Miskolc Mathematical Notes
Vol. 24 (2023), No. 1, pp. 139–152
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2023.3099

ANSWERS TO GEORGE-RADENOVIĆ-RESHMA-SHUKLA QUESTIONS IN RECTANGULAR b-METRIC SPACES

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Received 27 October, 2019

Abstract. In this paper, we prove two new general fixed point theorems in rectangular b-metric spaces. As applications, we answer two open questions in rectangular b-metric spaces posed in [12].

2010 *Mathematics Subject Classification:* 47H10; 54H25; 54D99; 54E99

Keywords: rectangular b-metric, fixed point

1. Introduction and preliminaries

There have been many generalizations of a metric. One of the most interesting generalizations is a quasi-distance D on a set X, where the triangle inequality is replaced by

$$D(x,z) \leq \kappa[D(x,y)+D(y,z)]$$

for all $x,y,z \in X$ and some constant $\kappa \geq 1$, see for example [21, page 257]. After that Czerwik used the name *b-metric space* for a set with a quasi-distance [6, 7]. In 2010 Khamsi and Hussain [17] reintroduced the notion of a b-metric under the name *metric-type*. Khamsi [16] also introduced another definition of a metric-type, that was called an *s-relaxed*$_p$ *metric* in [11, Definition 4.2]. The first fixed point theorems in b-metric spaces were proved by Bakhtin [1], and Czerwik [6]. Kirk and Shahzad essentially used Czerwik's technique to prove a general fixed point theorem [18, Theorem 12.2]. Recently, Kajántó and Lukács [15] pointed out and corrected an inaccuracy in the proof of [6, Theorem 1]. Kirk-Shahzad theorem was also used to answer the early stated question on transforming fixed point theorems in metric spaces to fixed point theorems in b-metric spaces [9, Theorem 2.1]. Bessenyei and Páles [2] introduced the notion of a triangle function and extended the Banach contraction principle in this spirit for such complete semi-metric spaces that fulfil an extra regularity property. Kirk and Shahzad [19] introduced a strengthening of a b-metric space, called a *strong b-metric space*, and examined instances in which this notion plays a critical role. Miculescu and Mihail also proved a fixed point theorem for φ-contraction but their main result requires the continuity of the given map [22, Theorem 3.1]. In fact, a very general result was proved by Bessenyei and Páles [2] in regular semi-metric spaces.

In 2000, Branciari [3] introduced the notion of a ν-generalized metric space. A 2-generalized metric space was also called a *generalized metric space*, or for short, *g.m.s* [3, Definition 1.1], or *rectangular metric space* [10, Definition 1]. ν-generalized metric spaces were investigated and fixed point theorems in such spaces were stated, see [13, 14] and references therein. For ν-generalized metrics that being not metrics, see [3, 3. An example], [8, Examples 1 & 2], [13, Examples 2.1 & 4.1].

Motivated by b-metric spaces and rectangular metric spaces, George *et al.* [12] introduced the notion of a rectangular b-metric space. This notion was also introduced independently by Roshan *et al.* in [25]. The convergence, Cauchy sequence and completeness in rectangular b-metric spaces were defined similarly to that in metric spaces.

Recently some results in b-metric spaces and in rectangular b-metric spaces were stated, see [5, 24] and the references therein. However there are open questions relating to such spaces, see Question 1 and Question 2 below. Note that Question 1 was answered positively recently in [23, Theorem 2.1] by direct proof. Also, a partial answer to Question 2 was presented on [20, Theorem 3.2] that an analogue of Reich contraction in rectangular b-metric spaces was proved. In the proof on [20, page 85], the author claimed $\lim\limits_{n\to\infty} d(x_{n+1}, Tx^*) = d(x^*, Tx^*)$ provided that $\lim\limits_{n\to\infty} x_n = x^*$. Unfortunately, this claim does not hold since the rectangular b-metric is not continuous. A similar flaw also appeared in the proof of [20, Theorem 3.1] for the case b-metric. Indeed, the conclusion in [20, Theorem 3.1] does not hold which was proved in [9, Remark 2.7].

In this paper, we are interested in studying fixed point theorems in rectangular b-metric spaces. We prove two fixed point theorems in rectangular b-metric spaces. Using these theorems, we give answers to Question 1 and Question 2.

Now we recall notions and properties which are useful in the latter.

Definition 1 ([7], page 263)**.** Let X be a nonempty set, $\kappa \geq 1$ and $D\colon X \times X \to [0,\infty)$ be a function such that for all $x, y, z \in X$,

(1) $D(x,y) = 0$ if and only if $x = y$.
(2) $D(x,y) = D(y,x)$.
(3) $D(x,z) \leq \kappa[D(x,y) + D(y,z)]$.

Then D is called a *b-metric* on X and (X, D, κ) is called a *b-metric space*.

Definition 2 ([16], Definition 2.7)**.** Let X be a nonempty set, $\kappa \geq 1$ and $D\colon X \times X \to [0,\infty)$ be a function such that for all $n \in \mathbb{N}$ and all $x, y_1, \ldots, y_n, z \in X$,

(1) $D(x,y) = 0$ if and only if $x = y$.
(2) $D(x,y) = D(y,x)$.
(3) $D(x,z) \leq \kappa[D(x,y_1) + \ldots + D(y_n,z)]$.

Then D is called a *metric-type* on X and (X, D, κ) is called a *metric-type space*.

Definition 3 ([12], Definition 1.3)**.** Let X be a nonempty set, $\kappa \geq 1$ and $D\colon X \times X \to [0,\infty)$ be a function such that for all $x, y \in X$, all distinct points $u, v \notin \{x, y\}$,

(1) $D(x,y) = 0$ if and only if $x = y$.
(2) $D(x,y) = D(y,x)$.
(3) $D(x,y) \leq \kappa[D(x,u) + D(u,v) + D(v,y)]$.

Then D is called a *rectangular b-metric* on X and (X,D,κ) is called a *rectangular b-metric space*.

Note that κ in Definitions 1-3 is always assumed to be the smallest possible value, and it is also called the *coefficient* of the corresponding distance function.

Definition 4 ([12], Definition 1.6)**.** Let (X,D,κ) be a rectangular b-metric space.

(1) A sequence $\{x_n\}$ is called *convergent* to x, written as $\lim\limits_{n\to\infty} x_n = x$, if $\lim\limits_{n\to\infty} D(x_n, x) = 0$.
(2) A sequence $\{x_n\}$ is called *Cauchy* if $\lim\limits_{n,m\to\infty} D(x_n, x_m) = 0$.
(3) (X,D,κ) is called *complete* if each Cauchy sequence is a convergent sequence.

Theorem 1 ([12], Theorem 2.1)**.** *Let (X,D,κ) be a complete rectangular b-metric space and $f\colon X \to X$ be a map such that for all $x, y \in X$ and for some $\lambda \in [0, \frac{1}{\kappa})$,*

$$D(f(x), f(y)) \leq \lambda D(x,y)$$

Then f has a unique fixed point $x^* \in X$.

Question 1 ([12], Open problem (1) on page 1012)**.** *In Theorem 1, can we extent the range of λ to the case $\frac{1}{\kappa} \leq \lambda < 1$?*

Question 2 ([12], Open problem (2) on page 1012)**.** *Prove the analogue of Chatterjea contraction, Reich contraction, Ćirić contraction and Hardy-Rogers contraction in rectangular b-metric spaces.*

Definition 5 ([2], page 516)**.** Let (X,d) be a semi-metric space. A function $\Phi\colon [0,\infty] \times [0,\infty] \longrightarrow [0,\infty]$ is called a *triangle function* for d if Φ is increasing in each of its variables, $\Phi(0,0) = 0$ and for all $x, y, z \in X$,

$$d(x,y) \leq \Phi(d(x,z), d(z,y)).$$

Lemma 1 ([2], page 516)**.** *Let (X,d) be a semi-metric space and for all $u, v \in [0,\infty]$,*

$$\Phi_d(u,v) = \sup\{d(x,y) : \exists p \in X, d(p,x) \leq u, d(p,y) \leq v\}.$$

Then Φ_d is a triangle function for d. Moreover, if Φ is a triangle function for d then $\Phi_d \leq \Phi$.

Definition 6 ([2], page 516)**.** Let (X,d) be a semi-metric space. Then the triangle function Φ_d defined as in Lemma 1 is called the *basic triangle function* and (X,d) is called *regular* if Φ_d is continuous at $(0,0)$.

Remark 1 ([2], page 516).

(1) Every metric space is a semi-metric space with the triangle function $\Phi(u,v) = u+v$.
(2) Every ultrametric space is a semi-metric space with the triangle function $\Phi(u,v) = \max\{u,v\}$.
(3) Every b-metric space is a semi-metric space with the triangle function $\Phi(u,v) = \kappa(u+v)$.

Theorem 2 ([2], Theorem 1)**.** *Let (X,D) be a complete regular semi-metric space and $f\colon X\to X$ be a map such that for all $x,y\in X$,*

$$D(f(x),f(y)) \le \varphi(D(x,y))$$

where $\varphi\colon [0,\infty)\to[0,\infty)$ is an increasing function and for each $t\in[0,\infty)$

$$\lim_{n\to\infty}\varphi^n(t) = 0.$$

Then f has a unique fixed point $x^*\in X$ and $\lim_{n\to\infty} f^n(x) = x^*$ for each $x\in X$.

Replacing regular semi-metric spaces in Theorem 2 by b-metric spaces we get the following result.

Theorem 3 ([18], Theorem 12.2)**.** *Let (X,D,κ) be a complete b-metric space and $f\colon X\to X$ be a map such that for all $x,y\in X$,*

$$D(f(x),f(y)) \le \varphi(D(x,y))$$

where $\varphi\colon [0,\infty)\to[0,\infty)$ is an increasing function and for each $t\in[0,\infty)$

$$\lim_{n\to\infty}\varphi^n(t) = 0.$$

Then f has a unique fixed point $x^*\in X$ and $\lim_{n\to\infty} f^n(x) = x^*$ for each $x\in X$.

2. Main results

We prove an analogue of Theorem 3 in rectangular b-metric spaces. Note that in the spirit of Remark 1, every rectangular b-metric space may not be a semi-metric space since the right side of Definition 3.(3) contains three terms while Φ is a two-variable function. So the following result may not be deduced directly from Theorem 2.

Theorem 4. *Let (X,D,κ) be a complete rectangular b-metric space and $f\colon X\to X$ be a map such that for all $x,y\in X$,*

$$D(f(x),f(y)) \le \varphi(D(x,y)) \tag{2.1}$$

where $\varphi\colon [0,\infty) \to [0,\infty)$ *is an increasing function and for each* $t \in [0,\infty)$

$$\lim_{n\to\infty} \varphi^n(t) = 0. \tag{2.2}$$

Then f *has a unique fixed point* $x^* \in X$ *and* $\lim_{n\to\infty} f^n(x) = x^*$ *for each* $x \in X$.

Proof. First we prove that $\varphi(t) < t$ for all $t > 0$. Indeed, if there exists $t_0 > 0$ such that $t_0 \le \varphi(t_0)$, then from the increasing property of φ we have for all n,

$$0 < t_0 \le \varphi(t_0) \le \varphi^2(t_0) \le \cdots \le \varphi^n(t_0).$$

It follows from $\lim_{n\to\infty} \varphi^n(t_0) = 0$ that $t_0 = 0$. It is a contradiction to $t_0 > 0$. So $\varphi(t) < t$ for all $t > 0$.

Now we prove that

$$\lim_{t\to 0^+} \varphi(t) = 0. \tag{2.3}$$

Indeed, since φ is increasing, there exists $\lim_{t\to 0^+} \varphi(t) = l \ge 0$. If $\lim_{t\to 0^+} \varphi(t) = l > 0$ then $\varphi(t) \ge l$ for all $t > 0$. In particular, $\varphi(l) \ge l$, a contradiction. So (2.3) holds. Then there exists n_0 such that

$$\varphi^{n_0}(1) < \frac{1}{3\kappa}. \tag{2.4}$$

Let $x \in X$. Put $g = f^{n_0}$ and put $x_m = g^m(x)$ for all $m \in \mathbb{N}$. By (2.1) we deduce that

$$D(x_{m+1}, x_m) = D(g^m(g(x)), g^m(x)) \le \ldots \le \varphi^m(D(g(x), x)) \tag{2.5}$$

and

$$D(x_{m+2}, x_m) = D(g^m(g^2(x)), g^m(x)) \le \ldots \le \varphi^m(D(g^2(x), x)). \tag{2.6}$$

Letting $m \to \infty$ in (2.5) and (2.6) we get

$$\lim_{m\to\infty} D(x_{m+1}, x_m) = \lim_{m\to\infty} D(x_{m+2}, x_m) = 0.$$

So there exists m_0 such that for all $m \ge m_0$,

$$D(x_{m+1}, x_m) < \frac{1}{3\kappa} \text{ and } D(x_{m+2}, x_m) < \frac{1}{3\kappa}. \tag{2.7}$$

Now for each $u \in B[x_{m_0}, 1]$ and by (2.4) we have

$$D(g(u), g(x_{m_0})) = D(f^{n_0}(u), f^{n_0}(x_{m_0})) \le \varphi^{n_0}(D(u, x_{m_0})) \le \varphi^{n_0}(1) < \frac{1}{3\kappa}. \tag{2.8}$$

We first show that there exists k such that g^k has a fixed point x^*. On the contrary, we have $g(x_{m_0}) \ne g(g(x_{m_0})) \ne x_{m_0}$. Let $u \in B[x_{m_0}, 1]$. If $g(x_{m_0}) = g(u)$ or $g(g(x_{m_0})) = g(u)$ then by (2.7) we get $D(g(u), x_{m_0}) < \frac{1}{3\kappa} < 1$. So $g(u) \in B[x_{m_0}, 1]$.

So we may assume that $g(x_{m_0}) \ne g(g(x_{m_0})) \notin \{x_{m_0}, g(u)\}$. In this case, from (2.7) and (2.8) we find that

$$D(g(u), x_{m_0}) \le \kappa[D(g(u), g(x_{m_0})) + D(g(x_{m_0}), g(g(x_{m_0}))) + D(g(g(x_{m_0})), x_{m_0})]$$

$$\leq \kappa \left[\frac{1}{3\kappa} + \frac{1}{3\kappa} + \frac{1}{3\kappa}\right] = 1.$$

So $g(u) \in B[x_{m_0}, 1]$.

Then we conclude that $g\colon B[x_{m_0}, 1] \to B[x_{m_0}, 1]$. For all $n, m \geq m_0$ and from the contrary assumption we get $x_{n+1} \neq x_{m_0} \notin \{x_n, x_m\}$. By (2.7) we have

$$\begin{aligned} D(x_n, x_m) &\leq \kappa[D(x_n, x_{n+1}) + D(x_{n+1}, x_{m_0}) + D(x_{m_0}, x_m)] \\ &\leq \kappa \left[\frac{1}{3\kappa} + 1 + 1\right] = \frac{1+6\kappa}{3}. \end{aligned}$$

By (2.1) we find that for $m \geq n \geq m_0$,

$$\begin{aligned} D(x_n, x_m) &= D(g^n(x), g^m(x)) \\ &= D(g^{n-m_0} g^{m_0}(x), g^{m-m_0} g^{m_0}(x)) \\ &= D(g^{n-m_0}(x_{m_0}), g^{m-m_0}(x_{m_0})) \\ &= D(f^{(n-m_0)n_0}(x_{m_0}), f^{(m-m_0)n_0}(x_{m_0})) \\ &\leq \varphi(D(f^{(n-m_0)n_0-1}(x_{m_0}), f^{(m-m_0)n_0-1}(x_{m_0}))) \\ &\leq \varphi^{(n-m_0)n_0}(D(x_{m_0}, f^{(m-m_0)n_0-(n-m_0)n_0}(x_{m_0}))) \\ &= \varphi^{(n-m_0)n_0}(D(x_{m_0}, f^{(m-n)n_0}(x_{m_0}))) \\ &= \varphi^{(n-m_0)n_0}(D(x_{m_0}, x_{m-n+m_0})) \\ &\leq \varphi^{(n-m_0)n_0}\left(\frac{1+6\kappa}{3}\right). \end{aligned} \tag{2.9}$$

Letting $n, m \to \infty$ in (2.9) and using (2.2) we find that $\lim\limits_{n,m\to\infty} D(x_n, x_m) = 0$. So $\{x_m\}$ is a Cauchy sequence in (X, D). Since (X, D) is complete, there exists $\lim\limits_{m\to\infty} x_m = x^*$.

It follows from (2.1) and (2.3) that f is continuous in the sense it preserves the limit of sequences. Therefore g is continuous. Then

$$x^* = \lim_{m\to\infty} x_m = \lim_{m\to\infty} x_{m+1} = \lim_{m\to\infty} g(x_m) = g(x^*).$$

So g has a fixed point. It is a contradiction to the contrary assumption.

Therefore, there exists k such that g^k has a fixed point x^*. From (2.1) we get

$$\begin{aligned} D(x^*, g^{km}(f(x))) &= D(g^{km}(x^*), g^{km}(f(x))) \\ &= D(f^{n_0km}(x^*), f^{n_0km}(f(x))) \\ &\leq \varphi^{n_0km}(D(x^*, f(x))) \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} D(x^*, g^{km}(x)) &= D(g^{km}(x^*), g^{km}(x)) \\ &= D(f^{n_0 km}(x^*), f^{n_0 km}(x)) \\ &\leq \varphi^{n_0 km}(D(x^*, x)). \end{aligned} \tag{2.11}$$

Letting $m \to \infty$ in (2.10) and (2.11) we get

$$\lim_{m\to\infty} D(x^*, g^{km}(f(x))) = \lim_{m\to\infty} D(x^*, g^{km}(x)) = 0.$$

Then $\lim_{m\to\infty} g^{km}(f(x)) = \lim_{m\to\infty} g^{km}(x) = x^*$ in (X, D). By the continuity of f we have

$$f(x^*) = \lim_{m\to\infty} f(g^{km}(x)) = \lim_{m\to\infty} f(f^{n_0 km}(x)) = \lim_{m\to\infty} g^{km}(f(x)) = x^*.$$

This proves that x^* is a fixed point of f.

We next prove the uniqueness of fixed points of f. On the contrary, let x^* and y^* be two distinct fixed points of f. Then $D(x^*, y^*) > 0$. Therefore

$$D(x^*, y^*) = D(f(x^*), f(y^*)) \leq \varphi(D(x^*, y^*)) < D(x^*, y^*).$$

It is a contradiction.

Finally, we show that $\lim_{n\to\infty} f^n(x) = x^*$. Note that $\lim_{m\to\infty} g^{km}(y) = x^*$ for all $y \in X$. For each $n \in \mathbb{N}$, there exists l_n such that $n = l_n k n_0 + r_n$ with $0 \leq r_n \leq kn_0 - 1$. So

$$f^n(x) = f^{l_n k n_0 + r_n}(x) = g^{l_n k}(f^{r_n}(x)).$$

Fix $r_n = r \in [0, kn_0 - 1]$. Then

$$\lim_{l_n\to\infty} f^{l_n k n_0 + r}(x) = \lim_{l_n\to\infty} g^{l_n k}(f^r(x)) = x^*.$$

It implies that $\lim_{n\to\infty} f^n(x) = x^*$. □

Now by using Theorem 4 with $\varphi(t) = \lambda t$ with $t \geq 0$ we get a positive answer to Question 1. Note that this question was answered recently in [23, Theorem 2.1] but by a different proof.

Corollary 1. *Let (X, D, κ) be a complete rectangular b-metric space and $f\colon X \to X$ be a map such that for all $x, y \in X$ and for some $\lambda \in [0, 1)$,*

$$D(f(x), f(y)) \leq \lambda D(x, y)$$

Then f has a unique fixed point $x^* \in X$ and $\lim_{n\to\infty} f^n(x) = x^*$ for each $x \in X$.

In 1974, Ćirić proved a very general fixed point theorem in metric spaces, see [4, Theorem 1]. Next, we prove Ćirić type fixed point theorem in rectangular b-metric spaces. The proof in rectangular b-metric spaces is more complicated than that in metric spaces since the inequality is only used for distinct points.

Theorem 5 (Ćirić type fixed point theorem in rectangular b-metric spaces)**.** *Let (X,D,κ) be a complete rectangular b-metric space and $f\colon X\to X$ be a map such that for some $\lambda\in[0,\frac{1}{\kappa})$ and all $x,y\in X$,*

$$\begin{aligned} &D(f(x),f(y))\\ &\le \lambda\max\big\{D(x,y),D(x,f(x)),D(y,f(y)),D(x,f(y)),D(y,f(x))\big\}. \end{aligned} \tag{2.12}$$

Then f has a unique fixed point x^* and $\lim\limits_{n\to\infty} f^n(x)=x^*$ for all $x\in X$.

Proof. For each $x\in X$ and $m\le n$ put

$$\begin{aligned} f(m,n)(x) &= \{f^i(x) : m\le i\le n\}\\ f(m,\infty)(x) &= \{f^i(x) : m\le i\}\\ D(f(m,n)(x)) &= \sup\{D(u,v) : u,v\in f(m,n)(x)\}\\ D(f(m,\infty)(x)) &= \sup\{D(u,v) : u,v\in f(m,\infty)(x)\} \end{aligned}$$

where f^0 is the identity map on X. For $m\le i\le n-1$ and $m\le j\le n$, from (2.12) we find that

$$\begin{aligned} &D(f^i(x),f^j(x)) = D(ff^{i-1}(x),ff^{j-1}(x))\\ &\le \lambda\max\big\{D(f^{i-1}(x),f^{j-1}(x)),D(f^{i-1}(x),ff^{i-1}(x)),\\ &\qquad D(f^{j-1}(x),ff^{j-1}(x)),D(f^{i-1}(x),ff^{j-1}(x)),\\ &\qquad D(f^{j-1}(x),ff^{i-1}(x))\big\}\\ &= \lambda\max\big\{D(f^{i-1}(x),f^{j-1}(x)),D(f^{i-1}(x),f^{i}(x)),D(f^{j-1}(x),f^{j}(x)),\\ &\qquad D(f^{i-1}(x),f^{j}(x)),D(f^{j-1}(x),f^{i}(x))\big\}. \end{aligned} \tag{2.13}$$

From (2.13), we get

$$D(f(m,n)(x)) \le \lambda D(f(m-1,n)(x)). \tag{2.14}$$

Since $0\le\lambda<1$, we see that

$$D(f(0,n)(x)) = \max\{D(x,f^i(x)) : 1\le i\le n\}. \tag{2.15}$$

We now consider the following two cases.

Case 1: There exists $m<n$ such that $f^m(x)=f^n(x)$. Note that $f^m(x)=f^n(x)$, so we have

$$\begin{aligned} D(f(m+1,n)(x)) &= \sup\{D(f^i(x),f^j(x)) : m+1\le i,j\le n\}\\ &= \sup\{D(f^i(x),f^j(x)) : m\le i,j\le n-1\}\\ &= D(f(m,n-1)(x)). \end{aligned} \tag{2.16}$$

Similarly, we have

$$\begin{aligned} D(f(m,n)(x)) &= \sup\{D(f^i(x), f^j(x)) : m \le i, j \le n\} \\ &= \sup\{D(f^i(x), f^j(x)) : m \le i, j \le n-1\} \\ &= D(f(m,n-1)(x)). \end{aligned} \tag{2.17}$$

It follows from (2.14), (2.16) and (2.17) that

$$\begin{aligned} D(f(m,n-1)(x)) &= D(f(m+1,n)(x)) \\ &\le \lambda D(f(m,n)(x)) \\ &= \lambda D(f(m,n-1)(x)). \end{aligned}$$

Since $0 \le \lambda < 1$, we get $D(f(m,n-1)(x)) = 0$ for all $n > m$. For $n = m+2$ we deduce that $D(f(m,m+1)(x)) = 0$. Then $x^* = f^m(x)$ is a fixed point of f. Moreover $D(f(m,\infty)(x)) = 0$. So $\lim_{n\to\infty} f^n(x) = x^*$.

Case 2: $f^m(x)$'s are all distinct. For each $n > 2$, by (2.15) there exists $1 \le k_n(x) \le n$ such that $D(x, f^{k_n(x)}(x)) = D(f(0,n)(x))$. If $k_n(x) \ge 3$ then by (2.14) we get

$$\begin{aligned} D(f(0,n)(x)) &= D(x, f^{k_n(x)}(x)) \\ &\le \kappa[D(x, f(x)) + D(f(x), f^2(x)) + D(f^2(x), f^{k_n(x)}(x))] \\ &\le 2\kappa D(f(0,2)(x)) + \kappa\lambda D(f(1,k_n(x))(x)) \\ &\le 2\kappa D(f(0,2)(x)) + \kappa\lambda^2 D(f(0,k_n(x))(x)) \\ &\le 2\kappa D(f(0,2)(x)) + \kappa\lambda^2 D(f(0,n)(x)). \end{aligned}$$

Then

$$D(f(0,n)(x)) \le \frac{2\kappa}{1-\kappa\lambda^2} D(f(0,2)(x)). \tag{2.18}$$

If $k_n(x) \le 2$ then (2.18) obviously holds. Therefore $\{D(f(0,n)(x))\}$ is bounded. So $D(f(0,\infty)(x)) < \infty$. By (2.14) we have

$$D(f(m,\infty)(x)) \le \lambda D(f(m-1,\infty)(x)) \le \ldots \le \lambda^m D(f(0,\infty)(x)).$$

Then $\lim_{m\to\infty} D(f(m,\infty)(x)) = 0$. Therefore the sequence $\{f^n(x)\}$ is a Cauchy sequence. Since X is complete, there exists $x^* \in X$ such that

$$\lim_{n\to\infty} f^n(x) = x^*. \tag{2.19}$$

By (2.12) we get

$$\begin{aligned} D(f^{n+1}(x), f(x^*)) &= D(ff^n(x), f(x^*)) \\ &\le \lambda \max\{D(f^n(x), x^*), D(f^n(x), f^{n+1}(x)), D(x^*, f(x^*)), \\ &\qquad D(f^n(x), f(x^*)), D(x^*, f^{n+1}(x))\}. \end{aligned}$$

Using (2.19) and $\{x_n\}$ being a Cauchy sequence, we obtain

$$\begin{aligned}
&\liminf_{n\to\infty} D(f^{n+1}(x), f(x^*)) \\
&\leq \lambda \max\left\{0, 0, D(x^*, f(x^*)), \liminf_{n\to\infty} D(f^n(x), f(x^*)), 0\right\} \\
&= \lambda \max\left\{D(x^*, f(x^*)), \liminf_{n\to\infty} D(f^n(x), f(x^*))\right\}.
\end{aligned} \tag{2.20}$$

From (2.20), we consider two following subcases.

Subcase 2.1: $\liminf_{n\to\infty} D(f^{n+1}(x), f(x^*)) \leq \lambda \liminf_{n\to\infty} D(f^n(x), f(x^*))$.

From $\liminf_{n\to\infty} D(f^n(x), f(x^*)) = \liminf_{n\to\infty} D(f^{n+1}(x), f(x^*))$ and $0 \leq \lambda < \frac{1}{\kappa}$ we have $\liminf_{n\to\infty} D(f^n(x), f(x^*)) = 0$. So there exists a subsetquence $\{f^{k_n}(x)\}$ of $\{f^n(x)\}$ such that

$$\lim_{n\to\infty} f^{k_n}(x) = f(x^*). \tag{2.21}$$

Note that all $f^n(x)$'s are distinct. So for n large enough we have

$$\begin{aligned}
&D(x^*, f(x^*)) \\
&\leq \kappa[D(x^*, f^n(x)) + D(f^n(x), f^{k_n}(x)) + D(f^{k_n}(x), f(x^*))].
\end{aligned} \tag{2.22}$$

Letting $n \to \infty$ in (2.22) and using (2.21), (2.19) we obtain $D(x^*, f(x^*)) = 0$. Then $x^* = f(x^*)$.

Subcase 2.2: $\liminf_{n\to\infty} D(f^{n+1}(x), f(x^*)) \leq \lambda D(x^*, f(x^*))$.

For n large enough we have $f^n(x)$'s are distinct and different from $f(x^*)$ and x^*. So we find that

$$\begin{aligned}
&D(x^*, f(x^*)) \\
&\leq \kappa[D(x^*, f^n(x)) + D(f^n(x), f^{n+1}(x)) + D(f^{n+1}(x), f(x^*))].
\end{aligned} \tag{2.23}$$

From (2.19) and (2.23) we deduce that

$$\liminf_{n\to\infty} D(f^{n+1}(x), f(x^*)) \geq \frac{1}{\kappa} D(x^*, f(x^*)). \tag{2.24}$$

On the contrary, suppose that $x^* \neq f(x^*)$. Note that $0 \leq \lambda < \frac{1}{\kappa}$. Then

$$\liminf_{n\to\infty} D(f^{n+1}(x), f(x^*)) \leq \lambda D(x^*, f(x^*)) < \frac{1}{\kappa} D(x^*, f(x^*)).$$

This is a contradiction with (2.24). Therefore $x^* = f(x^*)$.

By above Subcase 2.1 and Subcase 2.2, f has a fixed point x^* and by (2.19), $\lim_{n\to\infty} f^n(x) = x^*$.

By Case 1 and Case 2, f has a fixed point x^* and $\lim_{n\to\infty} f^n(x) = x^*$. We next show that the fixed point of f is unique. Indeed, let x^*, y^* be two fixed points of f. From (2.12) we have

$$\begin{aligned} D(x^*,y^*) &= D(f(x^*),f(y^*)) \\ &\leq \lambda \max\left\{D(x^*,y^*),D(x^*,f(x^*)),D(y^*,f(y^*)),D(x^*,f(y^*)),D(y^*,f(x^*))\right\} \\ &= \lambda D(x^*,y^*). \end{aligned}$$

Since $\lambda \in [0,\frac{1}{\kappa})$, we obtain $D(x^*,y^*) = 0$, that is, $x^* = y^*$. Then the fixed point of f is unique. □

From Theorem 5 we get the following corollaries since contraction conditions (2.25), (2.26), (2.27), (2.28) are particular cases of the contraction condition (2.12). Moreover, Theorem 5, Corollary 2, Corollary 3 and Corollary 4 are analogues of Ćirić contraction, Hardy-Rogers contraction, Reich contraction and Chatterjea contraction in rectangular b-metric spaces respectively, that are answers to Question 2.

Corollary 2 (Hardy-Rogers type fixed point theorem in rectangular b-metric spaces)**.** *Let (X,D,κ) be a complete rectangular b-metric space and $f\colon X \to X$ be a map such that there exist $a_i \geq 0$, $i = 1,\ldots,5$, $\sum_{i=1}^{5} a_i < \frac{1}{\kappa}$ and for all $x,y \in X$,*

$$\begin{aligned} &D(f(x),f(y)) \\ &\leq a_1 D(x,y) + a_2 D(x,f(x)) + a_3 D(y,f(y)) + a_4 D(x,f(y)) + a_5 D(y,f(x)). \end{aligned} \tag{2.25}$$

Then f has a unique fixed point x^* and $\lim_{n\to\infty} f^n(x) = x^*$ for all $x \in X$.

Corollary 3 (Reich type fixed point theorem in rectangular b-metric spaces)**.** *Let (X,D,κ) be a complete rectangular b-metric space and $f\colon X \to X$ be a map such that there exist $a,b,c \geq 0$, $a+b+c < \frac{1}{\kappa}$ and for all $x,y \in X$,*

$$D(f(x),f(y)) \leq aD(x,y) + bD(x,f(x)) + cD(y,f(y)). \tag{2.26}$$

Then f has a unique fixed point x^* and $\lim_{n\to\infty} f^n(x) = x^*$ for all $x \in X$.

Corollary 4 (Chatterjea type fixed point theorem in rectangular b-metric spaces)**.** *Let (X,D,κ) be a complete rectangular b-metric space and $f\colon X \to X$ be a map such that for some $a \in [0,\frac{1}{2\kappa})$ and for all $x,y \in X$,*

$$D(f(x),f(y)) \leq a[D(x,f(y)) + D(y,f(x))]. \tag{2.27}$$

Then f has a unique fixed point x^* and $\lim_{n\to\infty} f^n(x) = x^*$ for all $x \in X$.

Corollary 5 (Kannan type fixed point theorem in rectangular b-metric spaces)**.** *Let (X,D,κ) be a complete rectangular b-metric space and $f\colon X \to X$ be a map such that for some $a \in [0,\frac{1}{2\kappa})$ and for all $x,y \in X$,*

$$D(f(x),f(y)) \leq a[D(x,f(x)) + D(y,f(y))]. \tag{2.28}$$

Then f has a unique fixed point x^* and $\lim_{n\to\infty} f^n(x) = x^*$ for all $x \in X$.

Finally, the following example shows that the domain of contraction constant $[0, \frac{1}{\kappa})$ in Corollary 3 may not be relaxed to $[0,1)$. Then so may not the domains in Theorem 5.

Example 1. Let $X = \{0, 1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\}$, and

$$D(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y \in \{0,1\}, \\ |x-y| & \text{if } x \neq y \in \{0\} \cup \left\{\frac{1}{2n} : n = 1,2,\ldots\right\}, \\ \frac{1}{4} & \text{otherwise,} \end{cases}$$

and let $f\colon X \to X$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x = 0, \\ \frac{1}{10n} & \text{if } x = \frac{1}{n}, n = 1,2,\ldots. \end{cases}$$

Then

(1) (X, D, κ) is a complete rectangular b-metric space with the coefficient $\kappa = 4$.
(2) There exist $a, b, c \geq 0$, $\frac{1}{\kappa} \leq a + b + c < 1$ such that the contraction condition (2.26) holds for all $x, y \in X$.
(3) f is fixed point free.

Proof. By [9, Example 2.6], (X, D, κ) is a complete metric-type space with the coefficient $\kappa = 4$. Then (X, D, κ) is also a complete rectangular b-metric on X with the coefficient $\kappa = 4$. The remaining conclusions were proved in [9, Example 2.6]. □

3. ACKNOWLEDGMENTS

The authors would like to thank the anonymous reviewers for their comments to revise the paper better.

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