\varphi(\mu,\lambda)}}^{\lambda+\varphi(\mu,\lambda)}\frac{w(z)\,\vartheta(z)}{z^2}d\lambda.$$

Since $w(z)$ is harmonically symmetric with respect to $\frac{2\lambda(\lambda+\varphi(\mu,\lambda))}{2\lambda+\varphi(\mu,\lambda)}$, we have

$$\int_{\lambda}^{\frac{2\lambda(\lambda+\varphi(\mu,\lambda))}{2\lambda+\varphi(\mu,\lambda)}}\frac{w(z)}{z^2}dz=\int_{\frac{2\lambda(\lambda+\varphi(\mu,\lambda))}{2\lambda+\varphi(\mu,\lambda)}}^{\lambda+\varphi(\mu,\lambda)}\frac{w(z)}{z^2}dz=\frac{1}{2}\int_{\lambda}^{\lambda+\varphi(\mu,\lambda)}\frac{w(z)}{z^2}dz.$$

Thus, we have

$$\frac{\varphi(\mu,\lambda)}{4\lambda(\lambda+\varphi(\mu,\lambda))}(I_1+I_2)$$

$$= \frac{1}{8}\left(\frac{\lambda(\lambda+\varphi(\mu,\lambda))}{\varphi(\mu,\lambda)}\right)\left[\vartheta(\lambda)+6\vartheta\left(\frac{2\lambda(\lambda+\varphi(\mu,\lambda))}{2\lambda+\varphi(\mu,\lambda)}\right)+\vartheta(\lambda+\varphi(\mu,\lambda))\right]$$
$$\times \int_{\lambda}^{\lambda+\varphi(\mu,\lambda)} \frac{w(z)}{z^2}dz - \frac{\lambda(\lambda+\varphi(\mu,\lambda))}{\varphi(\mu,\lambda)} \int_{\lambda}^{\lambda+\varphi(\mu,\lambda)} \frac{w(z)\vartheta(z)}{z^2}dz.$$

Hence the proof of the theorem is done. □

Remark 1. Throughout this manuscript we will use the following notation for the sake of convenience

$$\vartheta(\lambda,\lambda+\varphi(\mu,\lambda);\vartheta,w) = \frac{1}{8}\left(\frac{\lambda(\lambda+\varphi(\mu,\lambda))}{\varphi(\mu,\lambda)}\right)$$
$$\times\left[\vartheta(\lambda)+6\vartheta\left(\frac{2\lambda(\lambda+\varphi(\mu,\lambda))}{2\lambda+\varphi(\mu,\lambda)}\right)+\vartheta(\lambda+\varphi(\mu,\lambda))\right]$$
$$\times \int_{\lambda}^{\lambda+\varphi(\mu,\lambda)} \frac{w(z)}{z^2}dz - \frac{\lambda(\lambda+\varphi(\mu,\lambda))}{\varphi(\mu,\lambda)} \int_{\lambda}^{\lambda+\varphi(\mu,\lambda)} \frac{w(z)\vartheta(z)}{z^2}dz.$$

Corollary 1. *Under the assumptions of Lemma 1, the following inequality holds*

$$\vartheta(\lambda,\lambda+\varphi(\mu,\lambda);\vartheta,w) \le \frac{\varphi(\mu,\lambda)}{4\lambda(\lambda+\varphi(\mu,\lambda))}\|w\|_{[\lambda,\lambda+\varphi(\mu,\lambda)],\infty} \tag{2.2}$$
$$\times\left\{\int_0^1\left(\frac{3}{4}-v\right)(U_1(v))^2\vartheta'(U_1(v))\,dv + \int_0^1\left(v-\frac{3}{4}\right)(U_2(v))^2\vartheta'(U_2(v))\,dv\right\}.$$

Proof. Proof follows from the fact that

$$\|w\|_{\left[\lambda,\frac{2\lambda(\lambda+\varphi(\mu,\lambda))}{2\lambda+\varphi(\mu,\lambda)}\right],\infty} \le \|w\|_{[\lambda,\lambda+\varphi(\mu,\lambda)],\infty}$$

and

$$\|w\|_{\left[\frac{2\lambda(\lambda+\varphi(\mu,\lambda))}{2\lambda+\varphi(\mu,\lambda)},\lambda+\varphi(\mu,\lambda)\right],\infty} \le \|w\|_{[\lambda,\lambda+\varphi(\mu,\lambda)],\infty}.$$

□

Theorem 1. *Let* $\vartheta : I_{h\varphi} \subseteq (0,\infty) \to \mathbb{R}$ *be a differentiable mapping on* $I_{h\varphi}^{\circ}$, *where* $\lambda, \lambda+\varphi(\mu,\lambda) \in I_{h\varphi}^{\circ}$ *with* $\varphi(\mu,\lambda) > 0$ *and let* $w : [\lambda,\lambda+\varphi(\mu,\lambda)] \to [0,\infty)$ *be a continuous positive mapping and harmonically symmetricto* $\frac{2\lambda(\lambda+\varphi(\mu,\lambda))}{2\lambda+\varphi(\mu,\lambda)}$.

If $\vartheta', w \in L_1([\lambda, \lambda+\varphi(\mu,\lambda)])$ *and* $\left|\vartheta'\right|^q$ *is harmonically-preinvex on* $[\lambda, \lambda+\varphi(\mu,\lambda)]$ *for* $q \geq 1$*, then*

$$\begin{aligned}
|\vartheta(\lambda, \lambda+\varphi(\mu,\lambda);\vartheta,w)| &\leq \frac{\varphi(\mu,\lambda)}{4\lambda(\lambda+\varphi(\mu,\lambda))}\|w\|_{[\lambda,\lambda+\varphi(\mu,\lambda)],\infty} \\
&\times\left(\frac{5}{16}\right)^{1-\frac{1}{q}} m_{\varphi}^{\frac{2}{q}}(\lambda,\mu)\left\{\left[o_1\left(n_{\varphi}(\lambda,\mu)\right)\left|\vartheta'(\lambda)\right|^q+o_2\left(n_{\varphi}(\lambda,\mu)\right)\left|\vartheta'(\mu)\right|^q\right]^{\frac{1}{q}}\right. \\
&\left.+\left[o_2\left(-n_{\varphi}(\lambda,\mu)\right)\left|\vartheta'(\lambda)\right|^q+o_1\left(-n_{\varphi}(\lambda,\mu)\right)\left|\vartheta'(\mu)\right|^q\right]^{\frac{1}{q}}\right\},
\end{aligned} \tag{2.3}$$

where

$$\begin{aligned}
o_1\left(n_{\varphi}(\lambda,\mu)\right) &= \frac{3n_{\varphi}^2(\lambda,\mu)-3n_{\varphi}(\lambda,\mu)-4}{8n_{\varphi}^2(\lambda,\mu)\left(1+n_{\varphi}(\lambda,\mu)\right)}+\frac{\left(n_{\varphi}(\lambda,\mu)-8\right)}{8n_{\varphi}^3(\lambda,\mu)}\ln\left(\frac{16\left(1+n_{\varphi}(\lambda,\mu)\right)}{\left(4+3n_{\varphi}(\lambda,\mu)\right)^2}\right), \\
o_2\left(n_{\varphi}(\lambda,\mu)\right) &= \frac{4+3n_{\varphi}(\lambda,\mu)}{8n_{\varphi}^2(\lambda,\mu)}+\frac{\left(8+7n_{\varphi}(\lambda,\mu)\right)}{8n_{\varphi}^3(\lambda,\mu)}\ln\left(\frac{16\left(1+n_{\varphi}(\lambda,\mu)\right)}{\left(4+3n_{\varphi}(\lambda,\mu)\right)^2}\right), \\
n_{\varphi}(\lambda,\mu) &= \frac{\varphi(\mu,\lambda)}{2\lambda+\varphi(\mu,\lambda)} \text{ and } m_{\varphi}(\lambda,\mu)=\frac{2\lambda(\lambda+\varphi(\mu,\lambda))}{2\lambda+\varphi(\mu,\lambda)}.
\end{aligned}$$

Proof. Taking the absolute value on both sides of (2.2) and using the power-mean inequality, we have

$$\begin{aligned}
|\vartheta(\lambda,\lambda+\varphi(\mu,\lambda);\vartheta,w)| &\leq \frac{\varphi(\mu,\lambda)}{4\lambda(\lambda+\varphi(\mu,\lambda))}\|w\|_{[\lambda,\lambda+\varphi(\mu,\lambda)],\infty} \\
&\times\left\{\left(\int_0^1\left|\frac{3}{4}-v\right|dv\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\frac{3}{4}-v\right|(U_1(v))^2\left|\vartheta'(U_1(v))\right|^q dv\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1\left|v-\frac{3}{4}\right|dv\right)^{1-\frac{1}{q}}\left(\int_0^1\left|v-\frac{3}{4}\right|(U_2(v))^2\left|\vartheta'(U_2(v))\right|^q dv\right)^{\frac{1}{q}}\right\}.
\end{aligned} \tag{2.4}$$

By using the harmonically-preinvexity of $\left|\vartheta'\right|^q$ on $[\lambda, \lambda+\varphi(\mu,\lambda)]$ for $q \geq 1$, we get

$$\int_0^1 \left|\frac{3}{4}-v\right| (U_1(v))^2 \left|\vartheta'(U_1(v))\right|^q dv \tag{2.5}$$

$$\leq \left|\vartheta'(\lambda)\right|^q \left[\int_0^{\frac{3}{4}} \left(\frac{3}{4}-v\right)\left(\frac{1+v}{2}\right)(U_1(v))^2 dv + \int_{\frac{3}{4}}^1 \left(v-\frac{3}{4}\right)\left(\frac{1+v}{2}\right)(U_1(v))^2 dv\right]$$

$$+\left|\vartheta'(\mu)\right|^q \left[\int_0^{\frac{3}{4}} \left(\frac{3}{4}-v\right)\left(\frac{1-v}{2}\right)(U_1(v))^2 dv + \int_{\frac{3}{4}}^1 \left(v-\frac{3}{4}\right)\left(\frac{1-v}{2}\right)(U_1(v))^2 dv\right]$$

$$= m_\varphi^2(\lambda,\mu)\left[\frac{3n_\varphi(\lambda,\mu)^2-3n_\varphi(\lambda,\mu)-4}{8n_\varphi^2(\lambda,\mu)\left(1+n_\varphi(\lambda,\mu)\right)} + \frac{\left(n_\varphi(\lambda,\mu)-8\right)}{8n_\varphi^3(\lambda,\mu)}\ln\left(\frac{16\left(1+n_\varphi(\lambda,\mu)\right)}{\left(4+3n_\varphi(\lambda,\mu)\right)^2}\right)\right]$$

$$\times \left|\vartheta'(\lambda)\right|^q + m_\varphi^2(\lambda,\mu)\left[\frac{4+3n_\varphi(\lambda,\mu)}{8n_\varphi^2(\lambda,\mu)} + \frac{\left(8+7n_\varphi(\lambda,\mu)\right)}{8n_\varphi^3(\lambda,\mu)}\ln\left(\frac{16\left(1+n_\varphi(\lambda,\mu)\right)}{\left(4+3n_\varphi(\lambda,\mu)\right)^2}\right)\right]\left|\vartheta'(\mu)\right|^q$$

and

$$\int_0^1 \left|v-\frac{3}{4}\right| (U_2(v))^2 \left|\vartheta'(U_2(v))\right|^q dv \tag{2.6}$$

$$\leq \left|\vartheta'(\lambda)\right|^q \left[\int_0^{\frac{3}{4}} \left(\frac{3}{4}-v\right)\left(\frac{1-v}{2}\right)(U_2(v))^2 dv. + \int_{\frac{3}{4}}^1 \left(v-\frac{3}{4}\right)\left(\frac{1-v}{2}\right)(U_2(v))^2 dv\right]$$

$$+\left|\vartheta'(\mu)\right|^q \left[\int_0^{\frac{3}{4}} \left(\frac{3}{4}-v\right)\left(\frac{1+v}{2}\right)(U_2(v))^2 dv + \int_{\frac{3}{4}}^1 \left(v-\frac{3}{4}\right)\left(\frac{1+v}{2}\right)(U_2(v))^2 dv\right]$$

$$= m_\varphi^2(\lambda,\mu)\left[\frac{4-3n_\varphi(\lambda,\mu)}{8n_\varphi^2(\lambda,\mu)} - \frac{\left(8-7n_\varphi(\lambda,\mu)\right)}{8n_\varphi^3(\lambda,\mu)}\ln\left(\frac{16\left(1-n_\varphi(\lambda,\mu)\right)}{\left(4-3n_\varphi(\lambda,\mu)\right)^2}\right)\right]\left|\vartheta'(\lambda)\right|^q$$

$$+ m_\varphi^2(\lambda,\mu)\left[\frac{3n_\varphi^2(\lambda,\mu)+3n_\varphi(\lambda,\mu)-4}{8n_\varphi^2(\lambda,\mu)\left(1-n_\varphi(\lambda,\mu)\right)} + \frac{\left(n_\varphi(\lambda,\mu)+8\right)}{8n_\varphi^3(\lambda,\mu)}\ln\left(\frac{16\left(1-n_\varphi(\lambda,\mu)\right)}{\left(4-3n_\varphi(\lambda,\mu)\right)^2}\right)\right]\left|\vartheta'(\mu)\right|^q,$$

where

$$n_\varphi(\lambda,\mu) = \frac{\varphi(\mu,\lambda)}{2\lambda+\varphi(\mu,\lambda)}$$

and

$$m_\varphi(\lambda,\mu) = \frac{2\lambda(\lambda+\varphi(\mu,\lambda))}{2\lambda+\varphi(\mu,\lambda)}.$$

Using (2.5) and (2.6) in (2.4) we get (2.3). □

Theorem 2. *Let $\vartheta : I_{h\varphi} \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable mapping on $I^{\circ}_{h\varphi}$, where $\lambda, \lambda+\varphi(\mu,\lambda) \in I^{\circ}_{h\varphi}$ with $\varphi(\mu,\lambda) > 0$ and let $w : [\lambda, \lambda+\varphi(\mu,\lambda)] \to [0,\infty)$ be a continuous positive mapping and harmonically symmetric to $\frac{2\lambda(\lambda+\varphi(\mu,\lambda))}{2\lambda+\varphi(\mu,\lambda)}$. If $\vartheta', w \in L_1([\lambda, \lambda+\varphi(\mu,\lambda)])$ and $\left|\vartheta'\right|^q$ is harmonically-preinvex on $[\lambda, \lambda+\varphi(\mu,\lambda)]$ for $q > 1$, then*

$$|\vartheta(\lambda, \lambda + \varphi(\mu,\lambda); \vartheta, w)| \leq \frac{m^2_{\varphi}(\lambda,\mu)\,\varphi(\mu,\lambda)}{4\lambda(\lambda+\varphi(\mu,\lambda))} \|w\|_{[\lambda,\lambda+\varphi(\mu,\lambda)],\infty} \left(\frac{1-q}{1+q}\right)^{1-\frac{1}{q}} \tag{2.7}$$
$$\times \left\{ \left[\rho_1\left(n_{\varphi}(\lambda,\mu)\right)\right]^{1-\frac{1}{q}} \left[\rho_2(q)\left|\vartheta'(\lambda)\right|^q + \rho_3(q)\left|\vartheta'(\mu)\right|^q\right]^{\frac{1}{q}} \right.$$
$$\left. + \left[\rho_1\left(-n_{\varphi}(\lambda,\mu)\right)\right]^{1-\frac{1}{q}} \left[\rho_3(q)\left|\vartheta'(\lambda)\right|^q + \rho_2(q)\left|\vartheta'(\mu)\right|^q\right]^{\frac{1}{q}} \right\},$$

where

$$\rho_1\left(n_{\varphi}(\lambda,\mu)\right) = n^{-1}_{\varphi}(\lambda,\mu)\left(\left(1+n_{\varphi}(\lambda,\mu)\right)^{\frac{1+q}{1-q}} - 1\right),$$
$$\rho_2(q) = \frac{2^{-2q-5}(8q+15) + 2^{-2q-5}\times 3^{q+1}(4q+11)}{(q+1)(q+2)},$$
$$\rho_3(q) = \frac{2^{-2q-5} + 2^{-2q-5}\times 3^{q+1}(4q+5)}{(q+1)(q+2)},$$
$$n_{\varphi}(\lambda,\mu) = \frac{\varphi(\mu,\lambda)}{2\lambda+\varphi(\mu,\lambda)} \text{ and } m_{\varphi}(\lambda,\mu) = \frac{2\lambda(\lambda+\varphi(\mu,\lambda))}{2\lambda+\varphi(\mu,\lambda)}.$$

Proof. Taking the absolute value on both sides of (2.2) and using the Hölder integral inequality, we have

$$|\vartheta(\lambda, \lambda + \varphi(\mu,\lambda); \vartheta, w)| \leq \frac{\varphi(\mu,\lambda)}{4\lambda(\lambda+\varphi(\mu,\lambda))} \|w\|_{[\lambda,\lambda+\varphi(\mu,\lambda)],\infty} \tag{2.8}$$
$$\times \left\{ \left(\int_0^1 (U_1(v))^{\frac{2q}{q-1}} dv\right)^{1-\frac{1}{q}} \left(\int_0^1 \left|\frac{3}{4} - v\right|^q \left|\vartheta'(U_1(v))\right|^q dv\right)^{\frac{1}{q}} \right.$$
$$\left. + \left(\int_0^1 (U_2(v))^{\frac{2q}{q-1}} dv\right)^{1-\frac{1}{q}} \left(\int_0^1 \left|v - \frac{3}{4}\right|^q \left|\vartheta'(U_2(v))\right|^q dv\right)^{\frac{1}{q}} \right\}.$$

By using the harmonically-preinvexity of $\left|\vartheta'\right|^q$ on $[\lambda, \lambda+\varphi(\mu,\lambda)]$ for $q \geq 1$, we get

$$\int_0^1 \left|\frac{3}{4}-v\right|^q \left|\vartheta'(U_1(v))\right|^q dv \tag{2.9}$$

$$\leq \left|\vartheta'(\lambda)\right|^q \left[\int_0^{\frac{3}{4}} \left(\frac{3}{4}-v\right)^q \left(\frac{1+v}{2}\right) dv + \int_{\frac{3}{4}}^1 \left(v-\frac{3}{4}\right)^q \left(\frac{1+v}{2}\right) dv\right]$$

$$+\left|\vartheta'(\mu)\right|^q \left[\int_0^{\frac{3}{4}} \left(\frac{3}{4}-v\right)^q \left(\frac{1-v}{2}\right) dv + \int_{\frac{3}{4}}^1 \left(v-\frac{3}{4}\right)^q \left(\frac{1-v}{2}\right) dv\right]$$

$$= \left(\frac{2^{-2q-5}\times 3^{q+1}(4q+11)+2^{-2q-5}(8q+15)}{(q+1)(q+2)}\right)\left|\vartheta'(\lambda)\right|^q$$

$$+\left(\frac{2^{-2q-5}+2^{-2q-5}\times 3^{q+1}(4q+5)}{(q+1)(q+2)}\right)\left|\vartheta'(\mu)\right|^q$$

and

$$\int_0^1 \left|v-\frac{3}{4}\right|^q \left|\vartheta'(U_2(v))\right|^q dv \tag{2.10}$$

$$\leq \left|\vartheta'(\lambda)\right|^q \left[\int_0^{\frac{3}{4}} \left(\frac{3}{4}-v\right)^q \left(\frac{1-v}{2}\right) dv + \int_{\frac{3}{4}}^1 \left(v-\frac{3}{4}\right)^q \left(\frac{1-v}{2}\right) dv\right]$$

$$+\left|\vartheta'(\mu)\right|^q \left[\int_0^{\frac{3}{4}} \left(\frac{3}{4}-v\right)^q \left(\frac{1+v}{2}\right) dv + \int_{\frac{3}{4}}^1 \left(v-\frac{3}{4}\right)^q \left(\frac{1+v}{2}\right) dv\right]$$

$$= \left(\frac{2^{-2q-5}\times 3^{q+1}(4q+5)+2^{-2q-5}}{(q+1)(q+2)}\right)\left|\vartheta'(\lambda)\right|^q$$

$$+\left(\frac{2^{-2q-5}(8q+15)+2^{-2q-5}\times 3^{q+1}(4q+11)}{(q+1)(q+2)}\right)\left|\vartheta'(\mu)\right|^q. \tag{2.11}$$

We also observe that

$$\int_0^1 (U_1(v))^{\frac{2q}{q-1}}\,dv = \left(\frac{1-q}{1+q}\right) m_\varphi^{\frac{2q}{q-1}}(\lambda,\mu)\, n_\varphi^{-1}(\lambda,\mu)\left(\left(1+n_\varphi(\lambda,\mu)\right)^{\frac{1+q}{1-q}}-1\right) \tag{2.12}$$

and

$$\int_0^1 (U_2(v))^{\frac{2q}{q-1}}\,dv = \left(\frac{1-q}{1+q}\right) m_\varphi^{\frac{2q}{q-1}}(\lambda,\mu)\, n_\varphi^{-1}(\lambda,\mu)\left(1-\left(1-n_\varphi(\lambda,\mu)\right)^{\frac{1+q}{1-q}}\right), \tag{2.13}$$

where

$$n_\varphi(\lambda,\mu) = \frac{\varphi(\mu,\lambda)}{2\lambda+\varphi(\mu,\lambda)} \text{ and } m_\varphi(\lambda,\mu) = \frac{2\lambda(\lambda+\varphi(\mu,\lambda))}{2\lambda+\varphi(\mu,\lambda)}.$$

Applying (2.9)-(2.13) in (2.8), we get (2.7). □

Theorem 3. *Let $\vartheta : I_{h\varphi} \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable mapping on $I_{h\varphi}^{\circ}$, where λ, $\lambda+\varphi(\mu,\lambda) \in I_{h\varphi}^{\circ}$ with $\varphi(\mu,\lambda) > 0$ and let $w : [\lambda,\lambda+\varphi(\mu,\lambda)] \to [0,\infty)$ be a continuous positive mapping and harmonically symmetric to $\frac{2\lambda(\lambda+\varphi(\mu,\lambda))}{2\lambda+\varphi(\mu,\lambda)}$. If $\vartheta', w \in L_1([\lambda,\lambda+\varphi(\mu,\lambda)])$ and $\left|\vartheta'\right|^q$ is harmonically-preinvex on $[\lambda,\lambda+\varphi(\mu,\lambda)]$ for $q>1$, then*

$$\begin{aligned} |\vartheta(\lambda,\lambda+\varphi(\mu,\lambda);\vartheta,w)| &\le \frac{m_\varphi^2(\lambda,\mu)\,\varphi(\mu,\lambda)}{4\lambda(\lambda+\varphi(\mu,\lambda))}\|w\|_{[\lambda,\lambda+\varphi(\mu,\lambda)],\infty}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}} \\ &\times\left[4^{-\frac{2q-1}{q-1}}\left(3^{\frac{2q-1}{q-1}}+1\right)\right]^{1-\frac{1}{q}}\left(\frac{1}{4(2q-1)(q-1)\mu^2}\right)^{\frac{1}{q}} \\ &\times\left\{\left[\rho_1\left(n_\varphi(\lambda,\mu)\right)\left|\vartheta'(\lambda)\right|^q+\rho_2\left(n_\varphi(\lambda,\mu)\right)\left|\vartheta'(\mu)\right|^q\right]^{\frac{1}{q}}\right. \\ &\left.+\left[\rho_2\left(-n_\varphi(\lambda,\mu)\right)\left|\vartheta'(\lambda)\right|^q+\rho_1\left(-n_\varphi(\lambda,\mu)\right)\left|\vartheta'(\mu)\right|^q\right]^{\frac{1}{q}}\right\}, \end{aligned} \tag{2.14}$$

where

$$\rho_1\left(n_\varphi(\lambda,\mu)\right) = 1+2(q-1)n_\varphi(\lambda,\mu)-\left(1+n_\varphi(\lambda,\mu)\right)^{1-2q}\left(1+(4q-3)n_\varphi(\lambda,\mu)\right),$$

$$\rho_2\left(n_\varphi(\lambda,\mu)\right) = \left(1+n_\varphi(\lambda,\mu)\right)^{2-2q}+2(q-1)n_\varphi(\lambda,\mu)-1$$

$$n_\varphi(\lambda,\mu) = \frac{\varphi(\mu,\lambda)}{2\lambda+\varphi(\mu,\lambda)} \text{ and } m_\varphi(\lambda,\mu) = \frac{2\lambda(\lambda+\varphi(\mu,\lambda))}{2\lambda+\varphi(\mu,\lambda)}.$$

Proof. Taking the absolute value on both sides of (2.2) and using the Hölder integral inequality, we have

$$
\begin{aligned}
|\vartheta(\lambda,\lambda+\varphi(\mu,\lambda);\vartheta,w)| \leq{} & \frac{\varphi(\mu,\lambda)}{4\lambda(\lambda+\varphi(\mu,\lambda))}\|w\|_{[\lambda,\lambda+\varphi(\mu,\lambda)],\infty} \\
& \times\left\{\left(\int_0^1\left|\frac{3}{4}-v\right|^{\frac{q}{q-1}}dv\right)^{1-\frac{1}{q}}\left(\int_0^1(U_1(v))^{2q}\left|\vartheta'(U_1(v))\right|^q dv\right)^{\frac{1}{q}}\right. \\
& \left.+\left(\int_0^1\left|v-\frac{3}{4}\right|^{\frac{q}{q-1}}dv\right)^{1-\frac{1}{q}}\left(\int_0^1(U_2(v))^{2q}\left|\vartheta'(U_2(v))\right|^q dv\right)^{\frac{1}{q}}\right\}.
\end{aligned}
\tag{2.15}
$$

By using the harmonically-preinvexity of $\left|\vartheta'\right|^q$ on $[\lambda,\lambda+\varphi(\mu,\lambda)]$ for $q\geq 1$, we get

$$
\begin{aligned}
&\int_0^1(U_1(v))^{2q}\left|\vartheta'(U_1(v))\right|^q dv \\
&\leq\left|\vartheta'(\lambda)\right|^q\int_0^1\left(\frac{1+v}{2}\right)(U_1(v))^{2q}dv+\left|\vartheta'(\mu)\right|^q\int_0^1\left(\frac{1-v}{2}\right)(U_1(v))^{2q}dv \\
&=\frac{m_\varphi^{2q}(\lambda,\mu)\left[1+2(q-1)n_\varphi(\lambda,\mu)-(1+n_\varphi(\lambda,\mu))^{1-2q}(1+(4q-3)n_\varphi(\lambda,\mu))\right]}{4(2q-1)(q-1)n_\varphi^2(\lambda,\mu)} \\
&\times\left|\vartheta'(\lambda)\right|^q+\frac{m_\varphi^{2q}(\lambda,\mu)\left[(1+n_\varphi(\lambda,\mu))^{2-2q}+2(q-1)n_\varphi(\lambda,\mu)-1\right]}{4(2q-1)(q-1)n_\varphi^2(\lambda,\mu)}\left|\vartheta'(\mu)\right|^q
\end{aligned}
\tag{2.16}
$$

and

$$
\begin{aligned}
&\int_0^1(U_2(v))^{2q}\left|\vartheta'(U_2(v))\right|^q dv \\
&\leq\left|\vartheta'(\lambda)\right|^q\int_0^1\left(\frac{1-v}{2}\right)(U_2(v))^{2q}dv+\left|\vartheta'(\mu)\right|^q\int_0^1\left(\frac{1+v}{2}\right)(U_2(v))^{2q}dv \\
&=\frac{m_\varphi^{2q}(\lambda,\mu)\left[(1-n_\varphi(\lambda,\mu))^{2-2q}+2(1-q)n_\varphi(\lambda,\mu)-1\right]}{4(2q-1)(q-1)n_\varphi^2(\lambda,\mu)}\left|\vartheta'(\lambda)\right|^q+m_\varphi^{2q}(\lambda,\mu) \\
&\times\frac{\left[1+2(1-q)n_\varphi(\lambda,\mu)-(1-n_\varphi(\lambda,\mu))^{1-2q}(1+(3-4q)n_\varphi(\lambda,\mu))\right]}{4(2q-1)(q-1)n_\varphi^2(\lambda,\mu)}\left|\vartheta'(\mu)\right|^q.
\end{aligned}
\tag{2.17}
$$

We notice that

$$\int_0^1 \left|\frac{3}{4}-v\right|^{\frac{q}{q-1}} dv = \int_0^1 \left|v-\frac{3}{4}\right|^{\frac{q}{q-1}} dv = 4^{-\frac{2q-1}{q-1}}\left(\frac{q-1}{2q-1}\right)\left(3^{\frac{2q-1}{q-1}}+1\right). \tag{2.18}$$

Applying (2.16)-(2.18) in (2.15), we get (2.14). □

Theorem 4. *Let $\vartheta : I_{h\varphi} \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable mapping on $I_{h\varphi}^{\circ}$, where λ, $\lambda+\varphi(\mu,\lambda) \in I_{h\varphi}^{\circ}$ with $\varphi(\mu,\lambda) > 0$ and let $w : [\lambda,\lambda+\varphi(\mu,\lambda)] \to [0,\infty)$ be a continuous positive mapping and harmonically symmetric to $\frac{2\lambda(\lambda+\varphi(\mu,\lambda))}{2\lambda+\varphi(\mu,\lambda)}$. If $\vartheta', w \in L_1([\lambda,\lambda+\varphi(\mu,\lambda)])$ and $\left|\vartheta'\right|^q$ is harmonically-preinvex on $[\lambda,\lambda+\varphi(\mu,\lambda)]$ for $q \geq 1$, then*

$$\begin{aligned} &|\vartheta(\lambda\,,\,\lambda+\varphi(\mu,\lambda);\vartheta,w)| \\ &\leq \frac{m_\varphi^2(\lambda,\mu)\,\varphi(\mu,\lambda)}{4\lambda(\lambda+\varphi(\mu,\lambda))}\|w\|_{[\lambda,\lambda+\varphi(\mu,\lambda)],\infty}\left\{\left[\sigma_1\left(n_\varphi(\lambda,\mu)\right)\right]^{1-\frac{1}{q}}\right. \\ &\times\left[o_1\left(n_\varphi(\lambda,\mu)\right)\left|\vartheta'(\lambda)\right|^q + o_2\left(n_\varphi(\lambda,\mu)\right)\left|\vartheta'(\mu)\right|^q\right]^{\frac{1}{q}} + \left[\sigma_1\left(-n_\varphi(\lambda,\mu)\right)\right]^{1-\frac{1}{q}} \\ &\times\left.\left[o_2\left(-n_\varphi(\lambda,\mu)\right)\left|\vartheta'(\lambda)\right|^q + o_1\left(-n_\varphi(\lambda,\mu)\right)\left|\vartheta'(\mu)\right|^q\right]^{\frac{1}{q}}\right\}, \end{aligned} \tag{2.19}$$

where

$$\sigma_1\left(n_\varphi(\lambda,\mu)\right) = \frac{3n_\varphi(\lambda,\mu)+2}{4n_\varphi(\lambda,\mu)(1+n_\varphi(\lambda,\mu))} + \frac{1}{n_\varphi^2(\lambda,\mu)}\ln\left(\frac{16(1+n_\varphi(\lambda,\mu))}{(4+3n_\varphi(\lambda,\mu))^2}\right),$$

$$n_\varphi(\lambda,\mu) = \frac{\varphi(\mu,\lambda)}{2\lambda+\varphi(\mu,\lambda)} \text{ and } m_\varphi(\lambda,\mu) = \frac{2\lambda(\lambda+\varphi(\mu,\lambda))}{2\lambda+\varphi(\mu,\lambda)}.$$

Proof. Taking the absolute value on both sides of (2.2) and using the power-mean inequality, we have

$$\begin{aligned} &|\vartheta(\lambda\,,\,\lambda+\varphi(\mu,\lambda);\vartheta,w)| \leq \frac{\varphi(\mu,\lambda)}{4\lambda(\lambda+\varphi(\mu,\lambda))}\|w\|_{[\lambda,\lambda+\varphi(\mu,\lambda)],\infty} \\ &\times\left\{\left(\int_0^1\left|\frac{3}{4}-v\right|(U_1(v))^2dv\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\frac{3}{4}-v\right|(U_1(v))^2\left|\vartheta'(U_1(v))\right|^q dv\right)^{\frac{1}{q}}\right. \\ &+\left.\left(\int_0^1\left|v-\frac{3}{4}\right|(U_2(v))^2dv\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\frac{3}{4}-v\right|(U_2(v))^2\left|\vartheta'(U_2(v))\right|^q dv\right)^{\frac{1}{q}}\right\}. \end{aligned} \tag{2.20}$$

By using the harmonically-preinvexity of $\left|\vartheta'\right|^q$ on $[\lambda, \lambda+\varphi(\mu,\lambda)]$ for $q \geq 1$, we get

$$\int_0^1 \left|\frac{3}{4}-v\right| (U_1(v))^2 \left|\vartheta'(U_1(v))\right|^q dv \tag{2.21}$$
$$\leq \left|\vartheta'(\lambda)\right|^q \left[\int_0^{\frac{3}{4}} \left(\frac{3}{4}-v\right)\left(\frac{1+v}{2}\right)(U_1(v)))^2 dv + \int_{\frac{3}{4}}^1 \left(v-\frac{3}{4}\right)\left(\frac{1+v}{2}\right)(U_1(v))^2 dv\right]$$
$$+\left|\vartheta'(\mu)\right|^q \left[\int_0^{\frac{3}{4}} \left(\frac{3}{4}-v\right)\left(\frac{1-v}{2}\right)(U_1(v))^2 dv + \int_{\frac{3}{4}}^1 \left(v-\frac{3}{4}\right)\left(\frac{1-v}{2}\right)(U_1(v))^2 dv\right]$$
$$=\left[\frac{3n_\varphi^2(\lambda,\mu)-3n_\varphi(\lambda,\mu)-4}{8n_\varphi(\lambda,\mu)^2\left(1+n_\varphi(\lambda,\mu)\right)}+\frac{\left(n_\varphi(\lambda,\mu)-8\right)}{8n_\varphi^3(\lambda,\mu)}\ln\left(\frac{16\left(1+n_\varphi(\lambda,\mu)\right)}{\left(4+3n_\varphi(\lambda,\mu)\right)^2}\right)\right]$$
$$\times m_\varphi^2(\lambda,\mu)\left|\vartheta'(\lambda)\right|^q + m_\varphi^2(\lambda,\mu)\left[\frac{4+3n_\varphi(\lambda,\mu)}{8n_\varphi^2(\lambda,\mu)}\right.$$
$$\left.+\frac{\left(8+7n_\varphi(\lambda,\mu)\right)}{8n_\varphi^3(\lambda,\mu)}\ln\left(\frac{16\left(1+n_\varphi(\lambda,\mu)\right)}{\left(4+3n_\varphi(\lambda,\mu)\right)^2}\right)\right]\left|\vartheta'(\mu)\right|^q$$

and

$$\int_0^1 \left|v-\frac{3}{4}\right| (U_2(v))^2 \left|\vartheta'(U_2(v))\right|^q dv$$
$$\leq \left|\vartheta'(\lambda)\right|^q \left[\int_0^{\frac{3}{4}} \left(\frac{3}{4}-v\right)\left(\frac{1-v}{2}\right)(U_2(v))^2 dv + \int_{\frac{3}{4}}^1 \left(v-\frac{3}{4}\right)\left(\frac{1-v}{2}\right)(U_2(v))^2 dv\right]$$
$$+\left|\vartheta'(\mu)\right|^q \left[\int_0^{\frac{3}{4}} \left(\frac{3}{4}-v\right)\left(\frac{1+v}{2}\right)(U_2(v))^2 dv + \int_{\frac{3}{4}}^1 \left(v-\frac{3}{4}\right)\left(\frac{1+v}{2}\right)(U_2(v))^2 dv\right]$$
$$=\left[\frac{4-3n_\varphi(\lambda,\mu)}{8n_\varphi^2(\lambda,\mu)}-\frac{\left(8-7n_\varphi(\lambda,\mu)\right)}{8n_\varphi^3(\lambda,\mu)}\ln\left(\frac{16\left(1-n_\varphi(\lambda,\mu)\right)}{\left(4-3n_\varphi(\lambda,\mu)\right)^2}\right)\right]$$

$$\times m_{\varphi}^{2}(\lambda,\mu)\left|\vartheta'(\lambda)\right|^{q}+m_{\varphi}^{2}(\lambda,\mu)\left[\frac{3n_{\varphi}(\lambda,\mu)^{2}+3n_{\varphi}(\lambda,\mu)-4}{8n_{\varphi}^{2}(\lambda,\mu)\left(1-n_{\varphi}(\lambda,\mu)\right)}\right. \tag{2.22}$$
$$\left.+\frac{\left(n_{\varphi}(\lambda,\mu)+8\right)}{8n_{\varphi}^{3}(\lambda,\mu)}\ln\left(\frac{16\left(1-n_{\varphi}(\lambda,\mu)\right)}{\left(4-3n_{\varphi}(\lambda,\mu)\right)^{2}}\right)\right]\left|\vartheta'(\mu)\right|^{q}.$$

We also have

$$\int_{0}^{1}\left|\frac{3}{4}-v\right|(U_1(v))^2dv=\int_{0}^{\frac{3}{4}}\left(\frac{3}{4}-v\right)(U_1(v))^2dv+\int_{\frac{3}{4}}^{1}\left(v-\frac{3}{4}\right)(U_1(v))^2dv \tag{2.23}$$
$$=m_{\varphi}^{2}(\lambda,\mu)\left[\frac{3n_{\varphi}(\lambda,\mu)+2}{4n_{\varphi}(\lambda,\mu)\left(1+n_{\varphi}(\lambda,\mu)\right)}+\frac{1}{n_{\varphi}^{2}(\lambda,\mu)}\ln\left(\frac{16(1+n_{\varphi}(\lambda,\mu))}{(4+3n_{\varphi}(\lambda,\mu))^{2}}\right)\right]$$

and

$$\int_{0}^{1}\left|\frac{3}{4}-v\right|(U_2(v))^2dv=\int_{0}^{\frac{3}{4}}\left(\frac{3}{4}-v\right)(U_2(v))^2dv+\int_{\frac{3}{4}}^{1}\left(v-\frac{3}{4}\right)(U_2(v))^2dv \tag{2.24}$$
$$=m_{\varphi}^{2}(\lambda,\mu)\left[\frac{3n_{\varphi}(\lambda,\mu)-2}{4n_{\varphi}(\lambda,\mu)\left(1-n_{\varphi}(\lambda,\mu)\right)}+\frac{1}{n_{\varphi}^{2}(\lambda,\mu)}\ln\left(\frac{16(1-n_{\varphi}(\lambda,\mu))}{(4-3n_{\varphi}(\lambda,\mu))^{2}}\right)\right],$$

where

$$n_{\varphi}(\lambda,\mu)=\frac{\varphi(\mu,\lambda)}{2\lambda+\varphi(\mu,\lambda)}\text{ and }m_{\varphi}(\lambda,\mu)=\frac{2\lambda(\lambda+\varphi(\mu,\lambda))}{2\lambda+\varphi(\mu,\lambda)}.$$

Using (2.21)-(2.24) in (2.20) we get (2.19). □

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