



THE ALGEBRA OF BESSEL SEQUENCES AND MEANS OF FRAMES IN HILBERT SPACES

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Abstract. In this paper, a one-to-one correspondence between Bessel sequences and bounded linear operators is provided. This leads to an algebra structure on the set of all Bessel sequences in a separable Hilbert space. Some kinds of frames as special classes of operators are considered. Also, normal Bessel sequences and positive frames are presented. Finally, power means of positive frames are introduced. These allow researches to construct a large number of new frames from existing frames.

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1. INTRODUCTION AND PRELIMINARIES

Frames in Hilbert spaces were introduced by Duffin and Schaeffer [5] as a part of their study in non-harmonic Fourier series. Frames did not find serious attention until 1986 when Daubechies, Grossmann and Meyer [2] brought it all back to life during their fundamental work on wavelets.

There are many generalizations of frame. Most of these generalizations have been found useful applications. Some generalizations of frame significance have been presented such as generalized frames [12, 13], continuous frames [7], continuous fusion frames [6] and K-frames [8]. In the sequel, we discuss results on one such generalization of frames, called K-frames.

In recent years many researchers studied some methods for constructing new frames from existing ones. In 2009, Obeidat et al. [11] studied sum of two frames.

A basic and frequently used tool in study and characterizations of frames is the frame operator. It is known that the frame operator of a given frame is a bounded positive invertible operator. Since two different frames may have the same frame operator, a natural question arises here: Is there a way to correspond each Bessel sequence to one and only one bounded linear operator such that its properties characterizes special kinds of frames?

In this paper, we find an appropriate correspondence between Bessel sequences (some kinds of frames) with bounded linear operators. Using this relation, we introduce an algebraic structure of the set of Bessel sequences in Hilbert spaces. Also, we study the means of frames.

Now, we recall some basic definitions and results that we need in the sequel. For background material on frame theory and the related topics, we refer to [1]. Throughout the paper \mathcal{H} is a separable Hilbert space with an arbitrary fixed orthonormal basis $\{e_j\}_{j \in J}$ where $J \subseteq \mathbb{N}$. We denote by $B(\mathcal{H})$ the space of all bounded linear operators on \mathcal{H} . For $A \in B(\mathcal{H})$, we denote $R(A)$ for range of A . A bounded operator $A \in B(\mathcal{H})$ is called positive if $\langle Af, f \rangle \geq 0$ for all $f \in \mathcal{H}$. Let $B_+(\mathcal{H})$ denote the set of all bounded and positive operators on \mathcal{H} . Every positive operator is clearly self-adjoint. For $A, B \in B(\mathcal{H})$, we write $A \leq B$ if $B - A$ is positive. If $A \in B(\mathcal{H})$ is positive, then there exists a unique positive operator B such that $B^2 = A$. This will be denoted by $B = A^{\frac{1}{2}}$. Moreover, if an operator C commutes with A , then C commutes with every operator in the C^* -algebra generated by A and I , specially C commutes with $A^{\frac{1}{2}}$.

Definition 1. A family $F = \{f_j\}_{j \in J}$ of elements in \mathcal{H} is called a *Bessel sequence* if there exists a constant $\mu > 0$ such that

$$\sum_{j \in J} |\langle f, f_j \rangle|^2 \leq \mu \|f\|^2, \quad \forall f \in \mathcal{H}.$$

For a Bessel sequence $\{f_j\}_{j \in J}$, the operator $T_F : \ell_2 \rightarrow \mathcal{H}$ defined by $T_F(\{c_j\}_{j \in J}) = \sum_{j \in J} c_j f_j$, is bounded. The operator T_F is called the *synthesis operator* (or, *pre-frame operator*). The adjoint of T_F , $T_F^* : \mathcal{H} \rightarrow \ell_2$ defined by $T_F^* f = \{\langle f, f_j \rangle\}_{j \in J}$ is called the *analysis operator*. By composing T_F and T_F^* , we obtain the *frame operator*

$$S_F f = T_F T_F^* f = \sum_{j \in J} \langle f, f_j \rangle f_j \quad \forall f \in \mathcal{H}. \quad (1.1)$$

Moreover, for each $f \in \mathcal{H}$,

$$\langle S_F f, f \rangle = \sum_{j \in J} |\langle f, f_j \rangle|^2.$$

Definition 2. The Bessel sequence $F = \{f_j\}_{j \in J}$ is a *frame* for \mathcal{H} if there exists a constant $\lambda > 0$ such that

$$\lambda \|f\|^2 \leq \sum_{j \in J} |\langle f, f_j \rangle|^2, \quad \forall f \in \mathcal{H}.$$

The constant λ is called the *lower frame bound* for the frame $\{f_j\}_{j \in J}$. Note that this bound is not unique.

For a given frame $F = \{f_j\}_{j \in J}$, the operator S_F is a bounded positive self-adjoint operator and hence S_F has a unique positive square root, denoted by $S_F^{1/2}$. The frame operator S_F of a given frame is invertible and, in fact, satisfies $\lambda I \leq S_F \leq \mu I$. For

normalized tight frames (that is, for frames with $\lambda = \mu = 1$), the frame operator is the identity operator.

The notion of K -frames has been introduced by Gavruta to study the atomic systems with respect to a bounded linear operator K in a separable Hilbert space. K -frames are more general than ordinary frames in the sense that the lower frame bound only holds for the elements in the range of K .

Definition 3 ([8]). Let $K \in B(\mathcal{H})$. A sequence $F = \{f_j\}_{j \in J}$ in \mathcal{H} is called a K -frame for \mathcal{H} if there exist constants $\lambda, \mu > 0$ such that

$$\lambda \|K^* f\|^2 \leq \sum_{j \in J} |\langle f, f_j \rangle|^2 \leq \mu \|f\|^2 \quad \forall f \in \mathcal{H}.$$

Definition 4 ([3]). Let $K \in B(\mathcal{H})$. A sequence $\{f_j\}_{j \in J}$ in \mathcal{H} is said a tight K -frame with bound λ if

$$\lambda \|K^* f\|^2 = \sum_{j \in J} |\langle f, f_j \rangle|^2, \quad \forall f \in \mathcal{H}. \quad (1.2)$$

If $\lambda = 1$ in (1.2), then the sequence $\{f_j\}_{j \in J}$ is called Parseval K -frame for \mathcal{H} .

Theorem 1 ([13]). Let $\{f_j\}_{j \in J}$ be a Bessel sequence in \mathcal{H} and $K \in B(\mathcal{H})$. Then $F = \{f_j\}_{j \in J}$ is a K -frame for \mathcal{H} if and only if there exists $\alpha > 0$ such that $S \geq \alpha K K^*$, where S is the frame operator for $\{f_j\}_{j \in J}$.

In the sequel we need the following theorem, which is known as Douglas' factorization theorem.

Theorem 2 ([4]). Let \mathcal{H} be a Hilbert spaces and $A, B \in B(\mathcal{H})$. Then the following conditions are equivalent.

- (1) $R(A) \subseteq R(B)$.
- (2) $AA^* \leq \alpha^2 BB^*$ for some $\alpha > 0$.
- (3) There exists $X \in B(\mathcal{H})$ such that $A = BX$.

2. BESSEL ALGEBRA $Bess(\mathcal{H})$

We seek a way to express each Bessel sequence (or frame) by one and only one operator. One natural way in this direction is using the frame operator.

Remark 1. Let $\{e_j\}_{j \in J}$ be an arbitrary orthonormal basis for \mathcal{H} and $U \in B_+(\mathcal{H})$. It is easily seen that $F = \{f_j\}_{j \in J} = \{U^{1/2} e_j\}_{j \in J}$ is a tight K -frame with $K = U^{1/2}$ for \mathcal{H} . Moreover, considering (1.1), for each $f \in \mathcal{H}$, we have

$$\begin{aligned} S_F f &= \sum_{j \in J} \langle f, f_j \rangle f_j = \sum_{j \in J} \langle f, U^{1/2} e_j \rangle U^{1/2} e_j \\ &= U^{1/2} \left(\sum_{j \in J} \langle U^{1/2} f, e_j \rangle e_j \right) = U^{1/2} (U^{1/2} f) \\ &= U f. \end{aligned}$$

That is the frame operator of the frame $\{U^{1/2}e_j\}_{j \in J}$ is S_F .

The following example shows that two different Bessel sequences (or frames) may have the same frame operator.

Example 1. Let $\{e_j\}_{j \in J}$ be an arbitrary orthonormal basis for \mathcal{H} . Define

$$f_1 = e_1, f_2 = e_1 - e_2, f_3 = e_1 + e_2, f_j = e_{j-1} \ (j \geq 4).$$

It is easy task to show that $\{f_j\}_{j \in J}$ is a frame with the frame operator S_F defined by

$$S_F e_1 = 3e_1, S_F e_2 = 2e_2, S_F e_j = e_j \ (j \geq 3).$$

Putting $g_j = S_F^{1/2} e_j$, we have

$$g_1 = \sqrt{3}e_1, g_2 = \sqrt{2}e_2, g_j = e_j \ (j \geq 3).$$

It follows from Remark 1 that the frame operator of $\{g_j\}_{j \in J}$ is also S_F .

This example shows that the frame operator is not appropriate for our goal. In the following theorem, we correspond each Bessel sequence to one and only one operator. We denote by $Bess(\mathcal{H})$, $Fr(\mathcal{H})$ and $KFr(\mathcal{H})$ the set of all Bessel sequences, frames and K -frames in \mathcal{H} , respectively.

Theorem 3. *Let $\{e_j\}_{j \in J}$ be an arbitrary orthonormal basis for \mathcal{H} . The Bessel sequences for \mathcal{H} are precisely the families $\{Ue_j\}_{j \in J}$, where U is a bounded operator on \mathcal{H} . Consequently, there is a one-to-one correspondence between the elements of $B(\mathcal{H})$ and $Bess(\mathcal{H})$.*

Proof. Let $\{\delta_j\}_{j \in J}$ be the canonical basis for $\ell^2(\mathbb{N})$ and $\{e_j\}_{j \in J}$ be an orthonormal basis for \mathcal{H} . Let $\Gamma : \mathcal{H} \rightarrow \ell^2(\mathbb{N})$ be the isometric isomorphism defined by $\Gamma e_j = \delta_j$. If $F = \{f_j\}_{j \in J}$ is a Bessel sequence, then the pre-frame operator T_F is bounded and $T_F \delta_j = f_j$. Putting $U_F = T_F \Gamma$, we have

$$\{f_j\}_{j \in J} = \{U_F e_j\}_{j \in J} \quad (2.1)$$

and U_F is bounded. If $U' \in B(\mathcal{H})$ is another operator such that $\{f_j\}_{j \in J} = \{U' e_j\}_{j \in J}$, then clearly $U' = U_F$. Therefore, U_F is unique.

On the other hand, if U is a bounded operator on \mathcal{H} , then for each $f \in \mathcal{H}$ we have

$$\sum_{j \in J} |\langle f, Ue_j \rangle|^2 = \sum_{j \in J} |\langle U^* f, e_j \rangle|^2 = \|U^* f\|^2 \leq \|U^*\|^2 \|f\|^2.$$

Hence $\{Ue_j\}_{j \in J}$ is a Bessel sequence for \mathcal{H} . Moreover, if S is the frame operator of $\{Ue_j\}_{j \in J}$, then for each $f \in \mathcal{H}$ we have

$$\begin{aligned} Sf &= \sum_{j \in J} \langle f, Ue_j \rangle Ue_j = \sum_{j \in J} \langle U^* f, e_j \rangle Ue_j \\ &= U \left(\sum_{j \in J} \langle U^* f, e_j \rangle e_j \right) = U (U^* f) \end{aligned}$$

$$= UU^* f \quad (2.2)$$

That is, the frame operator of the Bessel sequence $\{Ue_j\}_{j \in J}$ is UU^* . Now, define

$$\Theta : Bess(\mathcal{H}) \rightarrow B(\mathcal{H}) \quad \text{by} \quad \Theta(\{f_j\}_{j \in J}) = U_F \quad (2.3)$$

and

$$\Omega : B(\mathcal{H}) \rightarrow Bess(\mathcal{H}) \quad \text{by} \quad \Omega(U) = \{Ue_j\}_{j \in J}.$$

By the above mentioned argument, we see that Θ and Ω are well defined. Uniqueness of U_F together with (2.1) implies that

$$\Omega(\Theta(\{f_j\}_{j \in J})) = \Omega(U_F) = \{U_F e_j\}_{j \in J} = \{f_j\}_{j \in J}$$

and

$$\Theta(\Omega(U)) = \Theta(\{Ue_j\}_{j \in J}) = U.$$

This completes the proof. \square

For a Bessel sequence $F = \{f_j\}_{j \in J}$ in \mathcal{H} we denote by U_F the operator obtained in Theorem 3. As a consequence of Theorem 3 we have the following known result (see e.g. [1]).

Corollary 1. *Let \mathcal{H} be a Hilbert spaces with the Bessel sequence space $Bess(\mathcal{H})$ and Θ be as in (2.3). Then,*

$$KFr(\mathcal{H}) = \Omega\{A \in B(\mathcal{H}) : R(K) \subseteq R(A)\}$$

and so

$$Fr(\mathcal{H}) = \Omega\{A \in B(\mathcal{H}) : A \text{ is surjective}\}.$$

Proof. It follows from Theorem 1 and (2.2) that a Bessel sequence $F = \{f_j\}_{j \in J}$ is K -frame if and only if there exists $\alpha > 0$ such that $U_F U_F^* = S_F \geq \alpha K K^*$. By Douglas' factorization theorem (Theorem 2), it is equivalent to $R(K) \subseteq R(U_F)$. Moreover, taking $K = I$ (the identity operator) we obtain desired conclusion. \square

One-to-one correspondence in Theorem 3 motivate us to introduce the following notions.

Definition 5. Let $\{e_j\}_{j \in J}$ be an arbitrary orthonormal basis for \mathcal{H} and $\{f_j\}_{j \in J}$ be a Bessel sequence in \mathcal{H} . Adjoint of $\{f_j\}_{j \in J}$ is a Bessel sequence $\{g_j\}_{j \in J}$ in \mathcal{H} such that

$$\sum_{j \in J} \langle f, g_j \rangle \langle e_j, g \rangle = \sum_{j \in J} \langle f_j, g \rangle \langle f, e_j \rangle, \quad \forall f, g \in \mathcal{H}. \quad (2.4)$$

We denote by $\{f_j^*\}_{j \in J}$ the adjoint of $\{f_j\}_{j \in J}$. If

$$\sum_{j \in J} \langle f, f_j \rangle \langle e_j, g \rangle = \sum_{j \in J} \langle f_j, g \rangle \langle f, e_j \rangle, \quad \forall f, g \in \mathcal{H}, \quad (2.5)$$

we say that $\{f_j\}_{j \in J}$ is self-adjoint. Also, $\{f_j\}_{j \in J}$ is positive if

$$\sum_{j \in J} \langle f_j, f \rangle \langle f, e_j \rangle \geq 0, \quad \forall f \in \mathcal{H}. \quad (2.6)$$

A positive frame is a frame in which inequality (2.6) holds.

Theorem 4. *Let $\{f_j\}_{j \in J}$ be a Bessel sequence in \mathcal{H} . Then the followings hold:*

- (i) $\{f_j\}_{j \in J}$ has a unique adjoint in \mathcal{H} ;
- (ii) $\{f_j\}_{j \in J}$ is self-adjoint if and only if U_F is self-adjoint;
- (iii) $\{f_j\}_{j \in J}$ is positive if and only if U_F is positive.

Proof. (i) Let $\{f_j\}_{j \in J}$ be a Bessel sequence in \mathcal{H} . Consider the Bessel sequence $\{g_j\}_{j \in J} = \{U_F^* e_j\}_{j \in J}$. Now, we verify the equation (2.4). Let $f, g \in \mathcal{H}$. Then

$$\begin{aligned}
 \sum_{j \in J} \langle f, g_j \rangle \langle e_j, g \rangle &= \sum_{j \in J} \langle f, U_F^* e_j \rangle \langle e_j, g \rangle \\
 &= \sum_{j \in J} \langle U_F f, e_j \rangle \langle e_j, g \rangle \\
 &= \left\langle \sum_{j \in J} \langle U_F f, e_j \rangle e_j, g \right\rangle \\
 &= \langle U_F f, g \rangle \\
 &= \langle f, U_F^* g \rangle \\
 &= \left\langle f, \sum_{j \in J} \langle U_F^* g, e_j \rangle e_j \right\rangle \\
 &= \sum_{j \in J} \langle e_j, U_F^* g \rangle \langle f, e_j \rangle \\
 &= \sum_{j \in J} \langle U_F e_j, g \rangle \langle f, e_j \rangle = \sum_{j \in J} \langle f_j, g \rangle \langle f, e_j \rangle,
 \end{aligned}$$

so $\{U_F^* e_j\}_{j \in J}$ is an adjoint of $F = \{f_j\}_{j \in J}$. Uniqueness follows from uniqueness of U_F in Theorem 3 and its adjoint.

(ii) By a similar computation as in (i), we get that

$$\langle U_F f, g \rangle = \langle f, U_F^* g \rangle = \sum_{j \in J} \langle f_j, g \rangle \langle f, e_j \rangle, \quad \forall f, g \in \mathcal{H}, \quad (2.7)$$

and

$$\langle U_F^* f, g \rangle = \sum_{j \in J} \langle f, f_j \rangle \langle e_j, g \rangle, \quad \forall f, g \in \mathcal{H},$$

which proves (ii).

It follows from (2.7) that

$$\langle U_F f, f \rangle = \sum_{j \in J} \langle f_j, f \rangle \langle f, e_j \rangle, \quad \forall f \in \mathcal{H},$$

which gives us (iii). □

Corollary 2. *Let $\{f_j\}_{j \in J}$ be a Bessel sequence in \mathcal{H} .*

- (i) *If $\{f_j\}_{j \in J}$ is self-adjoint, then $\{U_F f_j\}_{j \in J}$ is positive.*

- (ii) If $V \in B(\mathcal{H})$ is self-adjoint, then $\{V^2 e_j\}_{j \in J}$ is positive.
 (iii) If $\{f_j\}_{j \in J}$ is positive, then there exists a self-adjoint operator $V \in B(\mathcal{H})$ such that $\{f_j\}_{j \in J} = \{V^2 e_j\}_{j \in J}$.

Proof. (i) We have

$$U_F f_j = U_F U_F e_j = U_F^2 e_j.$$

By Theorem 4, U_F is self-adjoint and so $U_F^2 = U_F U_F^* \geq 0$. Again, it follows from Theorem 4 that $\{U_F f_j\}_{j \in J}$ is positive.

(ii) Taking $\{f_j\}_{j \in J} = \{V e_j\}_{j \in J}$, we have $U_F = V$. Since V is self-adjoint, then $\{f_j\}_{j \in J}$ is self-adjoint. Hence using the previous part, $\{V^2 e_j\}_{j \in J} = \{V f_j\}_{j \in J}$ is positive.

(iii) Since $\{f_j\}_{j \in J}$ is positive, it follows from Theorem 4 that U_F is positive. Put $V = U_F^{\frac{1}{2}}$. Then, $\{f_j\}_{j \in J} = \{U_F e_j\}_{j \in J} = \{V^2 e_j\}_{j \in J}$. \square

Lemma 1. Let $F = \{f_j\}_{j \in J}$ be a Bessel sequence with the frame operator S_F . If $\{f_j\}_{j \in J}$ is positive, then $U_F = S_F^{\frac{1}{2}}$.

Proof. It follows from (2.2) that

$$S_F = U_F U_F^* = U_F^2.$$

By the uniqueness of positive square root we obtain the desired result. \square

The following lemma provides a basic tool to define product of Bessel sequences.

Lemma 2. Let $F = \{f_j\}_{j \in J}$ and $G = \{g_j\}_{j \in J}$ be two Bessel sequences in \mathcal{H} . Then, there exists a unique Bessel sequence $\{h_j\}_{j \in J}$ in \mathcal{H} such that

$$\sum_{j \in J} \langle f, h_j \rangle e_j = \sum_{j \in J} \sum_{i \in J} \langle f, f_i \rangle \langle e_i, g_j \rangle e_j, \quad \forall f \in \mathcal{H}. \quad (2.8)$$

Proof. Put $\{h_j\}_{j \in J} = \{U_F U_G e_j\}_{j \in J}$. For each $f \in \mathcal{H}$ we have

$$\begin{aligned} \sum_{j \in J} \langle f, h_j \rangle e_j &= \sum_{j \in J} \langle f, U_F U_G e_j \rangle e_j = \sum_{j \in J} \langle U_F^* f, g_j \rangle e_j \\ &= \sum_{j \in J} \left\langle \sum_{i \in J} \langle U_F^* f, e_i \rangle e_i, g_j \right\rangle e_j \\ &= \sum_{j \in J} \sum_{i \in J} \langle U_F^* f, e_i \rangle \langle e_i, g_j \rangle e_j \\ &= \sum_{j \in J} \sum_{i \in J} \langle f, U_F e_i \rangle \langle e_i, g_j \rangle e_j \\ &= \sum_{j \in J} \sum_{i \in J} \langle f, f_i \rangle \langle e_i, g_j \rangle e_j. \end{aligned}$$

To prove the uniqueness, let $H = \{h_j\}_{j \in J}$ and $M = \{m_j\}_{j \in J}$ be two Bessel sequences in \mathcal{H} for which the equality (2.8) holds. Hence,

$$\sum_{j \in J} \langle f, h_j \rangle e_j = \sum_{j \in J} \sum_{i \in J} \langle f, f_i \rangle \langle e_i, g_j \rangle e_j = \sum_{j \in J} \langle f, m_j \rangle e_j, \quad \forall f \in \mathcal{H}. \quad (2.9)$$

On the other hand, we have

$$U_H^* f = \sum_{j \in J} \langle U_H^* f, e_j \rangle e_j = \sum_{j \in J} \langle f, U_H e_j \rangle e_j = \sum_{j \in J} \langle f, h_j \rangle e_j, \quad \forall f \in \mathcal{H}. \quad (2.10)$$

Considering (2.9) and (2.10) we see that $U_H = U_M$ and so $\{h_j\}_{j \in J} = \{m_j\}_{j \in J}$. \square

For the Bessel sequences $\{f_j\}_{j \in J}$ and $\{g_j\}_{j \in J}$ in \mathcal{H} , we define product of them as the Bessel sequence obtained in Lemma 2.

Remark 2. By using the same argument as Theorem 4 and Lemma 2, we see that for the Bessel sequence $F = \{f_j\}_{j \in J}$ in \mathcal{H} , the operator U_F is normal if and only if

$$\sum_{j \in J} \sum_{i \in J} \langle f, f_i \rangle \langle e_i, f_j^* \rangle e_j = \sum_{j \in J} \sum_{i \in J} \langle f, f_i^* \rangle \langle e_i, f_j \rangle e_j, \quad \forall f \in \mathcal{H}.$$

In this case we say that the Bessel sequence $\{f_j\}_{j \in J}$ is normal.

Theorem 5. *The set $\text{Bess}(\mathcal{H})$ is an algebra with the following operations:*

$$\begin{aligned} \{f_j\}_{j \in J} + \{g_j\}_{j \in J} &:= \{f_j + g_j\}_{j \in J} = \{(U_F + U_G)e_j\}_{j \in J}, \\ \alpha \{f_j\}_{j \in J} &:= \{\alpha f_j\}_{j \in J} = \{(\alpha U_F)e_j\}_{j \in J} \\ \{f_j\}_{j \in J} \cdot \{g_j\}_{j \in J} &:= \{(U_F U_G)e_j\}_{j \in J}. \end{aligned}$$

The proof is straightforward.

Theorem 6. *The function*

$$\Theta : \text{Bess}(\mathcal{H}) \rightarrow B(\mathcal{H}) \quad \text{defined by} \quad \Theta(\{f_j\}_{j \in J}) = U_F$$

is an algebra positive homomorphism.

3. MEANS OF FRAMES

Recall that power mean of two positive operators A and B is defined by

$$\left(\frac{A^r + B^r}{2} \right)^{\frac{1}{r}} \quad r \in \mathbb{R} \setminus \{0\}.$$

We begin with Arithmetic mean of two positive frames.

Theorem 7. *Frame operator of Arithmetic mean of two positive frames is power mean of their frame operators for $r = \frac{1}{2}$.*

Proof. Let $F = \{f_j\}_{j \in J}$ and $G = \{g_j\}_{j \in J}$ be two positive frames in \mathcal{H} with the frame operators S_F and S_G , respectively. We know from Lemma 1 that

$$f_j = U_F e_j = S_F^{\frac{1}{2}} e_j, \quad g_j = U_G e_j = S_G^{\frac{1}{2}} e_j.$$

Since Arithmetic mean of $F = \{f_j\}_{j \in J}$ and $G = \{g_j\}_{j \in J}$ is

$$\left\{ \frac{f_j + g_j}{2} \right\}_{j \in J} = \left\{ \left(\frac{U_F + U_G}{2} \right) e_j \right\}_{j \in J},$$

considering (2.2), its frame operator is

$$\left(\frac{U_F + U_G}{2} \right) \left(\frac{U_F + U_G}{2} \right)^* = \left(\frac{S_F^{1/2} + S_G^{1/2}}{2} \right)^2.$$

□

To extend the idea to other means we need apply functional calculus on $Bess(\mathcal{H})$. Let A be a normal operator on \mathcal{H} , $C(Sp(A))$ be the set of all continuous functions on the spectrum of A and $C^*(A, I)$ be the abelian C^* -algebra generated by A and I . An immediate consequence of Gelfand's Theorem [10, Theorem 2.1.10] is that there is an isometric $*$ -isomorphism between $C(Sp(A))$ and $C^*(A, I)$. Hence, for any $\varphi \in C(Sp(A))$ the operator $\varphi(A)$ is in $C^*(A, I)$. Moreover, spectral mapping theorem [10, Theorem 2.1.14] guaranties that $\varphi(Sp(A)) = Sp(\varphi(A))$. So, if $\varphi(Sp(A)) \subseteq [0, +\infty)$, then $\varphi(A)$ is a positive operator.

Now, we define new Bessel sequences by using functional calculus. Let $F = \{f_j\}_{j \in J}$ be a Bessel sequence in \mathcal{H} for which U_F is normal operator and $\varphi \in C(Sp(U_F))$. We define

$$\varphi(F) = \{\varphi(U_F) e_j\}_{j \in J}.$$

For example, if the Bessel sequence $F = \{f_j\}_{j \in J}$ is positive, then $\varphi(t) = t^{\frac{1}{2}}$ is continuous on $Sp(U_F)$ and we have

$$\varphi(F) = \{\varphi(U_F) e_j\}_{j \in J} = \{U_F^{\frac{1}{2}} e_j\}_{j \in J}.$$

We denote this new Bessel sequences by $\{f_j\}_{j \in J}^{\frac{1}{2}}$.

Let \mathcal{H} and \mathcal{K} be two Hilbert spaces. Recall that a linear mapping $\Phi : B(\mathcal{H}) \rightarrow B(\mathcal{K})$ is called positive map if Φ maps positive elements to positive elements.

Definition 6. Let \mathcal{H} and \mathcal{K} be two separable Hilbert spaces. Every positive linear maps $\Phi : B(\mathcal{H}) \rightarrow B(\mathcal{K})$ induces positive linear map $\widehat{\Phi} : Bess(\mathcal{H}) \rightarrow Bess(\mathcal{K})$ such that

$$\widehat{\Phi}(F) = \{\Phi(U_F) \widehat{e}_j\}_{j \in J},$$

where $F = \{f_j\}_{j \in J}$ is a Bessel sequences in \mathcal{H} and $\{\widehat{e}_j\}_{j \in J}$ is an arbitrary and fixed orthonormal bases for \mathcal{K} .

Lemma 3. *Let \mathcal{H} and \mathcal{K} be two separable Hilbert spaces with the Bessel sequence spaces $Bess(\mathcal{H})$ and $Bess(\mathcal{K})$, respectively. For each positive linear map $\Phi : B(\mathcal{H}) \rightarrow B(\mathcal{K})$, the following diagram commutes:*

$$\begin{array}{ccc} Bess(\mathcal{H}) & \xrightarrow{\widehat{\Phi}} & Bess(\mathcal{K}) \\ \Theta \downarrow & & \downarrow \Theta \\ B(\mathcal{H}) & \xrightarrow{\Phi} & B(\mathcal{K}) \end{array}$$

where Θ is defined by (2.3).

Proof. Let $F = \{f_j\}_{j \in J}$ be a Bessel sequence in \mathcal{H} . Considering (2.3) we have $\Phi(\Theta\{f_j\}_{j \in J}) = \Phi(U_F)$. On the other hand,

$$\Theta(\widehat{\Phi}\{f_j\}_{j \in J}) = \Theta(\Phi(U_F)\widehat{e}_j) = \Phi(U_F).$$

□

Let $B_+(\mathcal{H})$ be the set of all positive operators of $B(\mathcal{H})$ and $P_N[B(\mathcal{H}), B(\mathcal{K})]$ be the set of all normalized positive linear maps from $B(\mathcal{H})$ to $B(\mathcal{K})$. Micic and Pecaric in [9] introduced the following weighted power means of positive operators and studied related inequalities. Let $\Phi_j \in P_N[B(\mathcal{H}), B(\mathcal{K})]$, $A_i \in B_+(\mathcal{H})$ with $Sp(A_i) \subseteq [m, M]$ for some scalars $0 < m < M$ and $\omega_i \in \mathbb{R}^+$ such that $\sum_{i=1}^k \omega_i = 1$ ($i = 1, \dots, k$). Define

$$M_k^{[r]}(A, \Phi, \omega) := \left(\sum_{i=1}^k \omega_i \Phi_i(A_i^r) \right)^{1/r} \quad \text{if } r \in \mathbb{R} \setminus \{0\}, \quad (3.1)$$

where $A = (A_1, \dots, A_k)$, $\Phi = (\Phi_1, \dots, \Phi_k)$ and $\omega = (\omega_1, \dots, \omega_k)$. Similarly, if $F_i = \{f_{ij}\}_{j \in J}$ are positive frames in \mathcal{H} , we define their weighted power means as

$$\mathcal{M}_k^{[r]}(F, \Phi, \omega) := \left(\sum_{i=1}^k \omega_i \widehat{\Phi}_i(\{f_{ij}\}_{j \in J}^r) \right)^{1/r} \quad \text{if } r \in \mathbb{R} \setminus \{0\}, \quad (3.2)$$

where $F = (F_1, \dots, F_k)$.

Theorem 8. *Frame operator of weighted power mean of positive frames for r is weighted power mean of their frame operators for $\frac{r}{2}$.*

Proof. Let $\{e_j\}_{j \in J}$ and $\{\widehat{e}_j\}_{j \in J}$ be arbitrary and fixed orthonormal bases for \mathcal{H} and \mathcal{K} , respectively. Let $F_i = \{f_{ij}\}_{j \in J}$ ($i = 1, \dots, k$) be positive frames in \mathcal{H} with the frame operators S_{F_i} 's. The weighted power mean of the positive frames $F_i = \{f_{ij}\}_{j \in J}$

is a positive frame in \mathcal{K} such that

$$\begin{aligned}\mathcal{M}_k^{[r]}(\mathbb{F}, \Phi, \omega) &= \left(\sum_{i=1}^k \omega_i \widehat{\Phi}_i(\{f_{ij}\}_{j \in J}^r) \right)^{1/r} \\ &= \left(\sum_{i=1}^k \omega_i \Phi_i(U_{F_i}^r) \right)^{1/r} \quad \widehat{e}_j = \left(\sum_{i=1}^k \omega_i \Phi_i(S_{F_i}^{\frac{r}{2}}) \right)^{1/r} \quad \widehat{e}_j := V \widehat{e}_j.\end{aligned}$$

Therefore, the frame operator of $\mathcal{M}_k^{[r]}(\mathbb{F}, \Phi, \omega)$ is

$$\begin{aligned}VV^* &= \left(\sum_{i=1}^k \omega_i \Phi_i(S_{F_i}^{\frac{r}{2}}) \right)^{1/r} \left(\sum_{i=1}^k \omega_i \Phi_i(S_{F_i}^{\frac{r}{2}}) \right)^{1/r} \\ &= \left(\sum_{i=1}^k \omega_i \Phi_i(S_{F_i}^{\frac{r}{2}}) \right)^{2/r} = M_k^{[\frac{r}{2}]}(S_F, \Phi, \omega),\end{aligned}$$

where $S_F = (S_{F_1}, \dots, S_{F_k})$. □

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