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NEW EXTENSIONS OF THE HERMITE-HADAMARD INEQUALITIES INVOLVING RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS

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Abstract. In this study, we establish the above and below bounds for the left and right hand sides of fractional Hermite-Hadamard inequalities by using functions whose second derivatives are bounded. We also give some refinements of fractional Hermite-Hadamard inequalities by using the functions that have the conditions $f'(a+b-t)-f'(t)\geq 0$, $t\in\left[a,\frac{a+b}{2}\right]$.

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1. Introduction

The Hermite-Hadamard inequality, which is the first fundamental result for convex mappings with a natural geometrical interpretation and many applications, has drawn attention much interest in elementary mathematics. A number of mathematicians have devoted their efforts to generalise, refine, counterpart and extend it for different classes of functions such as using convex mappings.

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are considerable significant in the literature (see, e.g., [18, p.137], [7]). These inequalities state that if $f:I\to\mathbb{R}$ is a convex function on the interval I of real numbers and $a,b\in I$ with $a<b$, then

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if f is concave.

In the following we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper.

Definition 1. Let $f\in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of order $\alpha>0$ with $a\geq 0$ are defined by

$$J_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt,\ \ x>a$$

and

$$J_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{b}(t-x)^{\alpha-1}f(t)dt,\ \ x<b$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J_{a+}^{0}f(x)=J_{b-}^{0}f(x)=f(x)$.

For more information about fraction calculus please refer to [9, 14, 16, 19].

In [23], Sarikaya et al. first give the following interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a positive function with $0\le a<b$ and $f\in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right)\le\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\le\frac{f(a)+f(b)}{2}\tag{1.2}$$

with $\alpha>0$.

Sarikaya and Yildirim also give the following Hermite-Hadamard type inequality for the Riemann-Lioville fractional integrals:

Theorem 2 ([24])**.** *Let $f:[a,b]\to\mathbb{R}$ be a positive function with $a<b$ and $f\in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right)\le\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{\left(\frac{a+b}{2}\right)^{+}}^{\alpha}f(b)+J_{\left(\frac{a+b}{2}\right)^{-}}^{\alpha}f(a)\right]\le\frac{f(a)+f(b)}{2}.\tag{1.3}$$

Moreover, Dragomir give the following another version of Hermite-Hadamard inequality for Riemann-Lioville fractional integrals:

Theorem 3 ([8])**.** *Let $f:[a,b]\to\mathbb{R}$ be a positive function with $a<b$ and $f\in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right)\le\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a^{+}}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b^{-}}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\le\frac{f(a)+f(b)}{2}\tag{1.4}$$

Over the years several papers devoted to fractional Hermite-Hadamard inequalities. One can refer to the references [1–4, 6, 10–13, 15, 17, 20–26] for some of them.

F. Chen prove the following inequalities which give the above and below bounds for the left and right hand sides of inequality (1.2).

Theorem 4 ([5])**.** *Let $f:[a,b]\to\mathbb{R}$ be positive, twice differentiable functions with $a<b$ and $f\in L_1[a,b]$. If f'' is bounded, then we have the inequalities*

$$\frac{m\alpha}{(b-a)^{\alpha}}\int_{a}^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^{2}\left[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}\right]dx$$

$$\leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b)+J_{b^-}^{\alpha} f(a)\right]-f\left(\frac{a+b}{2}\right)$$

$$\leq \frac{M\alpha}{(b-a)^{\alpha}} \int_{a}^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^{2}\left[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}\right] dx$$

and

$$\frac{-M\alpha}{2(b-a)^{\alpha}} \int_{a}^{\frac{a+b}{2}}(x-a)(b-x)\left[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}\right] dx$$

$$\leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b)+J_{b^-}^{\alpha} f(a)\right]-\frac{f(a)+f(b)}{2}$$

$$\leq \frac{-m\alpha}{2(b-a)^{\alpha}} \int_{a}^{\frac{a+b}{2}}(x-a)(b-x)\left[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}\right] dx$$

for $\alpha>0$, *where* $m=\inf_{t\in[a,b]} f''(t)$, $M=\sup_{t\in[a,b]} f''(t)$.

In this paper we establish extensions of inequalities (1.3) and (1.4).

2. Main Results

Firstly, we give the following inequalities which give the above and below bounds for the left and right hand sides of inequality (1.3).

Theorem 5. *Let* $f:[a,b]\rightarrow\mathbb{R}$ *be positive, twice differentiable functions with* $a<b$ *and* $f\in L_1[a,b]$. *If* f'' *is bounded, i.e.* $m\leq f''(t)\leq M$, $t\in[a,b]$, $m,M\in\mathbb{R}$, *then we have the inequalities*

$$\frac{m(b-a)^{2}}{4(\alpha+1)(\alpha+2)} \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{\left(\frac{a+b}{2}\right)^+}^{\alpha} f(b)+J_{\left(\frac{a+b}{2}\right)^-}^{\alpha} f(a)\right]-f\left(\frac{a+b}{2}\right) \leq \frac{M(b-a)^{2}}{4(\alpha+1)(\alpha+2)} \tag{2.1}$$

and

$$\frac{m(b-a)^{2}\alpha(\alpha+3)}{8(\alpha+1)(\alpha+2)} \leq \frac{f(a)+f(b)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{\left(\frac{a+b}{2}\right)^+}^{\alpha} f(b)+J_{\left(\frac{a+b}{2}\right)^-}^{\alpha} f(a)\right] \leq \frac{M(b-a)^{2}\alpha(\alpha+3)}{8(\alpha+1)(\alpha+2)}. \tag{2.2}$$

Proof. By using the change of variables we have

$$\begin{aligned}
&\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right]\\
&=\frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}}\left[\int_{\frac{a+b}{2}}^{b}(b-x)^{\alpha-1}f(x)dx+\int_{a}^{\frac{a+b}{2}}(x-a)^{\alpha-1}f(x)dx\right]\\
&=\frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}}\left[\int_{a}^{\frac{a+b}{2}}(b-x)^{\alpha-1}f(a+b-x)dx+\int_{a}^{\frac{a+b}{2}}(x-a)^{\alpha-1}f(x)dx\right]\\
&=\frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}}\int_{a}^{\frac{a+b}{2}}\left[f(x)+f(a+b-x)\right](x-a)^{\alpha-1}dx. \qquad (2.3)
\end{aligned}$$

By equality (2.3), we get

$$\begin{aligned}
&\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right]-f\left(\frac{a+b}{2}\right)\\
&=\frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}}\int_{a}^{\frac{a+b}{2}}\left[f(x)+f(a+b-x)\right](x-a)^{\alpha-1}dx-f\left(\frac{a+b}{2}\right)\\
&=\frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}}\int_{a}^{\frac{a+b}{2}}\left[f(x)+f(a+b-x)-2f\left(\frac{a+b}{2}\right)\right](x-a)^{\alpha-1}dx. \qquad (2.4)
\end{aligned}$$

Using the facts that

$$f(x)-f\left(\frac{a+b}{2}\right)=\int_{\frac{a+b}{2}}^{x}f'(t)dt$$

$$f\left(\frac{a+b}{2}\right)-f(a+b-x)=-\int_{\frac{a+b}{2}}^{a+b-x}f'(t)dt,$$

we have

$$f(x)+f(a+b-x)-2f\left(\frac{a+b}{2}\right)=\int_{\frac{a+b}{2}}^{a+b-x}f'(t)dt-\int_{x}^{\frac{a+b}{2}}f'(t)dt$$

$$= \int_x^{\frac{a+b}{2}} f'(a+b-u)du - \int_x^{\frac{a+b}{2}} f'(t)\,dt$$

$$= \int_x^{\frac{a+b}{2}} \left[f'(a+b-t) - f'(t)\right] dt. \tag{2.5}$$

We also have

$$f'(a+b-t) - f'(t) = \int_t^{a+b-t} f''(u)du. \tag{2.6}$$

By using equality (2.6) and $m < f''(u) < M$, $u \in [a,b]$, we obtain,

$$m \int_t^{a+b-t} du \leq \int_t^{a+b-t} f''(u)du \leq M \int_t^{a+b-t} du$$

i.e.

$$m(a+b-2t) \leq f'(a+b-t) - f'(t) \leq M(a+b-2t). \tag{2.7}$$

Integrating inequality (2.7) with respect to t on $\left[x, \frac{a+b}{2}\right]$, we get

$$m\left(\frac{a+b}{2} - x\right)^2 \leq \int_x^{\frac{a+b}{2}} \left[f'(a+b-t) - f'(t)\right] dt \leq M\left(\frac{a+b}{2} - x\right)^2.$$

By equality (2.5),

$$m\left(\frac{a+b}{2} - x\right)^2 \leq f(x) + f(a+b-x) - 2f\left(\frac{a+b}{2}\right) \leq M\left(\frac{a+b}{2} - x\right)^2. \tag{2.8}$$

Multiplying inequality (2.8) by $\frac{2^{\alpha-1}\alpha(x-a)^{\alpha-1}}{(b-a)^{\alpha}}$ and integrating the resultant inequality with respect to x on $\left[a, \frac{a+b}{2}\right]$, we establish

$$\frac{m2^{\alpha-1}\alpha}{(b-a)^{\alpha}} \int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - x\right)^2 (x-a)^{\alpha-1} dx$$

$$\leq \frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}} \int_a^{\frac{a+b}{2}} \left[f(x) + f(a+b-x) - 2f\left(\frac{a+b}{2}\right)\right] (x-a)^{\alpha-1} dx$$

$$\leq \frac{M2^{\alpha-1}\alpha}{(b-a)^{\alpha}} \int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - x\right)^2 (x-a)^{\alpha-1} dx.$$

By using equality (2.4) and

$$\int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2}-x\right)^2 (x-a)^{\alpha-1} dx = \left(\frac{b-a}{2}\right)^{\alpha+2} \frac{2}{\alpha(\alpha+1)(\alpha+2)},$$

then we obtain

$$\begin{aligned} \frac{m(b-a)^2}{4(\alpha+1)(\alpha+2)} &\leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^+} f(b) + J^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a)\right] - f\left(\frac{a+b}{2}\right) \\ &\leq \frac{M(b-a)^2}{4(\alpha+1)(\alpha+2)} \end{aligned}$$

which completes the proof of inequalities (2.1).

On the other hand, by equality (2.3), we have

$$\begin{aligned} &\frac{f(a)+f(b)}{2} - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^+} f(b) + J^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a)\right] \\ &= \frac{f(a)+f(b)}{2} - \frac{2^{\alpha-1}\alpha}{(b-a)^\alpha}\int_a^{\frac{a+b}{2}} \left[f(x)+f(a+b-x)\right](x-a)^{\alpha-1}dx \\ &= \frac{2^{\alpha-1}\alpha}{(b-a)^\alpha}\int_a^{\frac{a+b}{2}} \left[f(a)+f(b)-f(x)-f(a+b-x)\right](x-a)^{\alpha-1}dx. \end{aligned} \tag{2.9}$$

By using the equalities

$$f(x)-f(a) = \int_a^x f'(t)dt$$

and

$$f(b)-f(a+b-x) = \int_{a+b-x}^b f'(t)dt,$$

we get

$$\begin{aligned} f(a)+f(b)-f(x)-f(a+b-x) &= \int_{a+b-x}^b f'(t)dt - \int_a^x f'(t)dt \\ &= \int_a^x f'(a+b-u)du - \int_a^x f'(t)dt \\ &= \int_a^x \left[f'(a+b-t)dt - f'(t)\right]dt. \end{aligned} \tag{2.10}$$

By integrating inequality (2.7) with respect to t on $[a,x]$, we get

$$m\int_a^x (a+b-2t)\,dt \le \int_a^x \left[f'(a+b-t)-f'(t)\right]dt \le M\int_a^x (a+b-2t)\,dt.$$

That is,

$$m(x-a)(b-x) \le \quad f(a)+f(b)-f(x)-f(a+b-x) \le M(x-a)(b-x). \quad (2.11)$$

Multiplying inequality (2.11) by $\frac{2^{\alpha-1}\alpha(x-a)^{\alpha-1}}{(b-a)^{\alpha}}$ and integrating the resultant inequality with respect to x on $\left[a,\frac{a+b}{2}\right]$, we establish

$$\begin{aligned}
&\frac{m2^{\alpha-1}\alpha}{(b-a)^{\alpha}}\int_a^{\frac{a+b}{2}} (b-x)(x-a)^{\alpha-1}\,dx \\
&\le \frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}}\int_a^{\frac{a+b}{2}} \left[f(a)+f(b)-f(x)-f(a+b-x)\right](x-a)^{\alpha-1}\,dx \\
&\le \frac{M2^{\alpha-1}\alpha}{(b-a)^{\alpha}}\int_a^{\frac{a+b}{2}} (b-x)(x-a)^{\alpha-1}\,dx,
\end{aligned}$$

i.e.

$$\begin{aligned}
\frac{m(b-a)^2\alpha(\alpha+3)}{8(\alpha+1)(\alpha+2)} &\le \frac{f(a)+f(b)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right] \\
&\le \frac{M(b-a)^2\alpha(\alpha+3)}{8(\alpha+1)(\alpha+2)}.
\end{aligned}$$

which gives inequalities (2.2).

This completes the proof of the theorem. □

Now we give the following refinement of inequality (1.3).

Theorem 6. *Let $f:[a,b]\to\mathbb{R}$ be positive, twice differentiable functions with $a<b$ and $f\in L_1[a,b]$. If $f'(a+b-x)\ge f'(x)$ for all $x\in\left[a,\frac{a+b}{2}\right]$, then we have the inequalities*

$$f\left(\frac{a+b}{2}\right) \le \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right] \le \frac{f(a)+f(b)}{2}. \quad (2.12)$$

Proof. Since $f'(a+b-x)\ge f'(x)$ for all $x\in\left[a,\frac{a+b}{2}\right]$, by equalities (2.4) and (2.5), we have

$$\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right]-f\left(\frac{a+b}{2}\right)$$

$$= \frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}} \int_{a}^{\frac{a+b}{2}} \left[f(x) + f(a+b-x) - 2f\left(\frac{a+b}{2}\right) \right] (x-a)^{\alpha-1}\, dx$$

$$= \frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}} \int_{a}^{\frac{a+b}{2}} \left[\int_{x}^{\frac{a+b}{2}} \left[f'(a+b-t) - f'(t) \right] dt \right] (x-a)^{\alpha-1}\, dx$$

$$\geq 0$$

which gives the first inequality in (2.12).

Similarly, by equalities (2.9) and (2.10), we get

$$\frac{f(a)+f(b)}{2} - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}} f(b) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}} f(a) \right]$$

$$= \frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}} \int_{a}^{\frac{a+b}{2}} \left[f(a)+f(b) - f(x) - f(a+b-x) \right] (x-a)^{\alpha-1} dx$$

$$= \frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}} \int_{a}^{\frac{a+b}{2}} \left[\int_{a}^{x} \left[f'(a+b-t)dt - f'(t) \right] dt \right] (x-a)^{\alpha-1} dx$$

$$\geq 0.$$

This completes the proof. □

Now, we establish the following inequalities which give the above and below bounds for the left and right hand sides of inequality (1.4).

Theorem 7. *Let $f : [a,b] \to \mathbb{R}$ be positive, twice differentiable functions with $a < b$ and $f \in L_1[a,b]$. If f'' is bounded, i.e. $m \leq f''(t) \leq M$, $t \in [a,b]$, $m, M \in \mathbb{R}$, then we have the inequalities*

$$\frac{m(b-a)^2\alpha}{8(\alpha+2)} \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[J^{\alpha}_{a^+} f\left(\frac{a+b}{2}\right) + J^{\alpha}_{b^-} f\left(\frac{a+b}{2}\right) \right] - f\left(\frac{a+b}{2}\right) \leq \frac{m(b-a)^2\alpha}{8(\alpha+2)} \tag{2.13}$$

and

$$\frac{m(b-a)^2}{4(a+2)} \leq \frac{f(a)+f(b)}{2} - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[J^{\alpha}_{a^+} f\left(\frac{a+b}{2}\right) + J^{\alpha}_{b^-} f\left(\frac{a+b}{2}\right) \right] \leq \frac{m(b-a)^2}{4(a+2)}. \tag{2.14}$$

Proof. From the definition of Riemann-Liouville fractional integrals, we get

$$\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a^+}f\left(\frac{a+b}{2}\right)+J_{b^-}f\left(\frac{a+b}{2}\right)\right]$$
$$=\frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}}\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^{\alpha-1}f(x)dx+\int_{\frac{a+b}{2}}^{b}\left(x-\frac{a+b}{2}\right)^{\alpha-1}f(x)dx$$
$$=\frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}}\int_a^{\frac{a+b}{2}}[f(x)+f(a+b-x)]\left(\frac{a+b}{2}-x\right)^{\alpha-1}dx. \tag{2.15}$$

By equality (2.15), we have

$$\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{a^+}f\left(\frac{a+b}{2}\right)+J^{\alpha}_{b^-}f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)$$
$$=\frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}}\int_a^{\frac{a+b}{2}}[f(x)+f(a+b-x)]\left(\frac{a+b}{2}-x\right)^{\alpha-1}dx-f\left(\frac{a+b}{2}\right)$$
$$=\frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}}\int_a^{\frac{a+b}{2}}\left[f(x)+f(a+b-x)-2f\left(\frac{a+b}{2}\right)\right]\left(\frac{a+b}{2}-x\right)^{\alpha-1}dx. \tag{2.16}$$

Moreover, using the identities

$$f\left(\frac{a+b}{2}\right)-f(x)=\int_x^{\frac{a+b}{2}}f'(t)dt$$

and

$$f(a+b-x)-f\left(\frac{a+b}{2}\right)=\int_{\frac{a+b}{2}}^{a+b-x}f'(t)dt,$$

we obtain

$$f(x)+f(a+b-x)-2f\left(\frac{a+b}{2}\right)=\int_{\frac{a+b}{2}}^{a+b-x}f'(t)dt-\int_x^{\frac{a+b}{2}}f'(t)dt$$
$$=\int_x^{\frac{a+b}{2}}f'(a+b-u)du-\int_x^{\frac{a+b}{2}}f'(t)\,dt$$

$$= \int_{x}^{\frac{a+b}{2}} \left[f'(a+b-t) - f'(t) \right] dt. \tag{2.17}$$

By integrating inequality (2.7) with respect to t on $\left[x, \frac{a+b}{2}\right]$, we get

$$m\left(\frac{a+b}{2}-x\right)^2 \leq \int_{x}^{\frac{a+b}{2}} \left[f'(a+b-t) - f'(t) \right] dt \leq M\left(\frac{a+b}{2}-x\right)^2.$$

That is, by equality (2.17),

$$m\left(\frac{a+b}{2}-x\right)^2 \leq f(x)+f(a+b-x)-2f\left(\frac{a+b}{2}\right) \leq M\left(\frac{a+b}{2}-x\right)^2. \tag{2.18}$$

Multiplying inequality (2.18) by $\frac{2^{\alpha-1}\alpha(x-a)^{\alpha-1}}{(b-a)^{\alpha}}$ and integrating the resultant inequality with respect to x on $\left[a, \frac{a+b}{2}\right]$, we establish

$$\begin{aligned}
&\frac{m2^{\alpha-1}\alpha}{(b-a)^{\alpha}} \int_{a}^{\frac{a+b}{2}} \left(\frac{a+b}{2}-x\right)^{\alpha+1} dx \\
&\leq \frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}} \int_{a}^{\frac{a+b}{2}} \left[f(x)+f(a+b-x)-2f\left(\frac{a+b}{2}\right)\right] \left(\frac{a+b}{2}-x\right)^{\alpha-1} dx \\
&\leq \frac{M2^{\alpha-1}\alpha}{(b-a)^{\alpha}} \int_{a}^{\frac{a+b}{2}} \left(\frac{a+b}{2}-x\right)^{\alpha+1} dx.
\end{aligned}$$

It follows that

$$\begin{aligned}
\frac{m(b-a)^2\alpha}{8(\alpha+2)} &\leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[J_{a^+}^{\alpha} f\left(\frac{a+b}{2}\right) + J_{b^-}^{\alpha} f\left(\frac{a+b}{2}\right)\right] - f\left(\frac{a+b}{2}\right) \\
&\leq \frac{m(b-a)^2\alpha}{8(\alpha+2)}
\end{aligned}$$

which proves inequality (2.13).

On the other hand, using identity (2.15), we get

$$\begin{aligned}
&\frac{f(a)+f(b)}{2} - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[J_{a^+}^{\alpha} f\left(\frac{a+b}{2}\right) + J_{b^-}^{\alpha} f\left(\frac{a+b}{2}\right)\right] \\
&= \frac{f(a)+f(b)}{2} - \frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}} \int_{a}^{\frac{a+b}{2}} \left[f(x)+f(a+b-x)\right] \left(\frac{a+b}{2}-x\right)^{\alpha-1} dx
\end{aligned}$$

$$= \frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}} \int_a^{\frac{a+b}{2}} [f(a)+f(b)-f(x)-f(a+b-x)] \left(\frac{a+b}{2}-x\right)^{\alpha-1} dx. \qquad (2.19)$$

Using the facts that

$$f(x)-f(a) = \int_a^x f'(t)dt$$

and

$$f(b)-f(a+b-x) = \int_{a+b-x}^b f'(t)dt,$$

we get

$$\begin{aligned} f(a)+f(b)-f(x)-f(a+b-x) &= \int_{a+b-x}^b f'(t)dt - \int_a^x f'(t)dt \\ &= \int_a^x f'(a+b-u)du - \int_a^x f'(t)dt \\ &= \int_a^x \left[f'(a+b-t)dt - f'(t)\right] dt. \qquad (2.20) \end{aligned}$$

By integrating inequality (2.7) with respect to t on $[a,x]$, we get

$$\begin{aligned} m\left[\left(\frac{b-a}{2}\right)^2 - \left(\frac{a+b}{2}-x\right)^2\right] &\le f(a)+f(b)-f(x)-f(a+b-x) \\ &\le M\left[\left(\frac{b-a}{2}\right)^2 - \left(\frac{a+b}{2}-x\right)^2\right]. \qquad (2.21) \end{aligned}$$

Multiplying inequality (2.21) by $\frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}}\left(\frac{a+b}{2}-x\right)^{\alpha-1}$ and integrating the resultant inequality with respect to x on $\left[a, \frac{a+b}{2}\right]$, we establish

$$\begin{aligned} &\frac{m2^{\alpha-1}\alpha}{(b-a)^{\alpha}} \int_a^{\frac{a+b}{2}} \left[\left(\frac{b-a}{2}\right)^2 - \left(\frac{a+b}{2}-x\right)^2\right] \left(\frac{a+b}{2}-x\right)^{\alpha-1} dx \\ &\le \frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}} \int_a^{\frac{a+b}{2}} [f(a)+f(b)-f(x)-f(a+b-x)] \left(\frac{a+b}{2}-x\right)^{\alpha-1} dx \end{aligned}$$

$$\leq \frac{M2^{\alpha-1}\alpha}{(b-a)^{\alpha}} \int_{a}^{\frac{a+b}{2}} \left[\left(\frac{b-a}{2}\right)^2 - \left(\frac{a+b}{2}-x\right)^2 \right] \left(\frac{a+b}{2}-x\right)^{\alpha-1}.$$

By equality (2.19) and equality

$$\frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}} \int_{a}^{\frac{a+b}{2}} \left[\left(\frac{b-a}{2}\right)^2 - \left(\frac{a+b}{2}-x\right)^2 \right] \left(\frac{a+b}{2}-x\right)^{\alpha-1} dx = \frac{(b-a)^2}{4(a+2)},$$

we get the desired inequality (2.14).

This completes the proof. □

Theorem 8. *Let $f : [a,b] \to \mathbb{R}$ be positive, twice differentiable functions with $a < b$ and $f \in L_1[a,b]$. If $f'(a+b-x) \geq f'(x)$ for all $x \in \left[a, \frac{a+b}{2}\right]$, then we have the inequalities*

$$f\left(\frac{a+b}{2}\right) \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[J^{\alpha}_{a^+} f\left(\frac{a+b}{2}\right) + J^{\alpha}_{b^-} f\left(\frac{a+b}{2}\right) \right] \leq \frac{f(a)+f(b)}{2}. \tag{2.22}$$

Proof. From equalities (2.16) and (2.17), we have

$$\begin{aligned}
&\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[J^{\alpha}_{a^+} f\left(\frac{a+b}{2}\right) + J^{\alpha}_{b^-} f\left(\frac{a+b}{2}\right) \right] - f\left(\frac{a+b}{2}\right) \\
&= \frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}} \int_{a}^{\frac{a+b}{2}} \left[f(x) + f(a+b-x) - 2f\left(\frac{a+b}{2}\right) \right] \left(\frac{a+b}{2}-x\right)^{\alpha-1} dx \\
&= \frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}} \int_{a}^{\frac{a+b}{2}} \left[\int_{x}^{\frac{a+b}{2}} \left[f'(a+b-t) - f'(t) \right] dt \right] \left(\frac{a+b}{2}-x\right)^{\alpha-1} dx \\
&\geq 0
\end{aligned}$$

which proves the first inequality in (2.22).

Similarly, by equalities (2.19) and (2.20)

$$\begin{aligned}
&\frac{f(a)+f(b)}{2} - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[J^{\alpha}_{a^+} f\left(\frac{a+b}{2}\right) + J^{\alpha}_{b^-} f\left(\frac{a+b}{2}\right) \right] \\
&= \frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}} \int_{a}^{\frac{a+b}{2}} \left[f(a) + f(b) - f(x) - f(a+b-x) \right] \left(\frac{a+b}{2}-x\right)^{\alpha-1} dx \\
&= \frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}} \int_{a}^{\frac{a+b}{2}} \left[\int_{a}^{x} \left[f'(a+b-t)dt - f'(t) \right] dt \right] \left(\frac{a+b}{2}-x\right)^{\alpha-1} dx
\end{aligned}$$

$$\geq 0.$$

This completes the proof. □

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