Miskolc Mathematical Notes
Vol. 22 (2021), No. 1, pp. 245–258
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2021.3080

INEQUALITIES OF HERMITE–HADAMARD TYPE FOR EXTENDED HARMONICALLY (s,m)-CONVEX FUNCTIONS

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Received 18 October 2019

Abstract. In the paper, the authors introduce a new notion "extended harmonically (s,m)-convex function" and establish some integral inequalities of the Hermite–Hadamard type for extended harmonically (s,m)-convex functions in terms of hypergeometric functions.

2010 *Mathematics Subject Classification:* Primary 26D15; Secondary 26A51, 26D20, 33C05, 41A55

Keywords: extended harmonically (s,m)-convex function; integral inequality; Hermite–Hadamard type; hypergeometric function

1. INTRODUCTION

We recall some definitions on diverse convex functions in the literature.

Definition 1. A function $f : I \subseteq \mathbb{R} = (-\infty,\infty) \to \mathbb{R}$ is said to be convex if the inequality

$$f(tx+(1-t)y) \le tf(x)+(1-t)f(y)$$

holds for all $x,y \in I$ and $t \in [0,1]$.

Definition 2 ([2,11])**.** Let $s \in (0,1]$ be a real number. A function $f : \mathbb{R}_0 = [0,\infty) \to \mathbb{R}_0$ is said to be s-convex in the second sense if the inequality

$$f(tx+(1-t)y) \le t^s f(x)+(1-t)^s f(y)$$

holds for all $x,y \in I$ and $t \in [0,1]$.

Definition 3 ([25])**.** For $f : [0,b] \to \mathbb{R}$, $b > 0$, and $m \in (0,1]$, if the inequality

$$f(tx+m(1-t)y) \le tf(x)+m(1-t)f(y)$$

is valid for all $x,y \in [0,b]$ and $t \in [0,1]$, then we say that f is an m-convex function on $[0,b]$.

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Definition 4 ([28]). A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be extended s-convex if the inequality
$$f(tx+(1-t)y) \le t^s f(x) + (1-t)^s f(y)$$
holds for all $x, y \in I$ and $t \in (0,1)$ and for some fixed $s \in [-1,1]$.

Definition 5 ([13]). Let $I \subseteq \mathbb{R} \setminus \{0\}$ be a real interval. A function $I \to \mathbb{R}$ is said to be harmonically convex if the inequality
$$f\left(\frac{xy}{tx+(1-t)y}\right) \le tf(x) + (1-t)f(y) \tag{1.1}$$
holds for all $x, y \in$I and $t \in [0,1]$. If the inequality in (1.1) is reversed, then f is said to be harmonically concave.

Definition 6 ([17, Definition 2.6]). A function $I \subseteq \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ is said to be harmonically s-convex if the inequality
$$f\left(\frac{xy}{tx+(1-t)y}\right) \le t^s f(x) + (1-t)^s f(y)$$
is valid for $x, y \in I$, $t \in (0,1)$, and $s \in [-1,1]$.

Definition 7 ([29]). Let $f : (0,b] \to \mathbb{R}$ and let $m \in (0,1]$ be a constant. If the inequality
$$f\left(\left(\frac{t}{x} + m\frac{1-t}{y}\right)^{-1}\right) \le tf(x) + m(1-t)f(y)$$
is valid for all $x, y \in (0,b]$ and $t \in [0,1]$, then f is said to be an m-harmonic-arithmetically convex function or, simply speaking, an m-HA-convex function.

Definition 8 ([9]). Let $f : (0,b] \to \mathbb{R}$ and let $\alpha, m \in (0,1]$ be constants. If the inequality
$$f\left(\left(\frac{t}{x} + m\frac{1-t}{y}\right)^{-1}\right) \le t^\alpha f(x) + m(1-t^\alpha)f(y)$$
is valid for all $x, y \in (0,b]$ and $t \in [0,1]$, then f is said to be an (α,m)-harmonic-arithmetically convex function or, simply speaking, an (α,m)-HA-convex function.

In recent decades, establishing integral inequalities of the Hermite–Hadamard type for diverse convex functions has been being an active direction in mathematics. Some of these results can be reformulated as follows.

Theorem 1 ([5, Theorem 2.2]). *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then*
$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)(|f'(a)|+|f'(b)|)}{8}.$$

Theorem 2 ([20, Theorems 1 and 2]). *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable on I° and $a, b \in I$ with $a < b$. If $|f'|^q$ is convex on $[a,b]$ and $q \geq 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{1/q}$$

and

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{1/q}.$$

Theorem 3 ([6]). *Let $m \in (0,1]$ and $f : \mathbb{R}_0 = [0,\infty) \to \mathbb{R}$ be m-convex. If $f \in L_1([a,b])$ for $0 \leq a < b < \infty$, then*

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \min\left\{\frac{f(a)+mf(b/m)}{2}, \frac{mf(a/m)+f(b)}{2}\right\}.$$

Theorem 4 ([14]). *Let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be differentiable on I° and $a, b \in I$ with $a < b$. If $|f'|^q$ is s-convex on $[a,b]$ for some fixed $s \in (0,1]$ and $q \geq 1$, then*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\qquad \leq \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-1/q}\left[\frac{2+1/2^s}{(s+1)(s+2)}\right]^{1/q}\left[|f'(a)|^q + |f'(b)|^q\right]^{1/q}.\end{aligned}$$

Theorem 5 ([12]). *Let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be differentiable on I°, $a, b \in I$ with $a < b$, and $f' \in L[a,b]$. If $|f'|^q$ is s-convex on $[a,b]$ for some fixed $s \in (0,1]$ and $q > 1$, then*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left[\frac{1}{(s+1)(s+2)}\right]^{1/q}\left(\frac{1}{2}\right)^{1/p} \\ &\quad \times \left\{\left[|f'(a)|^q + (s+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}\right. \\ &\quad \left. + \left[|f'(b)|^q + (s+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}\right\},\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 6 ([24]). *Let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be differentiable on I°, $a, b \in I$ with $a < b$, and $f' \in L[a,b]$. If $|f'|$ is s-convex on $[a,b]$ for some $s \in (0,1]$ and $p > 1$, then*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\quad \leq \frac{(s-4)6^{s+1} + 2\times 5^{s+2} - 2\times 3^{s+2} + 2}{6^{s+2}(s+1)(s+2)}(b-a)\left(|f'(a)| + |f'(b)|\right),\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

For more information developed in recent decades on this topic, please refer to the papers [1, 3, 4, 7, 8, 10, 15, 16, 18, 19, 21–23, 26, 27, 30] and closely related references.

In this paper, we will introduce a new notion "extended harmonically (s,m)-convex function" and establish some new integral inequalities of the Hermite–Hadamard type for extended harmonically (s,m)-convex functions.

2. A DEFINITION AND A LEMMA

Now we introduce the notion "extended harmonically (s,m)-convex function".

Definition 9. For $m \in (0,1]$ and $s \in [-1,1]$, a function $f:(0,b] \to \mathbb{R}$ is said to be extended harmonically (s,m)-convex on $(0,b]$ if the inequality

$$f\left(\left(\frac{t}{x}+m\frac{1-t}{y}\right)^{-1}\right) \le t^s f(x)+m(1-t)^s f(y)$$

holds for all $x,y \in (0,b]$ and $t \in (0,1)$.

Example 1. Let $s \in [-1,1]$ and $f(x)=\frac{1}{x^r}$ for $x \in \mathbb{R}_+$ and $r \ge 1$. Since

$$f\left(\left(\frac{t}{x}+\frac{m(1-t)}{y}\right)^{-1}\right) \le \frac{ty^r+(1-t)(mx)^r}{(xy)^r} \le t^s f(x)+m(1-t)^s f(y)$$

for all $x,y \in \mathbb{R}_+$ and $t \in (0,1)$, the function $f(x)=\frac{1}{x^r}$ is extended harmonically (s,m)-convex on $\mathbb{R}_+$.

To establish some new integral inequalities of the Hermite–Hadamard type for extended harmonically (s,m)-convex functions, we need the following lemma.

Lemma 1. *Let $f: I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function on I° and $a,b \in I^\circ$ with $a<b$. If $f' \in L_1([a,b])$ and $0 \le \lambda,\mu \le 1$, then*

$$\begin{aligned}\frac{\lambda f(a)+\mu f(b)}{2}&+\frac{2-\lambda-\mu}{2}f(H(a,b))-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \qquad (2.1)\\ &=\frac{b-a}{4ab}\int_0^1\left[(1-\lambda-t)\left(\frac{t}{a}+\frac{1-t}{H(a,b)}\right)^{-2} f'\left(\left(\frac{t}{a}+\frac{1-t}{H(a,b)}\right)^{-1}\right)\right.\\ &\quad\left.+(\mu-1+t)\left(\frac{t}{b}+\frac{1-t}{H(a,b)}\right)^{-2} f'\left(\left(\frac{t}{b}+\frac{1-t}{H(a,b)}\right)^{-1}\right)\right]dt.\end{aligned}$$

In particular, if $\lambda=\mu=0$, then

$$\begin{aligned}f(H(a,b))&-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \qquad (2.2)\\ &=\frac{b-a}{4ab}\int_0^1\left[(1-t)\left(\frac{t}{a}+\frac{1-t}{H(a,b)}\right)^{-2} f'\left(\left(\frac{1+t}{2a}+\frac{1-t}{2b}\right)^{-1}\right)\right.\end{aligned}$$

$$-(1-t)\left(\frac{t}{b}+\frac{1-t}{H(a,b)}\right)^{-2} f'\left(\left(\frac{1+t}{2b}+\frac{1-t}{2a}\right)^{-1}\right)\Bigg] dt,$$

where $H(a,b)=\frac{2ab}{a+b}$.

Proof. Putting $x=\left(ta^{-1}+(1-t)[H(a,b)]^{-1}\right)^{-1}$ for $t\in[0,1]$ gives

$$\int_0^1 (1-\lambda-t)\left(\frac{t}{a}+\frac{1-t}{H(a,b)}\right)^{-2} f'\left(\left(\frac{t}{a}+\frac{1-t}{H(a,b)}\right)^{-1}\right) dt \tag{2.3}$$

$$=\frac{2ab}{b-a}[\lambda f(a)+(1-\lambda)f(H(a,b))]-\left(\frac{2ab}{b-a}\right)^2\int_a^{H(a,b)}\frac{f(x)}{x^2}dx.$$

Similarly, letting $x=\left(tb^{-1}+(1-t)[H(a,b)]^{-1}\right)^{-1}$ for $t\in[0,1]$ results in

$$\int_0^1 (\mu-1+t)\left(\frac{t}{b}+\frac{1-t}{H(a,b)}\right)^{-2} f'\left(\left(\frac{t}{b}+\frac{1-t}{H(a,b)}\right)^{-1}\right) dt \tag{2.4}$$

$$=\frac{2ab}{b-a}[\mu f(b)+(1-\mu)f([H(a,b)])]-\left(\frac{2ab}{b-a}\right)^2\int_{H(a,b)}^{b}\frac{f(x)}{x^2}dx.$$

Adding the equalities (2.3) and (2.4) leads to the equality (2.1). The proof of Lemma 1 is thus complete. □

3. Integral inequalities of Hermite–Hadamard type

Now we start out to establish some new integral inequalities of the Hermite–Hadamard type for extended harmonically (s,m)-convex functions.

Theorem 7. *Let* $f:(0,d]\to\mathbb{R}$ *be differentiable,* $a,b\in(0,d]$ *with* $a<b$*,* $f'\in L_1([a,b])$*, and* $0\le\lambda,\mu\le 1$*. If* $|f'|^q$ *for* $q\ge 1$ *is extended harmonically* (s,m)*-convex on* $(0,d]$ *for some fixed* $m\in(0,1]$ *and* $s\in[-1,1]$*, then*

(1) *when* $-1<s\le 1$,

$$\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f(H(a,b))-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right| \tag{3.1}$$

$$\le\frac{(b-a)^{(2-q)/q}}{2^{(q+1)/q}(ab)^{1/q}}\Big\{[T(a,b,\lambda)]^{1-1/q}\left(mK(a,H(a,b),s,\lambda)|f'(mH(a,b))|^q\right.$$

$$\left.+K(H(a,b),a,s,1-\lambda)|f'(a)|^q\right)^{1/q}+[T(b,a,\mu)]^{1-1/q}\left(mK(b,H(a,b),s,\mu)\right.$$

$$\left.\times|f'(mH(a,b))|^q+K(H(a,b),b,s,1-\mu)|f'(b)|^q\right)^{1/q}\Big\};$$

(2) *when* $s=-1$,

$$\left|f(H(a,b))-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|\le\frac{(ab)^{(q-2)/q}(a+b)^{(1-q)/q}}{2^{1/q}(b-a)^{(q-2)/q}}\{[(a+b)(\ln a$$

$$
\begin{aligned}
&-\ln H(a,b))+(b-a)]^{1-1/q}\left[\left(2b^2(\ln(2a)-\ln H(a,b))-bH(a,b)\right)|f'(a)|^q\right.\\
&\left.+maH(a,b)|f'(mb)|^q\right]^{1/q}+\left[m(2a^2(\ln(2b)-\ln H(a,b))-aH(a,b))|f'(mb)|^q\right.\\
&\left.+bH(a,b)|f'(a)|^q\right]^{1/q}[(a+b)(\ln b-\ln H(a,b))-(b-a)]^{1-1/q}\Big\},
\end{aligned}
$$

where

$$
\begin{aligned}
T(a,b,\lambda)=2ab(&\ln a+\ln H(a,b)\\
&-2\ln[(1-\lambda)H(a,b)+\lambda a])+(b-a)[(1-\lambda)H(a,b)-\lambda a]
\end{aligned}
$$

and

$$
\begin{aligned}
K(a,u,s,\lambda)=&\frac{2\lambda^{s+2}a^2}{(s+1)(s+2)}{}_2F_1\left(2,s+1,s+3,\frac{\lambda(u-a)}{u}\right)\\
&-\frac{\lambda a^2}{s+1}{}_2F_1\left(2,s+1,s+2,\frac{u-a}{u}\right)+\frac{a^2}{s+2}{}_2F_1\left(2,s+2,s+3,\frac{u-a}{u}\right)
\end{aligned}
$$

with the hypergeometric function

$$
{}_2F_1(c,d,e;z)=\frac{\Gamma(e)}{\Gamma(d)\Gamma(e-d)}\int_0^1 t^{d-1}(1-t)^{e-d-1}(1-zt)^{-c}dt \tag{3.2}
$$

for $e>d>0$, $|z|<1$, $c\in\mathbb{R}$, *and* $u>0$.

Proof. When $-1<s\le 1$, by virtue of Lemma 1 and the extended harmonic (s,m)-convexity of $|f'|^q$, we obtain

$$
\begin{aligned}
&\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f(H(a,b))-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right| \qquad (3.3)\\
&\le\frac{b-a}{4ab}\left[\left(\int_0^1|1-\lambda-t|\left(\frac{t}{a}+\frac{1-t}{H(a,b)}\right)^{-2}dt\right)^{1-1/q}\right.\\
&\quad\times\left(\int_0^1|1-\lambda-t|\left(\frac{t}{a}+\frac{1-t}{H(a,b)}\right)^{-2}\left|f'\left(\left(\frac{t}{a}+\frac{1-t}{H(a,b)}\right)^{-1}\right)\right|^q dt\right)^{1/q}\\
&\quad+\left(\int_0^1|1-\mu-t|\left(\frac{t}{b}+\frac{1-t}{H(a,b)}\right)^{-2}dt\right)^{1-1/q}\\
&\quad\left.\times\left(\int_0^1|1-\mu-t|\left(\frac{t}{b}+\frac{1-t}{H(a,b)}\right)^{-2}\left|f'\left(\left(\frac{t}{b}+\frac{1-t}{H(a,b)}\right)^{-1}\right)\right|^q dt\right)^{1/q}\right]\\
&\le\frac{b-a}{4ab}\left[\left(\int_0^1|1-\lambda-t|\left(\frac{t}{a}+\frac{1-t}{H(a,b)}\right)^{-2}dt\right)^{1-1/q}\left(\int_0^1|1-\lambda-t|\right.\right.
\end{aligned}
$$

$$\times\left(\frac{t}{a}+\frac{1-t}{H(a,b)}\right)^{-2}\left[t^s|f'(a)|^q+m(1-t)^s|f'(mH(a,b))|^q\right]dt\Bigg)^{1/q}$$

$$+\left(\int_0^1|1-\mu-t|\left(\frac{t}{b}+\frac{1-t}{H(a,b)}\right)^{-2}dt\right)^{1-1/q}\Bigg(\int_0^1|1-\mu-t|$$

$$\times\left(\frac{t}{b}+\frac{1-t}{H(a,b)}\right)^{-2}\left[t^s|f'(b)|^q+m(1-t)^s|f'(mH(a,b))|^q\right]dt\Bigg)^{1/q}\Bigg],$$

where we used the facts

$$\int_0^1|1-\lambda-t|\left(\frac{t}{a}+\frac{1-t}{H(a,b)}\right)^{-2}dt=\frac{2ab}{(b-a)^2}T(a,b,\lambda),$$

$$\int_0^1|1-\lambda-t|t^s\left(\frac{t}{a}+\frac{1-t}{H(a,b)}\right)^{-2}dt=K(H(a,b),a,s,1-\lambda),$$

and

$$\int_0^1|1-\lambda-t|(1-t)^s\left[\frac{t}{a}+\frac{1-t}{H(a,b)}\right]^{-2}dt=K(a,H(a,b),s,\lambda).$$

The inequality (3.1) is thus proved.

When $s=-1$, by the identity (2.2) and the extended harmonic (s,m)-convexity of $|f'|^q$, we have

$$\left|f(H(a,b))-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|$$

$$\le\frac{b-a}{2^{2-1/q}ab}\Bigg[\left(\int_0^1(1-t)\left(\frac{t}{a}+\frac{1-t}{H(a,b)}\right)^{-2}dt\right)^{1-1/q}$$

$$\times\Bigg(\int_0^1(1-t)\left(\frac{t}{a}+\frac{1-t}{H(a,b)}\right)^{-2}\left[(1+t)^{-1}|f'(a)|^q\right.$$

$$\left.+m(1-t)^{-1}|f'(mb)|^q\right]dt\Bigg)^{1/q}+\left(\int_0^1(1-t)\left(\frac{t}{b}+\frac{1-t}{H(a,b)}\right)^{-2}dt\right)^{1-1/q}$$

$$\times\Bigg(\int_0^1(1-t)\left(\frac{t}{b}+\frac{1-t}{H(a,b)}\right)^{-2}\left[(1-t)^{-1}|f'(a)|^q\right.$$

$$\left.+m(1+t)^{-1}|f'(mb)|^q\right]dt\Bigg)^{1/q}\Bigg] \tag{3.4}$$

$$=\frac{(ab)^{(q-2)/q}(a+b)^{(1-q)/q}}{2^{1/q}(b-a)^{(q-2)/q}}\Big\{[(a+b)(\ln a-\ln H(a,b))+(b-a)]^{1-1/q}$$

$$\times\left[\left(2b^2(\ln(2a)-\ln H(a,b))-bH(a,b)\right)|f'(a)|^q+maH(a,b)|f'(mb)|^q\right]^{1/q}$$
$$+\left[m\left(2a^2(\ln(2b)-\ln H(a,b))-aH(a,b)\right)|f'(mb)|^q+bH(a,b)|f'(a)|^q\right]^{1/q}$$
$$\times[(a+b)(\ln b-\ln H(a,b))-(b-a)]^{1-1/q}\Big\}.$$

The proof of Theorem 7 is thus complete. ☐

Corollary 1. *Under conditions of Theorem 7, when $q=1$,*

(1) *if $-1<s\le 1$, then*

$$\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f(H(a,b))-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|$$
$$\le\frac{b-a}{4ab}\big\{K(H(a,b),a,s,1-\lambda)|f'(a)|+K(H(a,b),b,s,1-\mu)|f'(b)|$$
$$+m[K(a,H(a,b),s,\lambda)+K(b,H(a,b),s,\mu)]|f'(mH(a,b))|\big\};$$

(2) *if $s=-1$, then*

$$\left|f(H(a,b))-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|$$
$$\le\frac{b-a}{ab}\left(b^2[\ln(2a)-\ln H(a,b)]|f'(a)|+ma^2[\ln(2b)-\ln H(a,b)]|f'(mb)|\right).$$

Corollary 2. *Under conditions of Theorem 7, when $q=s=1$ and $\lambda=\mu=\frac{1}{2}$, we have*

$$\left|\frac{f(a)+2f(H(a,b))+f(b)}{4}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dt\right|\le\frac{1}{4(b-a)^2}$$
$$\times\left\{\left[(20ab^2+12a^2b)\left(\ln\sqrt{aH(a,b)}-\ln H(a,H(a,b))\right)-a(b-a)^2\right]|f'(a)|\right.$$
$$+4mab\left[(7b+a)\left(\ln H(a,H(a,b))-\ln\sqrt{aH(a,b)}\right)\right.$$
$$\left.+(7a+b)\left(\ln\sqrt{bH(a,b)}-\ln H(b,H(a,b))\right)\right]|f'(mH(a,b))|$$
$$\left.+\left[\left(20a^2b+12ab^2\right)\left(\ln H(b,H(a,b))-\ln\sqrt{bH(a,b)}\right)+b(b-a)^2\right]|f'(b)|\right\}.$$

Theorem 8. *Let $f:(0,d]\to\mathbb{R}$ be differentiable, $a,b\in(0,d]$ with $a<b$, $f'\in L_1([a,b])$, and $0\le\lambda,\mu\le 1$. If $|f'|^q$ for $q>1$ is extended harmonically (s,m)-convex on $(0,d]$ for some fixed $s\in[-1,1]$ and $0<m\le 1$, then*

(1) *when $-1<s\le 1$,*

$$\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f(H(a,b))-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|\tag{3.5}$$
$$\le\frac{b-a}{4ab[(s+1)(s+2)]^{1/q}}\Big\{Q^{1-1/q}(a,H(a,b),\lambda)\left[(2(1-\lambda)^{s+2}+\lambda(s+2)\right.$$

$$-1)\left|f'(a)\right|^q+m\left(2\lambda^{s+2}+(1-\lambda)(s+2)-1\right)\left|f'(mH(a,b))\right|^q\Big]^{1/q}$$
$$+Q^{1-1/q}(b,H(a,b),\mu)\left[\left(2(1-\mu)^{s+2}+\mu(s+2)-1\right)\left|f'(b)\right|^q\right.$$
$$\left.+m\left(2\mu^{s+2}+(1-\mu)(s+2)-1\right)\left|f'(mH(a,b))\right|^q\right]^{1/q}\Big\};$$

(2) *when* $s=-1$,

$$\left|f(H(a,b))-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|\leq\frac{b-a}{2^{(2q-1)/q}ab} \tag{3.6}$$
$$\times\left\{\left[(2\ln 2-1)|f'(a)|^q+m|f'(mb)|^q\right]^{1/q}Q^{1-1/q}(a,H(a,b),0)\right.$$
$$\left.+\left[(2\ln 2-1)|f'(b)|^q+m|f'(ma)|^q\right]^{1/q}Q^{1-1/q}(b,H(a,b),0)\right\},$$

where, for $u>0$ *and* $u\neq a$

$$Q(a,u,\lambda)=\frac{(q-1)(au)^{2q/(q-1)}}{(q+1)(u-a)}\left\{(1-\lambda)a^{-(q+1)/(q-1)}-\lambda u^{-(q+1)/(q-1)}\right.$$
$$\left.-\frac{q-1}{2(u-a)}\left[a^{-2/(q-1)}-2[(1-\lambda)(u-a)+a]^{-2/(q-1)}+u^{-2/(q-1)}\right]\right\}.$$

Proof. If $-1<s\leq 1$, by the inequality (3.3) and the extended harmonic (s,m)-convexity of $|f'|^q$, we derive

$$\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f(H(a,b))-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right| \tag{3.7}$$
$$\leq\frac{b-a}{4ab}\left[\left(\int_0^1|1-\lambda-t|\left(\frac{t}{a}+\frac{1-t}{H(a,b)}\right)^{-2q/(q-1)}dt\right)^{1-1/q}\right.$$
$$\times\left(\int_0^1|1-\lambda-t|\left[t^s|f'(a)|^q+m(1-t)^s|f'(mH(a,b))|^q\right]dt\right)^{1/q}$$
$$+\left(\int_0^1|1-\mu-t|\left(\frac{t}{b}+\frac{1-t}{H(a,b)}\right)^{-2q/(q-1)}dt\right)^{1-1/q}$$
$$\left.\times\left(\int_0^1|1-\mu-t|\left[t^s|f'(b)|^q+m(1-t)^s|f'(mH(a,b))|^q\right]dt\right)^{1/q}\right],$$

where

$$\int_0^1|1-\lambda-t|\left(\frac{t}{a}+\frac{1-t}{H(a,b)}\right)^{-2q/(q-1)}dt=Q(a,H(a,b),\lambda),$$
$$\int_0^1|1-\lambda-t|t^s dt=\frac{2(1-\lambda)^{s+2}+\lambda(s+2)-1}{(s+1)(s+2)},$$

and

$$\int_0^1 |1-\lambda-t|(1-t)^s dt = \frac{2\lambda^{s+2}+(1-\lambda)(s+2)-1}{(s+1)(s+2)}. \tag{3.8}$$

Combining (3.7) with (3.8) gives the required inequality (3.5).

Similarly, by the inequality (3.4), we can prove the inequality (3.6). The proof of Theorem 8 is complete. □

Corollary 3. *Under assumptions of Theorem 8,*

(1) *if* $-1 < s \le 1$, *then*

$$\begin{aligned}
&\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2} f(H(a,b))-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \\
&\le \frac{b-a}{24ab}\left[\frac{6}{(s+1)(s+2)}\right]^{1/q}\Big\{\Big[\big(2(1-\lambda)^{s+2}+\lambda(s+2)-1\big)|f'(a)|^q \\
&\quad +m\left(2\lambda^{s+2}+(1-\lambda)(s+2)-1\right)|f'(mH(a,b))|^q\Big]^{1/q}\Big[\big(2(1-\lambda)^3 \\
&\quad +3\lambda-1\big)a^{2q/(q-1)}+(2\lambda^3-3\lambda+2)H^{2q/(q-1)}(a,b)\Big]^{1-1/q}+\Big[\big(2\,(1-\mu)^{s+2} \\
&\quad +\mu(s+2)-1\big)|f'(b)|^q+m\left(2\mu^{s+2}+(1-\mu)(s+2)-1\right)|f'(mH(a,b))|^q\Big]^{1/q} \\
&\quad \times\Big[(2(1-\mu)^3+3\mu-1)b^{2q/(q-1)}+\left(2\mu^3-3\mu+2\right)H^{2q/(q-1)}(a,b)\Big]^{1-1/q}\Big\};
\end{aligned}$$

(2) *if* $s=-1$, *then*

$$\begin{aligned}
&\left|f(H(a,b))-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \le \frac{b-a}{2\times 12^{(q-1)/q}ab}\Big\{\Big[(2\ln 2-1)|f'(a)|^q \\
&\quad +m|f'(mb)|^q\Big]^{1/q}\Big[a^{2q/(q-1)}+2H^{2q/(q-1)}(a,b)\Big]^{1-1/q}+\Big[|f'(a)|^q \\
&\quad +m(2\ln 2-1)|f'(mb)|^q\Big]^{1/q}\Big[b^{2q/(q-1)}+2H^{2q/(q-1)}(a,b)\Big]^{1-1/q}\Big\}.
\end{aligned}$$

Proof. Substituting

$$Q(a,H(a,b),\lambda) \le \frac{2(1-\lambda)^3+3\lambda-1}{6}a^{2q/(q-1)}+\frac{2\lambda^3-3\lambda+2}{6}H^{2q/(q-1)}(a,b)$$

into (3.7) yields Corollary 3. □

Theorem 9. *Let* $f:(0,d]\to\mathbb{R}$ *is extended harmonically* (s,m)*-convex for some fixed* $m\in(0,1]$, $s\in(-1,1]$, *and* $a,b\in(0,d]$ *with* $a<b$. *If* $f\in L_1([a,b])$, *then*

$$2^s f(H(a,b)) \le \frac{ab}{b-a}\int_a^b \frac{f(x)+mf(mx)}{x^2}dx \tag{3.9}$$

and

$$\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \le \min\left\{\frac{f(a)+mf(mb)}{s+1}, \frac{mf(ma)+f(b)}{s+1}\right\}.$$

In particular, when $m = 1$, we have

$$2^{s-1} f(H(a,b)) \le \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \le \frac{f(a)+f(b)}{s+1}. \tag{3.10}$$

Remark 1. The inequality (3.10) appeared in [17].

Proof. From the extended harmonic (s,m)-convexity f, it follows that

$$\begin{aligned} f(H(a,b)) &= \int_0^1 f\left(\frac{2}{ta^{-1}+(1-t)b^{-1}+tb^{-1}+(1-t)a^{-1}}\right)dt \\ &\le \frac{1}{2^s}\int_0^1 \left[f\left(\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-1}\right)+mf\left(m\left(\frac{t}{b}+\frac{1-t}{a}\right)^{-1}\right)\right]dt. \end{aligned}$$

Let $x = \left(\frac{t}{a}+\frac{1-t}{b}\right)^{-1}$ for $t \in [0,1]$. Then

$$\int_0^1 f\left(\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-1}\right)dt = \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx. \tag{3.11}$$

Similarly, we have

$$\int_0^1 f\left(m\left(\frac{t}{b}+\frac{1-t}{a}\right)^{-1}\right)dt = \frac{ab}{b-a}\int_a^b \frac{f(mx)}{x^2}dx. \tag{3.12}$$

From (3.11) and (3.12), the inequality (3.9) follows immediately.

Let $x = \left(\frac{t}{a}+\frac{1-t}{b}\right)^{-1}$ for $t \in [0,1]$. Then

$$\begin{aligned} \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx &= \int_0^1 f\left(\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-1}\right)dt \\ &\le \int_0^1 [t^s f(a)+m(1-t)^s f(mb)]\,dt = \frac{f(a)+mf(mb)}{s+1}. \end{aligned}$$

The proof of Theorem 9 is complete. □

Corollary 4. *Under assumptions of Theorem 9, if $s = m = 1$, then*

$$f(H(a,b)) \le \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \le \frac{f(a)+f(b)}{2}.$$

Theorem 10. *For $m \in (0,1]$ and $s \in (-1,1]$, let $f : (0,d] \to \mathbb{R}$ is extended harmonically (s,m)-convex and $a,b \in (0,d]$ with $a < b$. If $f \in L_1([a,b])$, then*

$$2^{s-2}H^2(a,b) f(H(a,b)) \le \frac{ab}{b-a}\int_a^b [f(x)+mf(mx)]dx$$

and

$$\frac{ab}{b-a}\int_a^b f(x)dx \le \frac{a^2}{s+1}\left[m \times {}_2F_1\left(2,1,s+2,1-ab^{-1}\right) f(ma)\right.$$

$$+{}_2F_1\left(2,s+1,s+2,1-ab^{-1}\right)f(b)\big].$$

where ${}_2F_1(a,b;c;z)$ *is the hypergeometric function defined by* (3.2).

Proof. By the extended harmonic (s,m)-convexity of f, we have

$$\begin{aligned}
H(a,b)]^2 f(H(a,b)) &\le 2^{2-s}\int_0^1\left[\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-2}\right]f\left(\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-1}\right)dt\\
&\quad+2^{2-s}m\int_0^1\left[\left(\tfrac{t}{b}+\tfrac{1-t}{a}\right)^{-2}\right]f\left(m\left(\tfrac{t}{b}+\tfrac{1-t}{a}\right)^{-1}\right)dt\\
&\le\frac{ab}{b-a}\int_a^b\frac{f(x)+mf(mx)}{2^{s-2}}dx
\end{aligned}$$

and

$$\begin{aligned}
\frac{ab}{b-a}\int_a^b f(x)dx &= \int_0^1\left(\frac{1-t}{a}+\frac{t}{b}\right)^{-2}f\left(\left(\frac{1-t}{a}+\frac{t}{b}\right)^{-1}\right)dt\\
&\le\int_0^1\left(\frac{1-t}{a}+\frac{t}{b}\right)^{-2}[m(1-t)^s f(ma)+t^s f(b)]dt\\
&=\frac{a^2}{s+1}\big[m\times{}_2F_1(2,1,s+2,1-ab^{-1})f(ma)\\
&\quad+{}_2F_1(2,s+1,s+2,1-ab^{-1})f(b)\big].
\end{aligned}$$

The proof of Theorem 10 is thus complete. □

Corollary 5. *Under assumptions of Theorem 10, if* $s=m=1$*, then*

$$\begin{aligned}
\frac{[H(a,b)]^2 f(H(a,b))}{4}&\le\frac{ab}{b-a}\int_a^b f(x)dx\\
&\le\frac{a^2b[b\ln(a^{-1}b)-(b-a)]}{(b-a)^2}f(a)+\frac{ab^2[(b-a)-a\ln(a^{-1}b)]}{(b-a)^2}f(b).
\end{aligned}$$

4. Funding

This work was partially supported by the National Natural Science Foundation of China (Grant No. 11901322), by the Natural Science Foundation of Inner Mongolia (Grant No. 2018MS01023), and by the Foundation of the Research Program of Science and Technology at Universities of Inner Mongolia of China (Grant No. NJZY20119).

5. Acknowledgements

The authors appreciate anonymous referees and the handling editor for their careful corrections to and valuable comments on the original version of this paper.

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