



## INITIAL COEFFICIENTS AND FOURTH HANKEL DETERMINANT FOR CERTAIN ANALYTIC FUNCTIONS

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*Dedicated to Professor V. Ravichandran*

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*Abstract.* The present work is an attempt to give partial proofs of certain conjectures on the fifth coefficient of certain normalized analytic functions. Further, bounds on the sixth and seventh coefficients for the starlike functions related to a lune are also investigated. The non-sharp bound on third and fourth Hankel determinants are also obtained.

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### 1. INTRODUCTION

Let  $\mathcal{A}$  be the class of analytic functions  $f$  defined on the unit disk  $\mathbb{D} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$  and satisfying the conditions  $f(0) = 0 = f'(0) - 1$ . Thus the functions in the class  $\mathcal{A}$  has the form  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ . The collection of functions in  $\mathcal{A}$ , which are one-one, is denoted by  $\mathcal{S}$ . It was proved, in 1985, by de Branges that “if  $f \in \mathcal{S}$ , then  $|a_n| \leq n$ .” The bound  $|a_n| \leq 2(1 - \alpha)$ ,  $0 \leq \alpha < 1$ , holds for the class of starlike and functions  $\mathcal{S}^*(\alpha) := \{f : f \in \mathcal{S} \text{ and } \Re(zf'(z)/f(z)) > \alpha\}$  of order  $\alpha$ . However, for the class of convex functions  $\mathcal{K}(\alpha) := \{f : f \in \mathcal{S} \text{ and } 1 + \Re(zf''(z)/f'(z)) > \alpha\}$  of order  $\alpha$ , we have the bound  $|a_n| \leq 2(1 - \alpha)$ ,  $0 \leq \alpha < 1$ . The classes  $\mathcal{S}^* := \mathcal{S}^*(0)$  and  $\mathcal{K} := \mathcal{K}(0)$  are known, respectively, as the classes of starlike and convex functions in  $\mathbb{D}$ . Finding bound on the coefficients of functions in a prescribed class has been one among the major problem in geometric function theory as they affects geometric properties. For example, the bound on the second coefficient gives the growth and distortion properties.

The Hankel determinant of the sequence of coefficients  $a_i$  of functions in a given class also gives the important information about the function. For given positive integers  $n$  and  $q$ , the Hankel determinant  $H_{q,n}(f)$  related to the function  $f \in \mathcal{A}$  is

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given by

$$H_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix}, \quad a_1 = 1.$$

For some special choices of  $n$  and  $q$ ;  $H_{2,1}(f) = a_3 - a_2^2$  is the famed Fekete-Szegő functional;  $H_{2,2}(f) = a_2a_4 - a_3^2$  is the second Hankel determinant, and  $H_{3,1}(f) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2)$  is the third Hankel determinant. The Hankel determinant  $H_{q,n}(f)$  for the class  $\mathcal{S}$  was investigated by Pommerenke [21] and Hayman [6]. There has been substantial progress in finding the bounds on the Hankel determinant for subclasses of  $\mathcal{S}$ , see [13].

In 2007, Janteng *et al.* [7] found the sharp estimate on the second Hankel determinant for the classes of starlike and convex functions. Non-sharp estimate on third Hankel determinant for classes of starlike and convex functions were investigated by Zaprawa [33] as  $|H_{3,1}(f)| \leq 1$  and  $|H_{3,1}(f)| \leq 49/540$ , respectively. Kowalczyk *et al.* [8], in 2018, proved that the bound  $|H_{3,1}(f)| \leq 4/135$  is sharp for the class of convex functions. However, the best known estimate for starlike functions is  $|H_{3,1}(f)| \leq 4/135$  due to Kwon *et al.* [11]. Later, in 2018, Lecko *et al.* [12] found the sharp bound  $|H_{3,1}(f)| \leq 1/9$  for starlike function of order  $1/2$ . Kowalczyk *et al.* [9] also found sharp bound of the third kind for the class  $T(\alpha) := \{f \in \mathcal{A} : \Re(f(z)/z) > \alpha; z \in \mathbb{D}\}$  for the cases when  $\alpha = 0$  and  $\alpha = 1/2$ .

In 2015, Raina and Sokół [24] introduced the class

$$\mathcal{S}_q^* := \left\{ f \in \mathcal{S}^* : \frac{zf'(z)}{f(z)} \prec \sqrt{1+z^2} + z =: q(z) \right\}$$

and discussed several properties of the class  $\mathcal{S}_q^*$ . Later, Raina and Sokół [23] conjectured that  $|a_4| \leq 5/12$ ,  $|a_5| \leq 2/9$  and  $|a_2a_4 - a_3^2| \leq 7/48$ . In 2017, Cho *et al.* [Coefficient bounds for certain subclasses of starlike functions, preprint] settled the conjecture  $|a_4| \leq 5/12$  posed by Raina and Sokół [23] for functions in the class  $\mathcal{S}_q^*$ .

Recently, Çağlar *et al.* [3] obtained upper bound for the second Hankel determinant for certain subclasses of analytic and bi-univalent functions. Later, Srivastava *et al.* [30] also discussed several properties of a newly-constructed subclass of bi-univalent functions in  $\mathbb{D}$  defined by using a symmetric basic (or  $q$ -) derivative operator. Further, they investigated bound on  $|a_2a_4 - a_3^2|$ . In [29] authors discussed bound on the Hankel determinant and the Toeplitz matrices for certain classes of analytic  $q$ -starlike functions. The authors [28] obtained non-sharp bound of the third Hankel determinant for a subclass of close-to-convex functions associated with the lemniscate of Bernoulli. They also investigated sharp bound on the initial coefficients. For more related results one may refer to the work [5, 16, 26, 27, 31].

Obradović and Ponnusamy [17] introduced the class

$$\mathcal{U}(\lambda) := \left\{ f \in \mathcal{A} : \left| \left( \frac{z}{f(z)} \right)^2 f'(z) - 1 \right| < \lambda, \lambda \in (0, 1], z \in \mathbb{D} \right\}.$$

The functions in the class  $\mathcal{U} := \mathcal{U}(0)$  are of the form  $f(z) = z/(1 - bz)$ ,  $|b| \leq 1$ . Note that if  $f \in \mathcal{U}(\lambda)$ , then

$$\frac{f(z)}{z} \prec \frac{1}{(1+z)(1+\lambda z)}, \quad z \in \mathbb{D}.$$

Obradović and co-authors have investigated several interesting properties of the class  $\mathcal{U}(\lambda)$ , see the works [17–19, 22] and the references cited therein.

In 2016, Obradović *et al.* [18] conjectured that the  $n^{\text{th}}$  coefficient of function  $f \in \mathcal{U}(\lambda)$  satisfies

$$|a_n| \leq \sum_{k=0}^{n-1} \lambda^k \quad (0 < \lambda \leq 1, n \geq 2).$$

This conjecture has been verified for  $n = 2$  first in [32] and a simpler proof was given in [18]. More recently, Obradović *et al.* [19] proved the conjecture for  $n = 3, 4$  with an alternate proof for the case  $n = 2$ , but it remains open for all  $n \geq 5$ . A partial solution to this conjecture was given in [22, Theorem 1, p. 91].

$$|a_n| \leq 1 + \lambda \sqrt{n-1} \sqrt{\sum_{k=0}^{n-2} \lambda^{2k}} \quad (0 < \lambda \leq 1, n \geq 2). \tag{1.1}$$

In the present paper, we improve the existing bound on  $|a_5|$  given in the above inequality for certain range of  $\lambda$  for  $f \in \mathcal{U}(\lambda)$ . We give non-sharp bound on  $|a_i|$  ( $i = 5, 6, 7$ ),  $|H_{3,1}(f)|$  and  $|H_{4,1}(f)|$  for  $f \in \mathcal{S}_q^*$ .

## 2. INITIAL COEFFICIENTS

The following theorem gives a refinement and improvement to the bound on  $|a_5|$  given in [22, Theorem 1, p. 91]:

**Theorem 1.** *Let  $f(z) = z + a_2z^2 + a_3z^3 + \dots \in \mathcal{U}(\lambda)$ . Then*

$$|a_5| \leq \frac{1}{2} (2 + 2\lambda + 5\lambda^2 + 6\lambda^3 + 4\lambda^4). \tag{2.1}$$

We need the following lemma to prove our result. Before proceeding further we recall that the class of functions  $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$  with  $\Re p(z) > 0, z \in \mathbb{D}$  is denoted by  $\mathcal{P}$  and they are related with the Schwarz function  $w : \mathbb{D} \rightarrow \mathbb{C}$  by means of the relation

$$w(z) = \frac{p(z) - 1}{p(z) + 1}.$$

**Lemma 1** ([25], Lemma 2.3, p. 507). *Let  $p \in \mathcal{P}$ . Then for all  $n, m \in \mathbb{N}$ ,*

$$|\mu p_n p_m - p_{m+n}| \leq \begin{cases} 2, & 0 \leq \mu \leq 1; \\ 2|2\mu - 1|, & \text{elsewhere.} \end{cases}$$

If  $0 < \mu < 1$ , then the inequality is sharp for the function

$$p(z) = (1 + z^{m+n}) / (1 - z^{m+n}).$$

In the other cases, the inequality is sharp for the function  $\hat{p}_0$ .

*Proof of Theorem 1.* Let  $f \in \mathcal{U}(\lambda)$ . Then, from [19, Eqn.(3)], we have

$$\frac{f(z)}{z} \prec \frac{1}{(1+z)(1+\lambda z)}.$$

Therefore, for  $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$  with  $\Re p(z) > 0, z \in \mathbb{D}$  such that

$$\frac{f(z)}{z} = \frac{1}{\left(1 + \frac{1-p(z)}{1+p(z)}\right) \left(1 + \lambda \frac{1-p(z)}{1+p(z)}\right)}.$$

On comparing coefficients, we have

$$a_2 = -\frac{p_1}{2}(\lambda + 1),$$

$$a_3 = \frac{1}{4}((\lambda^2 + 2\lambda + 2)p_1^2 - 2(\lambda + 1)p_2),$$

$$a_4 = \frac{1}{8}(-(\lambda^3 + 3\lambda^2 + 4\lambda + 4)p_1^3 + 4(\lambda^2 + 2\lambda + 2)p_1 p_2 - 4(\lambda + 1)p_3)$$

and

$$16a_5 = -8(1 + \lambda)p_4 + 4(2 + 2\lambda + \lambda^2) + 8(2 + 2\lambda + \lambda^2)p_1 p_3 - 6(4 + 4\lambda + 3\lambda^2 + \lambda^3)p_1^2 p_2 + (8 + 8\lambda + 7\lambda^2 + 4\lambda^3 + \lambda^4)p_1^4, \quad (2.2)$$

where  $A := 8 + 8\lambda + 7\lambda^2 + 4\lambda^3 + \lambda^4$ . Now rearranging the terms of (2.2), we have

$$\begin{aligned} \frac{16}{A}a_5 = & -\frac{8(1 + \lambda)}{A}p_4 + \frac{4(2 + 2\lambda + \lambda^2)}{A}p_2^2 + \frac{8(2 + 2\lambda + \lambda^2)}{A}p_1 p_3 \\ & - \frac{6(4 + 4\lambda + 3\lambda^2 + \lambda^3)}{A}p_1^2 p_2 + p_1^4. \end{aligned} \quad (2.3)$$

Now upon using triangle inequality, (2.3) gives

$$\begin{aligned} \left| \frac{16}{A}a_5 \right| \leq & |p_1^4 + 2p_1 p_3 + p_2^2 - 3p_1^2 p_2 - p_4| + \left| \frac{3\lambda^2(1 + \lambda)^2}{A}p_1^2 p_2 - \frac{(3 + 4\lambda + \lambda^2)\lambda^2}{A}p_2^2 \right| \\ & + \left| \frac{\lambda^2(\lambda^2 + 4\lambda + 7)}{A}p_4 - \frac{2\lambda^2(3 + 4\lambda + \lambda^2)}{A}p_1 p_3 \right|. \end{aligned} \quad (2.4)$$

To estimate bound on  $|a_5|$ , now we consider all three terms in the right hand side of (2.4) one by one and find the bound on them. It is well known form [4] (see also [15, Lemma 3, p. 227]) that

$$|p_1^4 + 2p_1p_3 + p_2^2 - 3p_1^2p_2 - p_4| \leq 2. \tag{2.5}$$

The following are obtained by using Lemma 1:

$$\begin{aligned} \left| \frac{3\lambda^2(1+\lambda)^2}{A} p_1^2 p_2 - \frac{\lambda^2(3+4\lambda+\lambda^2)}{A} p_2^2 \right| &= \left| \frac{(3+4\lambda+\lambda^2)\lambda p_2}{A} \left( \frac{3(1+\lambda)^2}{3+4\lambda+\lambda^2} p_1^2 - p_2 \right) \right| \\ &\leq \frac{4\lambda^2(5\lambda^2+8\lambda+3)}{A} \end{aligned} \tag{2.6}$$

$$\begin{aligned} \left| \frac{\lambda^2(\lambda^2+4\lambda+7)}{A} p_4 - \frac{2\lambda^2(\lambda^2+4\lambda+3)}{A} \right| &= \frac{\lambda^2}{A} |(\lambda^2+4\lambda+3)p_1p_3 - (\lambda^2+4\lambda+7)p_4| \\ &= \frac{(\lambda^2+4\lambda+7)\lambda^2}{A} \left| \frac{(\lambda^2+4\lambda+3)}{(\lambda^2+4\lambda+7)} p_1p_3 - p_4 \right| \\ &\leq \frac{2(\lambda^2+4\lambda+7)\lambda^2}{A}. \end{aligned} \tag{2.7}$$

Now, from (2.4), (2.5), (2.6) and (2.7), we have

$$\begin{aligned} |a_5| &\leq 2\frac{A}{16} + \left( \frac{4\lambda^2(5\lambda^2+8\lambda+3)}{A} + \frac{2(\lambda^2+4\lambda+7)\lambda^2}{A} \right) \\ &= \frac{A}{8} + \frac{4\lambda^2(5\lambda^2+8\lambda+3)}{4} + \frac{2(\lambda^2+4\lambda+7)\lambda^2}{8} \\ &= \frac{2+2\lambda+5\lambda^2+6\lambda^3+4\lambda^4}{2}. \end{aligned}$$

This completes the proof. □

*Remark 1.* In [22, Theorem 1, p. 91], the following non-sharp bound was proved:

$$|a_5| \leq 1 + 2\lambda\sqrt{(\lambda^2+1)(\lambda^4+1)}. \tag{2.8}$$

Now if  $x_0 \approx 0.305398$  is the root of  $48x^5 + 60x^4 + 76x^3 + 33x^2 + 20x - 12 = 0$ , then from (2.1) and (2.8), we have the following refined result:

$$|a_5| \leq \begin{cases} \frac{2+2\lambda+5\lambda^2+6\lambda^3+4\lambda^4}{2}, & 0 < \lambda \leq x_0; \\ 1 + 2\lambda\sqrt{(\lambda^2+1)(\lambda^4+1)}, & x_0 \leq \lambda < 1. \end{cases}$$

Thus, we see that our result improves over the result proved in [22, Theorem 1, p. 91] for the range  $0 < \lambda \leq x_0$ .

In the following theorem non-sharp bounds on  $|a_5|$ ,  $|a_6|$  and  $|a_7|$  for the class  $S_q^*$  are investigated.

**Theorem 2.** *Let  $f(z) = z + a_2z^2 + a_3z^3 + \dots \in S_q^*$ . Then*

- (i)  $|a_5| \leq \frac{1}{8} (1 + 2\sqrt{2}) \approx 0.478553$ ,  
(ii)  $|a_6| \leq \frac{5(711+32\sqrt{79})}{8532} \approx 0.341303$ ,  
(iii)  $|a_7| \leq \frac{2665}{2304} \approx 1.15668$ .

We need the following lemmas to prove the above theorem:

**Lemma 2** ([14], Theorem 4(b), p. 678). *A function  $p \in \mathcal{P}$  if and only if*

$$\sum_{j=0}^{\infty} \left\{ \left| 2z_j + \sum_{k=1}^{\infty} p_k z_{k+j} \right|^2 - \left| \sum_{k=0}^{\infty} p_{k+1} z_{k+j} \right|^2 \right\} \geq 0$$

for every sequence  $\{z_k\}_{k=1}^{\infty}$  of complex numbers that satisfies  $\lim_{k \rightarrow \infty} |z_k|^{1/k} < 1$ .

**Lemma 3** ([20], Proposition 6, p. 7). *Let  $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$ . Also, for any real numbers  $a, b$  and  $c$ , let the quantity  $Y(a, b, c) := \max_{z \in \overline{\mathbb{D}}} \{|a + bz + cz^2| + 1 - |z|^2\}$ . If  $ac \geq 0$ , then*

$$Y(a, b, c) = \begin{cases} |a| + |b| + |c| & (|b| \geq 2(1 - |c|)), \\ 1 + |a| + \frac{b^2}{4(1 - |c|)} & (|b| < 2(1 - |c|)). \end{cases}$$

Furthermore, if  $ac < 0$ , then

$$Y(a, b, c) = \begin{cases} 1 - |a| + \frac{b^2}{4(1 - |c|)} & (-4ac(c^{-2} - 1) \leq b^2; |b| < 2(1 - |c|)) \\ 1 + |a| + \frac{b^2}{4(1 + |c|)} & (b^2 < \min\{4(1 + |c|)^2, -4ac(c^{-2} - 1)\}) \\ R(a, b, c) & (\text{otherwise}), \end{cases}$$

where

$$R(a, b, c) = \begin{cases} |a| + |b| - |c| & (|c|(|b| + 4|a|) \leq |ab|) \\ -|a| + |b| + |c| & (|ab| \leq |c|(|b| - 4|a|)) \\ (|c| + |a|) \sqrt{1 - \frac{b^2}{4ac}} & (\text{otherwise}). \end{cases}$$

**Lemma 4** ([10], Lemma 1). *Let  $p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \dots \in \mathcal{P}$ . Then, for any real number  $\mu$ ,*

$$|\mu p_3 - p_1^3| \leq \begin{cases} 2|\mu - 4| & \left(\mu \leq \frac{4}{3}\right) \\ 2\mu\sqrt{\frac{\mu}{\mu - 1}} & \left(\frac{4}{3} < \mu\right). \end{cases}$$

*The result is sharp.*

**Lemma 5** ([1], Corollary 1, p. 68). *Let  $p \in \mathcal{P}$ . Then*

$$|p_3 - (\mu + 1)p_1p_2 + \mu p_1^3| \leq \begin{cases} 2, & 0 \leq \mu \leq 1; \\ 2|\mu - 1|, & \text{elsewhere.} \end{cases}$$

*Proof of Theorem 2.* (i) Since  $f \in \mathcal{S}_q^*$ , it follows that there exists a Schwarz function  $w$ , such that

$$\frac{zf'(z)}{f(z)} = w(z) + \sqrt{1 + w^2(z)}. \tag{2.9}$$

Writing

$$w(z) = \frac{p(z) - 1}{p(z) + 1}$$

with  $p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \dots \in \mathcal{P}$ , and equating the coefficients of like power terms, we have

$$a_5 = \frac{1}{384} (p_1^4 - 10p_1^2p_2 + 8p_1p_3 + 48p_4) \tag{2.10}$$

and therefore,

$$\begin{aligned} |a_5| &= \frac{1}{384} |p_1^4 - 10p_1^2p_2 + 8p_1p_3 + 48p_4| \\ &\leq \frac{1}{384} (|48p_4 - 10p_1^2p_2| + |p_1^4 + 8p_1p_3|) \\ &\leq \frac{1}{384} (48 + |48p_4 - 10p_1^2p_2|). \end{aligned} \tag{2.11}$$

We now choose a sequence

$$z_0 = \frac{5}{24}p_1p_2, z_1 = 0, z_2 = 0, z_3 = -1, z_k = 0 \ (k \geq 4)$$

to estimate the maximum value of  $|48p_4 - 10p_1^2p_2|$ .

Now an application of Lemma 2, for the above choice of sequence gives

$$\left| \frac{5}{24}p_1^2p_2 - p_4 \right|^2 \leq \left| p_3 - \frac{5}{12}p_1p_2 \right|^2 + 4 \leq 8.$$

Therefore,

$$|10p_1^2p_2 - 48p_4| \leq 96\sqrt{2}$$

and from Eq. (2.11), we have

$$|a_5| \leq \frac{1}{8} (1 + 2\sqrt{2}) \approx 0.478553.$$

This was the desired bound on  $|a_5|$ .

(ii) Proceeding as above, we have

$$a_6 = \frac{-p_1^5 + 42p_1^3p_2 - 80p_1^2p_3 - 80p_1p_2^2 + 48p_1p_4 - 32p_2p_3 + 384p_5}{3840} \quad (2.12)$$

and

$$a_7 = \frac{T}{184320}, \quad (2.13)$$

where

$$\begin{aligned} T := & -59p_1^6 - 392p_1^4p_2 + 1840p_1^3p_3 + 32p_1^2(95p_2^2 - 99p_4) \\ & - 64p_1(97p_2p_3 + 96p_5) - 320(3p_2^3 + 6p_2p_4 + 4p_3^2 - 48p_6). \end{aligned} \quad (2.14)$$

To find the estimate on  $|a_6|$ , by suitably rearranging the terms of (2.12) and using the triangle inequality, we have:

$$\begin{aligned} 3840|a_6| &= \left| -p_1^5 + 42p_1^3p_2 - 80p_1^2p_3 - 80p_1p_2^2 + 48p_1p_4 - 32p_2p_3 + 384p_5 \right| \\ &= \left| -80p_1^2 \left( p_3 - \frac{21}{40}p_1p_2 + \frac{19}{40}p_1^3 \right) + (48p_1p_4 - 80p_1p_2^2) + (384p_5 - 32p_2p_3) + 37p_1^5 \right| \\ &\leq 80 \left| p_1^2 \left( p_3 - \frac{21}{40}p_1p_2 + \frac{19}{40}p_1^3 \right) \right| + |48p_1p_4 - 80p_1p_2^2| + |384p_5 - 32p_2p_3| + 37|p_1|^5. \end{aligned} \quad (2.15)$$

Now using Lemma 1, Lemma 5 and the fact  $|p_i| \leq 2$ , we have

$$80 \left| p_1^2 \left( p_3 - \frac{21}{40}p_1p_2 + \frac{19}{40}p_1^3 \right) \right| \leq 640, \quad |48p_1p_4 - 80p_1p_2^2| \leq 320 \quad (2.16)$$

and

$$|384p_5 - 32p_2p_3| = 384 \left| \frac{1}{12}p_2p_3 - p_5 \right| \leq 768. \quad (2.17)$$

Now from (2.15), (2.16) and (2.17), we have

$$|a_6| \leq \frac{2272}{8532} \approx 0.341303.$$

This was the desired estimate on  $|a_6|$ .

(iii) From (2.14), by suitably rearranging the terms, we have

$$\begin{aligned} 184320a_7 = & -59p_1^6 - 392p_1^4p_2 + 1840p_1^3p_3 + 32p_1^2(95p_2^2 - 99p_4) \\ & - 64p_1(97p_2p_3 + 96p_5) - 320(3p_2^3 + 6p_2p_4 + 4p_3^2 - 48p_6) \end{aligned}$$



or

$$\begin{aligned} 184320|a_7| \leq & 320|6p_2p_4 - 48p_6| + |-392p_1^4p_2 + 1840p_1^3p_3| \\ & + |-3040p_1^2p_2^2 + 960p_2^3| + |59p_1^6 + 3168p_1^2p_4| \\ & + |64p_1(97p_2p_3 + 96p_5)| + |2180p_3^2|. \end{aligned} \quad (2.18)$$

Now applications of Lemma 1 and the fact  $|p_i| \leq 2$ , gives

$$320|6p_2p_4 - 48p_6| \leq 30720, \quad |-392p_1^4p_2 + 1840p_1^3p_3| \leq 29440, \quad (2.19)$$

$$|-3040p_1^2p_2^2 + 960p_2^3| \leq 40960. \quad (2.20)$$

and

$$|59p_1^6 + 3168p_1^2p_4| + |64p_1(97p_2p_3 + 96p_5)| + |2180p_3^2| \leq 108480. \quad (2.21)$$

Now from (2.18), (2.19), (2.20) and (2.21), we have

$$|a_7| \leq \frac{2665}{2304} \approx 1.15668.$$

This completes the proof.  $\square$

### 3. THIRD AND FOURTH HANKEL DETERMINANTS

#### 3.1. Third Hankel determinant

In this section, we find an estimate on the third Hankel determinant for functions in the class  $\mathcal{S}_q^*$ . As before, for functions in the class  $\mathcal{S}_q^*$ , we have

$$a_2 = \frac{p_1}{2}, \quad a_3 = \frac{1}{16}(p_1^2 + 4p_2) \quad \text{and} \quad a_4 = \frac{1}{96}(16p_3 + 4p_1p_2 - p_1^3). \quad (3.1)$$

A computation using (2.10), (2.12), (2.13) and (3.1), gives

$$H_{3,1}(f) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2) = \frac{S}{36864}, \quad (3.2)$$

where

$$\begin{aligned} S := & -55p_1^6 + 128p_1^4p_2 + 368p_1^3p_3 + 1216p_1p_2p_3 \\ & - 32p_1^2(11p_2^2 + 27p_4) - 64(9p_2^3 + 16p_3^2 - 18p_2p_4). \end{aligned}$$

By rearranging terms and using the triangle inequality from (3.2), we have

$$\begin{aligned} 36864|H_{3,1}(f)| \leq & |-55p_1^6 + 128p_1^4p_2| + |1152p_2p_4 - 864p_1^2p_4| \\ & + |1216p_1p_2p_3 - 1024p_3^2| + |368p_1^3p_3 - 352p_1^2p_2^2| + |-576p_2^3|. \end{aligned} \quad (3.3)$$

Using Lemma 1 and the fact  $|p_i| \leq 2$ , we have

$$|-55p_1^6 + 128p_1^4p_2| \leq 4096, \quad |1152p_2p_4 - 864p_1^2p_4| \leq 4608, \quad (3.4)$$

$$|1216p_1p_2p_3 - 1024p_3^2| \leq 5632 \quad \text{and} \quad |-576p_2^3| \leq 4608. \quad (3.5)$$

Now in order to get the desired bound, we claim that  $|368p_1^3p_3 - 352p_1^2p_2^2| \leq 5632$ . To establish our claim, we shall prove that

$$\left| \frac{23}{22}p_1p_3 - p_2^2 \right| \leq 4. \quad (3.6)$$

We use the following:

**Lemma 6** ([15], Libera and Zlotkiewicz). *Let  $p(z) = 1 + p_1z + p_2z^2 + \dots \in \mathcal{P}$  with  $p_1 \geq 0$ . Then*

$$2p_2 = p_1^2 + x(4 - p_1^2) \quad (3.7)$$

and

$$4p_3 = p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)y \quad (3.8)$$

for some  $x$  and  $y$  such that  $|x| \leq 1$  and  $|y| \leq 1$ .

To prove (3.6), we consider  $(23/22)p_1p_3 - p_2^2$ . After substituting expression for  $p_2$ ,  $p_3$  and simplifying, from Lemma 6, we have

$$\begin{aligned} I &:= \frac{23}{22}p_1p_3 - p_2^2 \\ &= \frac{23}{88}p_1(p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)y) \\ &\quad - \frac{1}{4}(p_1 + x(4 - p_1^2))^2. \end{aligned} \quad (3.9)$$

Using the invariant property for the class  $\mathcal{S}_q^*$  under rotation, we can assume that  $s := p_1 \in [0, 2]$  and thus (3.9) becomes

$$I = \frac{1}{88}s^4 + \frac{1}{88}(4 - s^2)\frac{s^2}{2} + \frac{1}{4}(4 - s^2)\left(\frac{45}{22}s^2 - 4\right)x^2 + \frac{23}{44}s(4 - s^2)(1 - |x|^2)y. \quad (3.10)$$

Now from (3.10), we have

$$I = \begin{cases} 4, & s = 0; \\ \frac{6}{11}, & s = 2. \end{cases} \quad (3.11)$$

For  $s \in (0, 2)$  consider the expression:

$$I = \frac{23s}{44}(4 - s^2)(a + bx + cx^2 + 1 - |x|^2),$$

where

$$a := \frac{s^3}{23(4 - s^2)}, \quad b := \frac{s}{23} \quad \text{and} \quad c := \frac{45s^2 - 88}{46s}.$$

For suitability, we divide the calculation in five cases:

**Case (I):** Here  $a > 0$  always and  $c \geq 0$  if and only if  $1.39841 \approx (2/3)\sqrt{22/5} =:$

$s_1 \leq s < 2$ . The condition  $|b| > 2(1 - |c|) = 2(1 - |c|)$  holds if  $s \geq 1.97073 \approx (23 + \sqrt{4577})/46 =: s_2$ . Thus, for  $s \in [s_2, 2)$ , we have

$$I = \frac{23s}{44}(4 - s^2)(|a| + |b| + |c|) = \frac{(4 - s^2)(45s^4 - 276s^2 + 352)}{88(s^2 - 4)} =: f_1(s).$$

Now computation shows that  $f_1$  attains its maximum at  $s_0 = \sqrt{46/15}$  and

$$f_1(s) \leq f_1(s_0) = \frac{89}{110}, \quad s \in (s_1, 2). \tag{3.12}$$

**Case (II):** The condition  $|b| < 2(1 - |c|) = 2(1 - c)$  holds for  $s \in [s_1, s_2)$  and for this range, we have

$$I = \frac{s(91s^4 - 1980s^3 - 2024s^2 + 8280s + 8096)}{88(45s + 44)} = f_2(s).$$

Computation reveals that  $f_2$  never vanish in  $[s_1, s_2)$ , therefore,

$$f_2(s) \leq \max \{f_2(s_1), f_2(s_2)\} = f_1(s_1) = \frac{2(14520 + 23827\sqrt{110})}{334125} \approx 1.58276. \tag{3.13}$$

**Case (III):** Now computation shows that

$$\min \{4(1 + c^2), -4ac(c^{-2} - 1)\} = \begin{cases} -4ac(c^{-2} - 1), & 0 < s \leq s_3; \\ 4(1 + c^2), & s_3 < s < s_1, \end{cases}$$

where  $s_3$  is a root of  $95175x^6 - 443252x^4 + 859232x^2 - 681472 = 0$  in  $(0, s_1)$ . Further,  $b^2 < -4ac(c^{-2} - 1)$  holds if and only if

$$\frac{s^2}{529} \leq -\frac{2s^2(2025s^2 - 1936)}{529(45s^2 - 88)}$$

or equivalently

$$0.983378 \approx 2\sqrt{\frac{22}{91}} =: s_4 = < s < s_1.$$

Moreover,  $b^2 < 4(1 + c^2)$  holds for all  $s > 0$ . Thus, for  $s \in (s_4, s_1)$ , we have

$$I = \frac{s(91s^4 - 1980s^3 - 2024s^2 + 8280s + 8096)}{88(45s + 44)} = f_3(s).$$

$f_3'(s) = 0$  for  $s = s^* \approx 1.1977 \in (s_4, s_1)$  and  $f_3''(s^*) < 0$ , so

$$f_3(s) \leq \max \{f_3(s^*), f_3(s_1), f_3(4)\} = f_3(s^*) \approx 1.65372, \quad s \in (s_4, s_1). \tag{3.14}$$

**Case (IV):** It can be verified that  $ab \leq -c(b - 4a)$  holds for  $s \in (0, s_5]$ , where

$$s_5 := \sqrt{\frac{2}{223} (155 - 3\sqrt{489})}.$$

In this case, we have

$$I = \frac{41s^4}{88} - \frac{65s^2}{22} + 4 =: f_4(s).$$

Now a computation shows that, for  $s \in (0, s_5]$ ,

$$\begin{aligned} f_4(s) &\leq \max\{f_4(0), f_4(s_5)\} \\ &= f_4(s_5) \\ &= \frac{4(11911 + 555\sqrt{489})}{49729} \approx 1.94526. \end{aligned} \quad (3.15)$$

**Case (V):** In view of Lemma 3, for the case,  $s \in (s_5, s_4]$ , we have

$$I = \frac{1}{88} \sqrt{\frac{180 - 91s^2}{176 - 90s^2}} (47s^4 - 268s^2 + 352) =: f_5(s).$$

Now a computation shows that

$$\begin{aligned} f_5(s) &\leq \max\{f_5(s_4), f_5(s_5)\} = f_5(s_5) \\ &= \frac{\sqrt{4930 + 138\sqrt{489}}(6139 + 261\sqrt{489})}{547019} \approx 1.94526, \quad s \in (s_5, s_4]. \end{aligned} \quad (3.16)$$

From (3.11)-(3.16), we conclude that  $I \leq 4$ , that is

$$\left| \frac{23}{22} p_1 p_3 - p_2^2 \right| \leq 4. \quad (3.17)$$

The above inequality (3.17) is sharp for the function  $p(\tilde{z}) = (1 + z^3)/(1 - z^3)$  and therefore,

$$|368p_1^3 p_3 - 352p_1^2 p_2^2| = 352|p_1^2| \left| \frac{23}{22} p_1 p_3 - p_2^2 \right| \leq 5632. \quad (3.18)$$

From (3.3), (3.4), (3.5) and (3.18), we have

$$|H_{3,1}(f)| \leq \frac{2}{3} \approx 0.67.$$

Thus, we have the following:

**Theorem 3.** Let  $f(z) = z + a_2 z^2 + a_3 z^3 + \dots \in S_q^*$ . Then

$$|H_{3,1}(f)| \leq \frac{2}{3} \approx 0.67.$$

3.2. Fourth Hankel determinant

The fourth Hankel determinant can be expressed as, see also [2]:

$$H_{4,1}(f) = a_7H_{3,1}(f) - a_6\Delta_1 + a_5\Delta_2 - a_4\Delta_3,$$

where

$$\Delta_1 = a_3a_6 - a_4a_5 - a_2(a_2a_6 - a_3a_5) + a_4(a_2a_4 - a_3^2),$$

$$\Delta_2 = (a_4a_6 - a_5^2) - a_2(a_3a_6 - a_4a_5) + a_3(a_3a_5 - a_4^2)$$

and

$$\Delta_3 = a_2(a_4a_6 - a_5^2) - a_3(a_3a_6 - a_4a_5) + a_4(a_3a_5 - a_4^2).$$

To compute the fourth Hankel determinant for the functions in class  $\mathcal{S}_q^*$ , we substitute the vales of  $a_i$  in terms of  $p_j$  as given in (2.10), (2.12), (2.13) and (3.1) and after simplification we have

$$\begin{aligned} 122880\Delta_1 &= 31p_1^7 - 400p_1^5p_2 + 240p_1^4p_3 + 32p_1^3(18p_2^2 + 11p_4) \\ &\quad - 64p_1(15p_2^3 - 20p_3^2 - 26p_2p_4) + 256p_1^2(2p_2p_3 - 9p_5) \\ &\quad - 512(3p_2^2p_3 + 5p_3p_4 - 6p_2p_5). \end{aligned} \tag{3.19}$$

Rearranging the terms of (3.19) and applying Lemmas 1 and 4, we have

$$\begin{aligned} 122880|\Delta_1| &\leq |31p_1^7 - 400p_1^5p_2| + |576p_1^3p_2^2 - 960p_1p_2^3| + |3072p_2p_5 - 1536p_2^2p_3| \\ &\quad + 256|p_1^2(2p_2p_3 - 9p_5)| + |352p_1^3p_4 - 2560p_3p_4| \\ &\quad + 64|p_1||20p_3^2p_1 + 26p_2p_4| \\ &\leq \frac{1024}{69} (5451 + 40\sqrt{345}). \end{aligned}$$

Therefore, we have

$$|\Delta_1| \leq \frac{5451 + 40\sqrt{345}}{8280} \approx 0.748064. \tag{3.20}$$

In a similar line, we have

$$\begin{aligned} -491520\Delta_2 &= (-552p_1^5p_3 + 3p_1^8) + (656p_4p_1^4 - 576p_1^2p_2p_4) \\ &\quad + (-1920p_1^3p_2p_3 + 1216p_1p_2^2p_3) + (7680p_4^2 - 3584p_1p_3p_4) \\ &\quad + (4096p_1^2p_2p_5 - 8192p_3p_5) + (1216p_1p_2^2p_3 - 3840p_2^2p_4). \end{aligned} \tag{3.21}$$

From (3.21) and applying Lemmas 1 and 4, we have

$$|\Delta_2| \leq \frac{118279 + 92\sqrt{8418}}{117120} \approx 1.08197. \tag{3.22}$$

Similarly rearranging the terms of

$$8847360\Delta_3 = -29p_1^9 - 78p_1^7p_2 + 768p_1^6p_3 - 72p_1^5(17p_2^2 + 74p_4)$$

$$\begin{aligned}
& -32p_1^3(299p_2^3 + 180p_3^2 - 864p_2p_4) + 288p_1^4(57p_2p_3 - 28p_5) \\
& -1536p_1^2(17p_2^2p_3 - 6p_3p_4 + 6p_2p_5) \\
& + 512(9p_2^3p_3 - 80p_3^3 + 180p_2p_3p_4 - 108p_2^2p_5) \\
& + 768p_1(15p_2^4 - 28p_2p_3^2 + 21p_2^2p_4 - 90p_4^2 + 96p_3p_5)
\end{aligned}$$

suitably, we have

$$\begin{aligned}
8847360\Delta_3 &= (768p_1^6p_3 - 29p_1^9) + (27648p_1^3p_2p_4 - 5328p_4p_1^5) \\
&+ (-78p_1^7p_2 + 16416p_1^4p_2p_3) + 1536p_1^2(-17p_2^2p_3 + 6p_3p_4) \\
&+ (73728p_1p_3p_5 - 9216p_1^2p_2p_5) + (92160p_2p_3p_4 - 21504p_1p_2p_3^2) \\
&+ 768p_1(21p_2^2p_4 - 90p_4^2) + 512(9p_2^3p_3 - 108p_2^2p_5) \\
&+ (11520p_1p_2^4 - 9568p_1^3p_2^3) \\
&+ (-1224p_1^5p_2^2 - 5760p_3^2p_1^3 - 8064p_1^4p_5 - 40960p_3^3). \quad (3.23)
\end{aligned}$$

Using triangle inequality in (3.23) and applying Lemmas 1 and 4, we have

$$|\Delta_3| \leq \frac{3720606449 + 33460224\sqrt{717} + 2942568\sqrt{51737}}{5622886080} \approx 0.940065. \quad (3.24)$$

Now using the triangle inequality and substituting the bounds on initial coefficients  $|a_i|$  ( $i = 4, 5, 6, 7$ ) and  $|\Delta_j|$  ( $j = 1, 2, 3$ ), we have

$$\begin{aligned}
|H_{4,1}(f)| &\leq |a_7||H_{3,1}(f)| + |a_6||\Delta_1| + |a_5||\Delta_2| + |a_4||\Delta_3| \\
&\leq 1.93977.
\end{aligned}$$

Thus, we have the following:

**Theorem 4.** *Let  $f(z) = z + a_2z^2 + a_3z^3 + \dots \in \mathcal{S}_q^*$ . Then  $|H_{4,1}(f)| \leq 1.93977$ .*

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