ON ALMOST EVERYWHERE CONVERGENCE OF THE GENERALIZED MARCIENKIWICZ MEANS WITH RESPECT TO TWO DIMENSIONAL VILENKIN-LIKE SYSTEMS

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Abstract. In this paper we investigate the almost everywhere convergence of two dimensional Marcinkiewicz-like means of two variable integrable functions which is given by

$$t_n^\alpha f = \frac{1}{n} \sum_{k=0}^{n-1} S_{\alpha_1(|n|,k), \alpha_2(|n|,k)} f$$

$(M_{|n|} \leq n < M_{|n|+1})$ and give a sufficient condition for functions $\alpha : N^2 \mapsto N^2$ in order to have the a.e. relation $t_n^\alpha f \to f$ for all $f \in L^1(G_m^2)$ with respect to two dimensional bounded Vilenkin-like systems. Finally, we give an application of the main result with respect to triangular summability of Vilenkin-like-Fourier series.

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1. INTRODUCTION

In 1939 for the two-dimensional trigonometric Fourier partial sums $S_{j,j}f$ Marcinkiewicz [14] proved that for all $f \in L \log L([0, 2\pi]^2)$ the a.e. relation

$$\frac{1}{n} \sum_{j=1}^{n} S_{j,j}f \to f \tag{1.1}$$

holds as $n \to \infty$. Zhizhiashvili [18] improved this result for $f \in L([0, 2\pi]^2)$. Dyachenko [3] proved this result for dimensions greater than 2. In 2003 Goginava [10] proved this result with respect to the d-dimensional Walsh-Paley system. The case $d = 2$ is due to Weisz [17]. In 2012 Gát [7] proved this result for generalized Marcinkiewicz means with respect to the two dimensional Walsh system and in 2016 [8] for bounded two-dimensional Vilenkin systems. The aim of this paper is to generalize this result of Gát with respect to two-dimensional generalized Vilenkin-like systems.

Besides, we give an application of the main result of this paper, that is, Theorem 3 with respect to triangular summability of Vilenkin-Fourier series.

Denote by $\mathbb{N}$ the set of natural numbers and $\mathbb{P}$ the set of positive integers. Denote $m := (m_k : k \in \mathbb{N})$ a sequence of positive integers such that $m_k \geq 2$, $k \in \mathbb{N}$ and G_{m_k} a set of cardinality m_k. Suppose that each (coordinate) set has the discrete topology and measure μ_k which maps every singleton of G_{m_k} to $\frac{1}{m_k}$ ($\mu_k(G_{m_k}) = 1$), $k \in \mathbb{N}$. Let G_m be the compact set formed by the complete direct product of G_{m_k} with the product of the topologies and measures (μ). Thus, each $x \in G_m$ is a sequence $x := (x_0, x_1, ...)$, where $x_k \in G_{m_k}$, $k \in \mathbb{N}$. G_m is called a Vilenkin space. G_m is a compact totally disconnected space, with normalized regular Borel measure μ, $\mu(G_m) = 1$. The Vilenkin space G_m is said to be bounded if the generating system m is a bounded one. *Throughout this paper the boundedness of G_m is supposed.* In this paper C, C_p denote absolute constants (the latter may depend on p), which may vary from line to line.

A base for the neighborhoods of G_m can be given as follows

$$I_0(x) := G_m, \quad I_n(x) := \{y = (y_i, i \in \mathbb{N}) \in G_m : y_i = x_i \text{ for } i < n\}$$

for $x \in G_m, n \in \mathbb{P}$.

$$I := \{I_n(x) : n \in \mathbb{N}, x \in G_m\}$$

is the set of intervals on G_m.

Denote by $L^p(G_m)$ the usual Lebesgue spaces ($\|.\|_p$ the corresponding norms) ($1 \leq p \leq \infty$), $\mathcal{A}_n$ the σ algebra generated by the sets $I_n(x)$ ($x \in G_m$) and E_n the conditional expectation operator with respect to $\mathcal{A}_n$ ($n \in \mathbb{N}$).

Let $L^0(G_m)$ be the space of measurable functions on the Vilenkin space G_m. We say that an operator $T : L^1(G_m) \to L^0(G_m)$ is of type (L^p, L^p) (for $1 \leq p \leq \infty$) if $\|Tf\|_p \leq C_p \|f\|_p$ for all $f \in L^p(G_m)$, where the constant C_p depends only on p. We say that T is of weak type (L^1, L^1) if $\mu(|Tf| > \lambda) \leq C\|f\|_1/\lambda$ for all $f \in L^1(G_m)$ and $\lambda > 0$.

Let $M_0 := 1$ and $M_{k+1} := m_k M_k$, for $k \in \mathbb{N}$ be the so-called generalized powers. Then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_k M_k$, $0 \leq n_k < m_k$, $n_k \in \mathbb{N}$. This allows one to say that the sequence $(n_0, n_1, ...)$ is the expansion of n with respect to m. We often use the following notations. Let $|n| := \max\{k \in \mathbb{N} : n_k \neq 0\}$ (that is, $M_{|n|} \leq n < M_{|n|+1}$) and $n^{(k)} = \sum_{j=k}^{\infty} n_j M_j$. Next we introduce on G_m an orthonormal system called a Vilenkin-like system.

For $k \in \mathbb{N}$ and $x \in G_m$ denote by r_k the k-th generalized Rademacher function:

$$r_k(x) := \exp(2\pi\iota \frac{x_k}{m_k}) \quad (x \in G_m, \; \iota := \sqrt{-1}, \; k \in \mathbb{N}).$$

We introduce a so-called Vilenkin-like (or $\psi\alpha$) system (see [5], [2]). Let functions

$$\psi_n, \alpha_n, \alpha_j^k : G_m \to \mathbb{C} \; (n, j, k \in \mathbb{N})$$

satisfy :

$$\alpha_k^j \text{ is measurable with respect to } \mathcal{A}_k \ (j,k \in \mathbb{N}), \qquad (1.2)$$

$$|\alpha_k^j| = \alpha_k^j(0) = \alpha_0^j = \alpha_k^0 = 1 \ (j,k \in \mathbb{N}), \qquad (1.3)$$

$$\alpha_n := \prod_{k=0}^{\infty} \alpha_k^{n^{(k)}}, \psi_n := \prod_{k=0}^{\infty} r_k^{n_k}, \ n^{(k)} := \sum_{i=k}^{\infty} n_i M_i \ (n \in \mathbb{N}). \qquad (1.4)$$

Let $\chi_n := \psi_n \alpha_n \ (n \in \mathbb{N})$. The system $\chi := \{\chi_n : n \in \mathbb{N}\}$ is called a Vilenkin-like (or $\psi\alpha$) system (see [5], [2]). We also introduce the two-variable functions:

$$\chi_n(y,x) := \chi_n(y)\bar{\chi}_n(x), \quad r_n(y,x) := r_n(y)\bar{r}_n(x) \quad (n \in \mathbb{N}, y, x \in G_m).$$

This will not cause misunderstand by clearly making a difference between $\chi_n(x)$ and $\chi_n(y,x)$. The following lemma is an easy consequence of properties (1.2), (1.3), (1.4). We also need the fact that the generalized Rademacher functions form an orthonormal system (see e.g. [1, 5]).

Lemma 1 ([5]). *Let $t, n, l \in \mathbb{N}, u \in G_m$. Then we have that*

$$\int_{I_{t+1}(u)} \chi_n(x)\bar{\chi}_l(x)d\mu(x) \neq 0$$

implies $n^{(t+1)} = l^{(t+1)}$.

Example A, the Vilenkin and the Walsh system

Let $G_{m_k} := Z_{m_k}$ be the m_k-th ($2 \leq m_k \in \mathbb{N}$) discrete cyclic group ($k \in \mathbb{N}$). That is Z_{m_k} can be represented by the set $\{0, 1, ..., m_k - 1\}$, where the group operation is the mod m_k addition and every subset is open. The group operation on G_m (+) is the coordinate-wise addition. G_m is called a Vilenkin group. The Vilenkin group for which $m_k = 2$ for all $k \in \mathbb{N}$ is the Walsh-Paley group. In this case let $\alpha_k^j(x) := 1$, where $j, k \in \mathbb{N}, x \in G_m$. The system $\chi := (\chi_n : n \in \mathbb{N})$ is the Vilenkin system, where $\chi_n := \prod_{k=0}^{\infty} r_k^{n_k} \alpha_k^{n^{(k)}} = \prod_{k=0}^{\infty} r_k^{n_k}$. In the case of the Vilenkin group, $m_k = 2$ for all $k \in \mathbb{N}$, we get the Walsh-Paley system. Properties (1.2), (1.3), (1.4) are trivial fulfilled. For more on Vilenkin and Walsh system and group see e.g. [1] and [6]. For this special case the result of this paper can be found in [8].

Example B, the group of 2-adic (m-adic) integers

Let $G_{m_k} := \{0, 1, ..., m_k - 1\}$ for all $k \in \mathbb{N}$. Define on G_m the following (commutative) addition: Let $x, y \in G_m$. Then $x + y = z \in G_m$ is defined in a recursive way. $x_0 + y_0 = t_0 m_0 + z_0$, where (of course) $z_0 \in \{0, 1, ..., m_0 - 1\}$ and $t_0 \in \mathbb{N}$. Suppose that $z_0, ..., z_k$ and $t_0, ..., t_k$ have been defined. Then write $x_{k+1} + y_{k+1} + t_k = t_{k+1} m_{k+1} + z_{k+1}$, where $z_{k+1} \in \{0, 1, ..., m_{k+1} - 1\}$ and $t_{k+1} \in \mathbb{N}$. Then G_m is called

the group of m-adic integers (if $m_k = 2$ for all $k \in \mathbb{N}$, then 2-adic integers). In this case let

$$\alpha_k^j(x) := \left(\exp\left(2\pi \iota \left(\frac{x_{k-1}}{m_k m_{k-1}} + \cdots + \frac{x_0}{m_k m_{k-1} \ldots m_0} \right) \right) \right)^j.$$

Let $\chi_n := \prod_{k=0}^{\infty} r_k^{n_k} \alpha_k^{n^{(k)}}$. Then the system $\chi := (\chi_n : n \in \mathbb{N})$ is the character system of the group of m-adic (if $m_k = 2$ for each $k \in \mathbb{N}$ then 2-adic) integers. Conditions (1.2), (1.3), (1.4) are trivially fulfilled. For more on the group of m-adic (if $m_k = 2$ for each $k \in \mathbb{N}$ then 2-adic) integers see e.g. [2] or [13]. For the case when $m_k = 2 \, (k \in \mathbb{N})$ the a.e. convergence of the ordinary Marcinkiewicz means were discussed by Blahota and Gát in [2]. That is, the results of this paper are new on the two-dimensional group of m-adic integers. Not only with respect to the general case $\alpha : \mathbb{N}^2 \to \mathbb{N}^2$ but also for $\alpha_1(n) = \alpha_2(n) = n$. Besides, the same can be said in the situation of the Example C below.

Example C, a system in the field of number theory

Let

$$\chi_n(x) := \exp\left(2\pi \iota \sum_{j=0}^{\infty} \frac{n_j}{M_{j+1}} \sum_{i=0}^{\infty} x_i M_i \right)$$

for $n \in \mathbb{N}$ and $x \in G_m$. Then

$$\chi_n(x) = \exp\left(2\pi \iota \left(\sum_{k=0}^{\infty} \frac{n_k x_k}{m_k} + \sum_{k=0}^{\infty} \frac{n_k}{M_{k+1}} \sum_{i=0}^{k-1} x_i M_i \right) \right) = \psi_n(x) \alpha_n(x),$$

where $\alpha_k^{n^{(k)}}(x) = \exp\left(2\pi \iota \frac{n_k}{M_{k+1}} \sum_{i=0}^{k-1} x_i M_i \right)$. Then, $\chi := (\chi_n : n \in \mathbb{N})$ is a Vilenkin-like system (introduced in [5]) which is a useful tool in the approximation theory of limit periodic, almost even arithmetical functions [5] and [4]. Again, properties (1.2), (1.3), (1.4) are trivially fulfilled. This system (on Vilenkin groups) was a new tool in order to investigate limit periodic arithmetical functions. For the definition of these arithmetical functions see also the book of Mauclaire [15].

Define the Fourier coefficients, the partial sums of the Fourier series and the Dirichlet kernels with respect to the Vilenkin-like system χ as follows

$$\hat{f}(n) := \int_{G_m} f \bar{\chi}_n d\mu, \quad S_n f := \sum_{k=0}^{n-1} \hat{f}(k) \chi_k,$$

$$D_n(y,x) := \sum_{k=0}^{n-1} \chi_k(y) \bar{\chi}_k(x) = \sum_{k=0}^{n-1} \chi_k(y,x) \quad (n \in \mathbb{N}, \quad y,x \in G_m, f \in L^1(G_m)).$$

It is well-known that

$$S_n f(y) = \int_{G_m} f(x) D_n(y,x) d\mu(x), \quad (n \in \mathbb{N}, y \in G_m, f \in L^1(G_m)).$$

It is also well-known [5] that

$$D_{M_n}(y,x) = \begin{cases} M_n, & \text{if } y \in I_n(x) \\ 0, & \text{if } y \notin I_n(x) \end{cases}$$

$$D_n(y,x) = \chi_n(y)\bar{\chi}_n(x) \sum_{j=0}^{\infty} D_{M_j}(y,x) \sum_{p=m_j-n_j}^{m_j-1} r_j^p(y)\bar{r}_j^p(x),$$

$$S_{M_n}f(x) = M_n \int_{I_n(x)} f d\mu = E_n f(x) \quad (f \in L^1(G_m), n \in \mathbb{N}), \tag{1.5}$$

$$D_n(y,x) = \chi_n(y)\bar{\chi}_n(x) \left(\sum_{j=0}^{t-1} n_j M_j + M_t \sum_{i=m_t-n_t}^{m_t-1} r_t^i(y)\bar{r}_t^i(x) \right),$$

$$y \in I_t(x) \setminus I_{t+1}(x), t \in \mathbb{N}.$$

Next, we introduce some notation with respect to the theory of two-dimensional Vilenkin-like systems. Let $\tilde{m}$ be a sequence like m. The relation between the sequence $(\tilde{m}_n)$ and $(\tilde{M}_n)$ is the same as between sequence (m_n) and (M_n). The group $G_m \times G_{\tilde{m}}$ is called a two-dimensional Vilenkin group. The normalized Haar measure is denoted by μ, just as in the one-dimensional case. It will not cause any misunderstood. In this paper we also suppose that $m = \tilde{m}$. The two-dimensional Fourier coefficients, the rectangular partial sums of the Fourier series, the Dirichlet kernels, the Marcinkiewicz means, and the Marcinkiewicz kernels with respect to the two-dimensional Vilenkin-like system are defined as follows:

$$\hat{f}(n_1,n_2) := \int_{G_m \times G_{\tilde{m}}} f(x^1,x^2)\bar{\chi}_{n_1}(x^1)\bar{\chi}_{n_2}(x^2)d\mu(x^1,x^2),$$

$$S_{n_1,n_2}f(y^1,y^2) := \sum_{k_1=0}^{n_1-1}\sum_{k_2=0}^{n_2-1} \hat{f}(k_1,k_2)\chi_{k_1}(y^1)\chi_{k_2}(y^2),$$

$$D_{n_1,n_2}(y,x) = D_{n_1}(y^1,x^1)D_{n_2}(y^2,x^2)$$

$$:= \sum_{k_1=0}^{n_1-1}\sum_{k_2=0}^{n_2-1} \chi_{k_1}(y^1)\chi_{k_2}(y^2)\bar{\chi}_{k_1}(x^1)\bar{\chi}_{k_2}(x^2),$$

$$\sigma_n f := \frac{1}{n}\sum_{j=0}^{n-1} S_{j,j}f,$$

$$K_n(y,x) := \frac{1}{n}\sum_{j=0}^{n-1} D_{j,j}(y,x),$$

$$(y = (y^1,y^2), x = (x^1,x^2) \in G_m \times G_m, n \in \mathbb{N}).$$

It is also well-known that

$$\sigma_n f(y) = \int_{G_m \times G_m} f(x) K_n(y,x) d\mu(x) =: f * K_n(y).$$

2. RESULT

Now, turn our attention to the generalization of Marcienkiewicz means with respect to two dimensional bounded Vilenkin-like systems. The two-dimensional generalized Marcinkiewicz kernels and Marcinkiewicz means, with respect to the two-dimensional Vilenkin-like system are defined as follows: Let $\alpha = (\alpha_1, \alpha_2) : \mathbb{N}^2 \to \mathbb{N}^2$ be a function. (From now functions α_1, α_2 play the role of indices. We know that in the first part of the article the function α_n appeared in the definition of the Vilenkin-like (or $\psi\alpha$ system), but this will not cause any misunderstanding.) Define the following generalized Marcinkiewicz kernels and means respectively:

$$M_n^\alpha(y,x) := \frac{1}{n} \sum_{k=0}^{n-1} D_{\alpha_1(|n|,k)}(y^1,x^1) D_{\alpha_2(|n|,k)}(y^2,x^2),$$

$$t_n^\alpha f := f * M_n^\alpha \quad (f \in L^1(G_m^2), n \in \mathbb{P}).$$

This concept of Marcinkiewicz-like kernels and means is due to Gát [7].
The main aim of this paper is to give a class of functions α for we have the a.e. convergence relation $t_n^\alpha f \to f$ for each integrable two variable function with respect to two dimensional bounded Vilenkin-like systems. To investigate this the following properties play a prominent role (x0 $Car(B)$ denotes cardinality of the set B),

$$Car\{l \in \mathbb{N} : \alpha_j(|n|,l) = \alpha_j(|n|,k), l < n\} \leq C \quad (k < n, n \in \mathbb{P}, j = 1,2), \quad (2.1)$$

$$\max\{\alpha_j(|n|,k) : k < n\} \leq Cn \quad (n \in \mathbb{P}, j = 1,2). \quad (2.2)$$

We emphasize that the function α has the properties (2.1) and (2.2) everywhere in this paper. Our first aim is to prove that the operator $t_*^\alpha := \sup_{n \in \mathbb{P}} |t_n^\alpha|$ is of weak type (L^1, L^1). In order to do this we need a sequence of lemmas. The second lemma is the base of the proof of Theorems 1 and 2. The Walsh-Paley version of Theorems 1 and 2 are due to Gát [7]. That is, we generalize a result of the first author. Moreover, techniques of papers [7] and [8] will also be used in the proof of the forthcoming lemmas.

Denote for $k \in \mathbb{N}, x \in G_m, J_k(x) = I_k(x) \setminus I_{k+1}(x)$ and recall also that

$$n^{(s)} = \sum_{k=s}^{\infty} n_k M_k, \quad n^{(0)} = n, \quad n^{(|n|+1)} = 0 \quad (n, s \in \mathbb{N}).$$

Besides, for $x, y \in G_m^2, A, n, s, j, k \in \mathbb{N}$ let

$$\Phi(A, n^{(s+1)} + jM_s + k, y, x) = D_{\alpha_1(A, n^{(s+1)}+jM_s+k)}(y^1, x^1) D_{\alpha_2(A, n^{(s+1)}+jM_s+k)}(y^2, x^2).$$

Lemma 2. *Let* $t^1, t^2, A, s \in \mathbb{N}, s \leq A$ *and* $y \in G_m^2$. *Then,*

$$\int_{J_{t^1}(y^1) \times J_{t^2}(y^2)} \sup_{|n|=A} \Big| \sum_{j=0}^{n_s-1} \sum_{k=0}^{M_s-1} \Phi(A, n^{(s+1)} + jM_s + k, y, x) \Big| d\mu(x) \leq C(M_A M_{t^1})^{\frac{1}{2}}.$$

Proof. For fixed $t = (t^1, t^2), s, A$ we discuss the integral

$$\int_{J_{t^1}(y^1) \times J_{t^2}(y^2)} \sup_{|n|=A} \Big| \sum_{j=0}^{n_s-1} \sum_{k=0}^{M_s-1} \Phi(A, n^{(s+1)} + jM_s + k, y, x) \Big| d\mu(x).$$

Check the function $\sum_{j=0}^{n_s-1} \sum_{k=0}^{M_s-1} \Phi(A, n^{(s+1)} + jM_s + k, y, x)$ on the set $J_{t^1}(y^1) \times J_{t^2}(y^2)$. Since we have $x^2 \in J_{t^2}(y^2)$, then by (1.5) we have $|D_j(y^2, x^2)| \leq CM_{t^2}$ for each $j \in \mathbb{N}$ (this inequality is due to the fact that the Vilenkin space is bounded) and consequently by (1.5) $|D_{\alpha_2(A, n^{(s+1)} + jM_s + k)}(y^2, x^2)| \leq CM_{t^2}$.

On the other hand, again by (1.5) for $x^1 \in J_{t^1}(y^1)$ we have

$$D_{\alpha_1(A, n^{(s+1)} + jM_s + k)}(y^1, x^1) = \chi_{[\alpha_1(A, n^{(s+1)} + jM_s + k)]^{(t^1)}}(y^1)$$

$$\bar{\chi}_{[\alpha_1(A, n^{(s+1)} + jM_s + k)]^{(t^1)}}(x^1) \times \Bigg[\sum_{w=0}^{t^1-1} [\alpha_1(A, n^{(s+1)} + jM_s + k)]_w M_w$$

$$+ \sum_{i=m_{t^1} - (\alpha_1(A, n^{(s+1)} + jM_s + k))_{t^1}}^{m_{t^1}-1} r_{t^1}^i(y^1) \bar{r}_{t^1}^i(x^1) M_{t^1} \Bigg].$$

The function

$$\sum_{w=0}^{t^1-1} [\alpha_1(A, n^{(s+1)} + jM_s + k)]_w M_w + \sum_{i=m_{t^1} - (\alpha_1(A, n^{(s+1)} + jM_s + k))_{t^1}}^{m_{t^1}-1} r_{t^1}^i(y^1) \bar{r}_{t^1}^i(x^1) M_{t^1}$$

depends only on $x_{t^1}^1$ as x varies (and not on the other coordinates of x^1), so its absolute value is bounded by CM_{t^1}. Besides,

$$\left| \sum_{j=0}^{n_s-1} \sum_{k=0}^{M_s-1} D_{\alpha_1(A, n^{(s+1)} + jM_s + k)}(y^1, x^1) D_{\alpha_2(A, n^{(s+1)} + jM_s + k)}(y^2, x^2) \right|^2$$

$$= \sum_{j,h=0}^{n_s-1} \sum_{k,l=0}^{M_s-1} D_{\alpha_1(A, n^{(s+1)} + jM_s + k)}(y^1, x^1) \overline{D}_{\alpha_1(A, n^{(s+1)} + hM_s + l)}(y^1, x^1)$$

$$\times D_{\alpha_2(A, n^{(s+1)} + jM_s + k)}(y^2, x^2) \overline{D}_{\alpha_2(A, n^{(s+1)} + hM_s + l)}(y^2, x^2).$$

Therefore, apply the Cauchy-Bunyakovsky-Schwarz inequality:

$$\int_{J_{t^2}(y^2)}\left[\int_{J_{t^1}(y^1)}\sup_{|n|=A}\left|\sum_{j=0}^{n_s-1}\sum_{k=0}^{M_s-1}\Phi(A,n^{(s+1)}+jM_s+k,y^1,y^2,x^1,x^2)\right|d\mu(x^1)\right]d\mu(x^2)$$

$$\leq\int_{J_{t^2}(y^2)}M_{t^1}^{\frac{-1}{2}}\left[\int_{J_{t^1}(y^1)}\sup_{|n|=A}\left|\sum_{j=0}^{n_s-1}\sum_{k=0}^{M_s-1}\Phi(A,n^{(s+1)}+jM_s\right.\right.$$
$$\left.\left.+k,y^1,y^2,x^1,x^2)\right|^2d\mu(x^1)\right]^{\frac{1}{2}}d\mu(x^2)$$

$$\leq\int_{J_{t^2}(y^2)}M_{t^1}^{\frac{-1}{2}}\left[\int_{J_{t^1}(y^1)}\sup_{|n|=A}\sum_{j,h=0}^{n_s-1}\sum_{k,l=0}^{M_s-1}D_{\alpha_1(A,n^{(s+1)}+jM_s+k)}(y^1,x^1)\overline{D}_{\alpha_1(A,n^{(s+1)}+hM_s+l)}(y^1,x^1)\right.$$

$$\left.\times D_{\alpha_2(A,n^{(s+1)}+jM_s+k)}(y^2,x^2)\overline{D}_{\alpha_2(A,n^{(s+1)}+hM_s+l)}(y^2,x^2)d\mu(x^1)\right]d\mu(x^2)$$

$$=\int_{J_{t^2}(y^2)}M_{t^1}^{\frac{-1}{2}}\left[\int_{J_{t^1}(y^1)}\sup_{|n|=A}\sum_{j,h=0}^{n_s-1}\sum_{k,l=0}^{M_s-1}\chi_{\left[\alpha_1(A,n^{(s+1)}+jM_s+k)\right]^{(t^1)}}(y^1,x^1)\right.$$
$$\overline{\chi}_{\left[\alpha_1(A,n^{(s+1)}+hM_s+l)\right]^{(t^1)}}(y^1,x^1)$$
$$\times\left(\sum_{w=0}^{t^1-1}\alpha_1(A,n^{(s+1)}+jM_s+k)_wM_w+\sum_{a=m_{t1}-[\alpha_1(A,n^{(s+1)})+jM_s+k]_{t1}}^{m_{t1}-1}r_{t1}^a(y^1,x^1)\right)$$
$$\times\left(\sum_{v=0}^{t^1-1}\alpha_1(A,n^{(s+1)}+hM_s+l)_vM_v+\sum_{b=m_{t1}-[\alpha_1(A,n^{(s+1)})+hM_s+l]_{t1}}^{m_{t1}-1}\overline{r_{t1}^b}(y^1,x^1)\right)$$
$$\left.D_{\alpha_2(A,n^{(s+1)}+jM_s+k)}(x^2)\overline{D}_{\alpha_2(A,n^{(s+1)}+hM_s+l)}(y^2,x^2)d\mu(x^1)\right]^{\frac{1}{2}}d\mu(x^2)=:B^1.$$

Since $n^{(s+1)}$ depends only on natural numbers $n_{s+1},...,n_{A-1},n_A$, then the supreme $\sup_{n\in\mathbb{N}:|n|=A}$ above also depends only on $n_s,n_{s+1},...,n_{A-1},n_A$. Thus, by $|n|=A$

$$B^1\leq\int_{J_{t^2}(y^2)}M_{t^1}^{-\frac{1}{2}}\left[\sum_{n_A=1}^{m_A-1}\sum_{n_{A-1}=0}^{m_{A-1}-1}\cdots\sum_{n_{s+1}=0}^{m_{s+1}-1}\sum_{n_s=0}^{m_s-1}\int_{J_{t^1}(y^1)}\sum_{j,h=0}^{n_s-1}\sum_{k,l=0}^{M_s-1}\right.$$
$$\chi_{\left[\alpha_1(A,n^{(s+1)}+jM_s+k)\right]^{(t^1)}}(y^1,x^1)\overline{\chi}_{\left[\alpha_1(A,n^{(s+1)}+hM_s+l)\right]^{(t^1)}}(y^1,x^1)$$
$$\times\left(\sum_{w=0}^{t^1-1}\alpha_1(A,n^{(s+1)}+jM_s+k)_wM_w+\sum_{a=m_{t1}-[\alpha_1(A,n^{(s+1)})+jM_s+k]_{t1}}^{m_{t1}-1}r_{t1}^a(y^1,x^1)\right)$$

$$\times \left(\sum_{v=0}^{t^1-1} \alpha_1(A,n^{(s+1)}+hM_s+l)_v M_v + \sum_{b=m_{t1}-[\alpha_1(A,n^{(s+1)})+hM_s+l]_{t1}}^{m_{t1}-1} \overline{r_{t_1}^b}(y^1,x^1) \right)$$

$$\times D_{\alpha_2(A,n^{(s+1)}+jM_s+k)}(y^1,x^1)\overline{D}_{\alpha_2(A,n^{(s+1)}+hM_s+l)}(y^2,x^2)d\mu(x^1)\Bigg]^{\frac{1}{2}} d\mu(x^2)$$

$$= \int_{J_{t^2}(y^2)} M_{t^1}^{-\frac{1}{2}} \left[\sum_{n_A=1}^{m_A-1} \sum_{n_{A-1}=0}^{m_{A-1}} \cdots \sum_{n_{s+1}=0}^{m_{s+1}-1} \sum_{n_s=0}^{m_s-1} \sum_{j,h=0}^{n_s-1} \sum_{k,l=0}^{M_s-1} \right.$$

$$\int_{J_{t^1}(y^1)} \left(\sum_{w=0}^{t^1-1} \alpha_1(A,n^{(s+1)}+jM_s+k)_w M_w + \sum_{a=m_{t1}-[\alpha_1(A,n^{(s+1)})+jM_s+k]_{t1}}^{m_{t1}-1} r_{t_1}^a(y^1,x^1) \right)$$

$$\times \left(\sum_{v=0}^{t^1-1} \alpha_1(A,n^{(s+1)}+hM_s+l)_v M_v + \sum_{b=m_{t1}-[\alpha_1(A,n^{(s+1)})+hM_s+l]_{t1}}^{m_{t1}-1} \overline{r_{t_1}^b}(y^1,x^1) \right)$$

$$\times \chi_{\left[\alpha_1(A,n^{(s+1)}+jM_s+k)\right]^{(t^1)}}(y^1,x^1)\overline{\chi}_{\left[\alpha_1(A,n^{(s+1)}+hM_s+l)\right]^{(t^1)}}(y^1,x^1)d\mu(x^1)$$

$$\times D_{\alpha_2(A,n^{(s+1)}+jM_s+k)}(y^2,x^2)\overline{D}_{\alpha_2(A,n^{(s+1)}+hM_s+l)}(y^2,x^2)\Bigg]^{\frac{1}{2}} d\mu(x^2)$$

$$= \int_{J_{t^2}(y^2)} M_{t^1}^{-\frac{1}{2}} \left[\sum_{n_A=1}^{m_A-1} \sum_{n_{A-1}=0}^{m_{A-1}} \cdots \sum_{n_{s+1}=0}^{m_{s+1}-1} \sum_{n_s=0}^{m_s-1} \sum_{j,h=0}^{n_s-1} \sum_{k,l=0}^{M_s-1} \right.$$

$$\left(\sum_{w=0}^{t^1-1} \sum_{v=0}^{t^1-1} \alpha_1(A,n^{(s+1)}+jM_s+k)_w M_w \alpha_1(A,n^{(s+1)}+hM_s+l)_v M_v \right.$$

$$\times \int_{J_{t^1}(y^1)} \chi_{\left[\alpha_1(A,n^{(s+1)}+jM_s+k)\right]^{(t^1)}}(y^1,x^1)\overline{\chi}_{\left[\alpha_1(A,n^{(s+1)}+hM_s+l)\right]^{(t^1)}}(y^1,x^1)d\mu(x^1)$$

$$+ \sum_{w=0}^{t^1-1} \sum_{b=m_{t1}-[\alpha_1(A,n^{(s+1)})+hM_s+l]_{t1}}^{m_{t1}-1} \alpha_1(A,n^{(s+1)}+jM_s+k)_w M_w$$

$$\times \int_{J_{t^1}(y^1)} \chi_{\left[\alpha_1(A,n^{(s+1)}+jM_s+k)\right]^{(t^1)}}(x^1)\overline{\chi}_{\left[\alpha_1(A,n^{(s+1)}+hM_s+l)\right]^{(t^1)}}(x^1)\overline{r_{t_1}^b}(y^1,x^1)d\mu(x^1)$$

$$+ \sum_{a=m_{t1}-[\alpha_1(A,n^{(s+1)})+jM_s+k]_{t1}}^{m_{t1}-1} \sum_{v=0}^{t^1-1} \alpha_1(A,n^{(s+1)}+hM_s+l)_v M_v$$

$$\times \int_{J_{t^1}(y^1)} \chi_{\left[\alpha_1(A,n^{(s+1)}+jM_s+k)\right]^{(t^1)}}(y^1,x^1)\overline{\chi}_{\left[\alpha_1(A,n^{(s+1)}+hM_s+l)\right]^{(t^1)}}(x^1)r_{t_1}^a(y^1,x^1)d\mu(x^1)$$

$$+ \sum_{a=m_{t^1}-[\alpha_1(A,n^{(s+1)})+jM_s+k]_{t^1}}^{m_{t^1}-1} \sum_{b=m_{t^1}-[\alpha_1(A,n^{(s+1)})+hM_s+l]_{t^1}}^{m_{t^1}-1}$$

$$\int_{J_{t^1}(y^1)} \chi_{[\alpha_1(A,n^{(s+1)}+jM_s+k)]^{(t^1)}}(y^1,x^1) \overline{\chi}_{[\alpha_1(A,n^{(s+1)}+hM_s+l)]^{(t^1)}}(y^1,x^1)$$

$$r_{t^1}^a(y^1,x^1)\overline{r_{t^1}^b}(y^1,x^1)d\mu(x^1) \Bigg) \times D_{\alpha_2(A,n^{(s+1)}+jM_s+k)}(y^2,x^2)$$

$$\overline{D}_{\alpha_2(A,n^{(s+1)}+hM_s+l)}(y^2,x^2) \Bigg]^{\frac{1}{2}} d\mu(x^2) =: \sum_{q=1}^{4} B^{2,q} =: B^2.$$

Before we estimate B^2, (that is, any of $B^{2,q}$) let us determine the possible number of k, $l's$ such that the integral

$$\int_{J_{t^1}(y^1)} \chi_{[\alpha_1(A,n^{(s+1)}+jM_s+k)]^{(t^1)}}(y^1,x^1) \overline{\chi}_{[\alpha_1(A,n^{(s+1)}+hM_s+l)]^{(t^1)}}(y^1,x^1)$$

$$\times r_{t^1}^a(y^1,x^1)\overline{r_{t^1}^b}(y^1,x^1)d\mu(x^1) \neq 0.$$

(Recall that in the case of $B^{2,1}$ we have $a = b = 0$, in the case of $B^{2,2}$ we have $a = 0$ and in the case of $B^{2,3}$ we have $b = 0$.) Suppose the integral,

$$\int_{J_{t^1}(y^1)} \chi_{[\alpha_1(A,n^{(s+1)}+jM_s+k)]^{(t^1)}}(y^1,x^1) \overline{\chi}_{[\alpha_1(A,n^{(s+1)}+hM_s+l)]^{(t^1)}}(y^1,x^1)$$

$$\times r_{t^1}^a(y^1,x^1)\overline{r_{t^1}^b}(y^1,x^1)d\mu(x^1) \neq 0$$

for some $a, b \in \{0, 1, ..., m_{t_1} - 1\}$. Then use Lemma 1 and (1.2)-(1.4). Consequently, the $(t^1+1)^{th}$, $(t^1+2)^{th}$, ..., coordinates of $\alpha_1(A, n^{(s+1)} + jM_s + k)$ and $\alpha_1(A, n^{(s+1)} + hM_s + l)$ should be equal. The absolute value of this integral is not greater than $1/M_{t^1}$. Since by (2.1) we have that for every k, there exist a bounded number of $l's$ for which $\alpha_1(A, n^{(s+1)} + jM_s + k) = \alpha_1(A, n^{(s+1)} + hM_s + l)$. This gives that for every k, there exist at most CM_{t^1} number of $l's$ for which this integral is not zero. Consequently, by

$$\sum_{w=0}^{t^1-1} [\alpha_1(A, n^{(s+1)} + jM_s + k)]_w M_w \leq CM_{t^1}, \quad |D_{\alpha_2(A,n^{(s+1)}+jM_s+k)}(y^2,x^2)| \leq CM_{t^2}$$

we have

$$B^{2,1} \leq C \int_{J_{t^2}(y^2)} M_{t^1}^{-\frac{1}{2}} \left[\sum_{n_A=1}^{m_A-1} \sum_{n_{A-1}=0}^{m_A-1} \cdots \sum_{n_{s+1}=0}^{m_{s+1}-1} \sum_{n_s=0}^{m_s-1} \sum_{j,h=0}^{n_s-1} \sum_{k=0}^{M_s-1} M_{t^1}M_{t^1}^2 \frac{1}{M_{t^1}} M_{t^2}^2 \right]^{\frac{1}{2}} d\mu(x^2).$$

(Keep in mind that the Vilenkin space is bounded. That is, $m_j \leq C$ for all $j \in \mathbb{N}$.) By the very same steps we get the identical upper bound also for $B^{2,2}, B^{2,3}$ and $B^{2,4}$. Then, also for their sum. That is,

$$B^2 \leq C \int_{J_{t^2}(y^2)} M_{t^1}^{-\frac{1}{2}} \left[\sum_{n_A=1}^{m_A-1} \sum_{n_{A-1}=0}^{m_{A-1}-1} \cdots \sum_{n_{s+1}=0}^{m_{s+1}-1} \sum_{n_s=0}^{m_s-1} ((M_{t^1} M_{t^2})^2 M_s M_{t^1} M_{t^1}^{-1}) \right]^{\frac{1}{2}} d\mu$$

$$\leq C M_{t^2}^{-1} M_{t^1}^{\frac{-1}{2}} \left[\frac{M_A}{M_s} M_{t^1}^2 M_{t^2}^2 M_s \right]^{\frac{1}{2}} \leq C (M_A M_{t^1})^{\frac{1}{2}}.$$

This means,

$$\int_{J_{t^1}(y^1) \times J_{t^2}(y^2)} \sup_{|n|=A} \left| \sum_{j=0}^{n_s-1} \sum_{k=0}^{M_s-1} \Phi(A, n^{(s+1)} + jM_s + k, y, x) \right| d\mu(x)$$

$$\leq C (M_A M_{t^1})^{\frac{1}{2}}.$$

This completes the proof of Lemma 2. $\qquad\qquad\square$

Lemma 3. *Let $a \in \mathbb{N}, y \in G_m^2$. Then,*

$$\sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{\infty} \int_{J_{t^1}(y^1) \times J_{t^2}(y^2)} \sup_{A \geq a} \sup_{|n|=A} \frac{1}{M_A} \sum_{s=t^1}^{A} \sum_{j=0}^{n_s-1} \left| \sum_{k=0}^{M_s-1} \Phi(A, n^{(s+1)} + jM_s + k, y, x) \right| d\mu(x) \leq C.$$

Proof. The inequality in Lemma 2 immediately gives the following result:

$$\sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{\infty} \int_{J_{t^1}(y^1) \times J_{t^2}(y^2)} \sup_{A \geq \max(a,t^2-c)} \sup_{|n|=A} \frac{1}{M_A} \sum_{s=t^1}^{A} \sum_{j=0}^{n_s-1}$$

$$\left| \sum_{k=0}^{M_s-1} \Phi(A, n^{(s+1)} + jM_s + k, y, x) \right| d\mu(x)$$

$$\leq C \sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{\infty} \sum_{A=\max(a,t^2-c)}^{\infty} \sum_{s=t^1}^{A} (M_{t^1} M_A^{-1})^{\frac{1}{2}}$$

$$= C \sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{\infty} \sum_{A=\max(a,t^2-c)}^{\infty} (A - t^1 + 1)(M_{t^1} M_A^{-1})^{\frac{1}{2}}$$

$$\leq C \sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{\infty} (\max(a,t^2) - t^1)(M_{t^1} M_{\max(a,t^2-c)}^{-1})^{\frac{1}{2}}$$

$$\leq C \sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{\infty} (\max(a,t^2) - t^1)(M_{t^1} M_{\max(a,t^2)}^{-1})^{\frac{1}{2}}$$

$$\leq C \sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{a-1} (a - t^1)(M_{t^1} M_a^{-1})^{\frac{1}{2}} + C \sum_{t^1=0}^{a-1} \sum_{t^2=a+1}^{\infty} (t^2 - t^1)(M_{t^1} M_{t^2}^{-1})^{\frac{1}{2}} \leq C.$$

This estimation used the fact that $M_{\max(a,t^2)} \leq CM_{\max(a,t^2-c)}$ which from the boundedness of the system (i.e. the Vilenkin space). This inequality shows that if we want to complete the proof of this lemma, then we have to discuss also the case when $\sup_{t^2-c>A\geq a}$. This follows that t^2 should be at least $a+c$. That is, we have to prove that the following integral is bounded

$$\sum_{t^1=0}^{a-1}\sum_{t^2=a+c}^{\infty}\int_{J_{t^1}(y^1)\times J_{t^2}(y^2)}\sup_{t^2-c>A\geq a}\sup_{|n|=A}\frac{1}{M_A}\sum_{s=t^1}^{A}\sum_{j=0}^{n_s-1}$$
$$\left|\sum_{k=0}^{M_s-1}\Phi(A,n^{(s+1)}+jM_s+k,x^1,x^2)\right|d\mu(x)$$
$$:=B^3.$$

The method we are going to use in order to discuss B^3 is the same as we used for the investigation of B^1. The only difference is that in the situation of B^1 we used the estimation $|D_{\alpha_2(A,n^{(s+1)}+jM_s+k)}(y^2,x^2)| \leq CM_{t^2}$ and in the case of B^3 we use the formula of the Dirichlet kernel D_n (1.5) and the estimation $|D_{\alpha_2(A,n^{(s+1)}+jM_s+k)}(y^2,x^2)| \leq CM_A$. The other steps of this process are the same. That is,

$$B^3 \leq C\sum_{t^1=0}^{a-1}\sum_{t^2=a+c}^{\infty}\int_{J_{t^2}(y^2)}\sum_{A=a}^{t^2-c}\frac{1}{M_A}\sum_{s=t^1}^{A}M_{t^1}^{-\frac{1}{2}}$$

$$\times\left[\sum_{n_A=1}^{m_A-1}\sum_{n_{A-1}=0}^{m_{A-1}-1}\cdots\sum_{n_{s+1}=0}^{m_{s+1}-1}(M_{t^1}M_A)^2M_sM_{t^1}^{-1}\right]^{\frac{1}{2}}d\mu(x^2)$$

$$=C\sum_{t^1=0}^{a-1}\sum_{t^2=a+c}^{\infty}\sum_{A=a}^{t^2-c}\sum_{s=t^1}^{A}M_{t^2}^{-1}M_A^{-1}M_{t^1}^{-\frac{1}{2}}(M_AM_s^{-1}M_{t^1}^2M_A^2M_s)^{\frac{1}{2}}$$

$$\leq C\sum_{t^1=0}^{a-1}\sum_{t^2=a+c}^{\infty}\sum_{A=a}^{t^2-c}\sum_{s=t^1}^{A}\frac{(M_AM_{t^1})^{\frac{1}{2}}}{M_{t^2}}$$

$$\leq C\sum_{t^1=0}^{a-1}\sum_{t^2=a+c}^{\infty}\sum_{A=a}^{t^2-c}(A-t^1+1)M_A^{\frac{1}{2}}M_{t^1}^{\frac{1}{2}}M_{t^2}^{-1}$$

$$\leq C\sum_{t^1=0}^{a-1}\sum_{t^2=a+c}^{\infty}(t^2-t^1+1)M_{t^1}^{\frac{1}{2}}M_{t^2}^{-\frac{1}{2}}\leq C.$$

This completes the proof.

$\square$

In the sequel we step further and with the application of Lemma 3, we prove the main tool with respect to the maximal generalized Marcinkiwicz kernel in order to prove that the maximal operator $t_*^\alpha := \sup_{n \in \mathbb{P}} |t_n^\alpha f|$ is quasi-local (for the definition of quasi-locality, see e.g. [16, page 262]) and then it is of weak type (L^1, L^1).

Lemma 4. *Let* $u \in G_m^2, a \in \mathbb{N}, y \in I_a(u^1) \times I_a(u^2)$. *Then we have*

$$\int_{G_m^2(\backslash I_a(u^1) \times I_a(u^2))} \sup_{n \geq M_{a-c}} |M_n^\alpha(y,x)| d\mu(x) \leq C.$$

Proof. For $t^1 \leq a-1, t^2 \geq t^1$ and $x \in J_{t^1}(y^1) \times J_{t^2}(y^2)$ by (1.5) and (2.2) it is clear that

$$|\Phi(A, n^{(s+1)} + jM_s + k, y, x)|$$
$$= |D_{\alpha_1(A,n^{(s+1)}+jM_s+k)}(y^1,x^1) D_{\alpha_2(A,n^{(s+1)}+jM_s+k)}(y^2,x^2)| \leq CM_{t^1} M_{\min(t^2,A)}.$$

This gives,

$$\sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{\infty} \int_{J_{t^1}(y^1) \times J_{t^2}(y^2)} \sup_{A \geq a} \sup_{|n|=A} \frac{1}{M_A} \sum_{s=0}^{t^1} \sum_{j=0}^{n_s-1} \left| \sum_{k=0}^{M_s-1} \Phi(A, n^{(s+1)} + jM_s + k, y, x) \right| d\mu(x)$$

$$\leq C \sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{\infty} \int_{J_{t^1}(y^1) \times J_{t^2}(y^2)} \sup_{A \geq a-c} \frac{1}{M_A} \sum_{s=0}^{t^1} M_s M_{t^1} M_{\min(t^2,A)} d\mu(x)$$

$$\leq C \sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{a-c} \frac{1}{M_{t^1} M_{t^2}} \sup_{A \geq a-c} \frac{1}{M_A} M_{t^1}^2 M_{t^2} + C \sum_{t^1=0}^{a-1} \sum_{t^2=a-c}^{\infty} \frac{1}{M_{t^1} M_{t^2}} M_{t^1}^2$$

$$\leq C \sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{a-c} M_a^{-1} M_{t^1} + C \sum_{t^1=0}^{a-1} \sum_{t^2=a-c}^{\infty} M_{t^1} M_{t^2}^{-1} \leq C.$$

This by equality

$$M_n^\alpha(y,x) = \frac{1}{n} \sum_{s=0}^{A} \sum_{j=0}^{n_s-1} \sum_{k=0}^{M_s-1} \Phi(A, n^{(s+1)} + jM_s + k, y, x)$$

and by Lemma 3 immediately follows that

$$\sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{\infty} \int_{J_{t^1}(y^1) \times J_{t^2}(y^2)} \sup_{\{n:|n|=A \geq a-c\}} |M_n^\alpha(y,x)| d\mu(x) \leq C.$$

Now, we prove for each $y \in I_a(u^1) \times I_a(u^2)$ the almost everywhere relation

$$G_m^2 \setminus (I_a(u^1) \times I_a(u^2))$$
$$\subset \left(\bigcup_{t^1=0}^{a-1} \bigcup_{t^2=t^1}^{\infty} J_{t^1}(y^1) \times J_{t^2}(y^2) \right) \cup \left(\bigcup_{t^1=0}^{a-1} \bigcup_{t^1=t^2}^{\infty} J_{t^1}(y^1) \times J_{t^2}(y^2) \right)$$
$$=: J^1(y) \cup J^2(y).$$

Let $x = (x^1, x^2) \in G_m^2 \setminus (I_a(u^1) \times I_a(u^2))$. Then, either $x^1 \notin I_a(u^1)$ or $x^2 \notin I_a(u^2)$ (or both). Say, x^1 is not element of $I_a(u^1)$. Then, $x^1 \in J_{t^1}(u^1) = J_{t^1}(y^1)$ for some $t^1 < a$.

If $x^2 \in I_a(u^2) = I_a(y^2)$ and $x^2 \neq y^2$, then $x \in J^1(y)$. If $x^1 \in J_{t^1}(u^1) = J_{t^1}(y^1)$ and x^2 is not element of $I_a(u^2) = I_a(y^2)$, then, $x^1 \in J_{t^1}(u^1) = J_{t^1}(y^1)$ and $x^2 \in J_{t^2}(u^2) = J_{t^2}(y^2)$ for some $t^1, t^2 < a$.

For $t^2 \geq t^1$ we have $x \in J^1(y)$ and for $t^1 \geq t^2$ we have $x \in J^2(y)$. This procedure can be done if x^1, x^2 different from y^1 and y^2 respectively. The set of the points $x = (x^1, x^2)$, where either $x^1 = y^1$ or $x^2 = y^2$ is a zero measure set, so this can be supposed and the a.e. relation $G_m^2 \setminus (I_a(u^1) \times I_a(u^2)) \subset J^1(y) \cup J^2(y)$ is proved for each $y \in I_a(u^1) \times I_a(u^2)$. Therefore, the proof of Lemma 4 is complete. $\qquad \square$

Corollary 1. *Let $y \in G_m, n \in \mathbb{P}$. Then, $||M_n^\alpha(y, \cdot)||_1 \leq C$.*

Proof. By Lemma 4, we have $\int_{G_m^2 \setminus (I_{|n|}(y^1) \times I_{|n|}(y^2))} |M_n^\alpha(y, x)| d\mu(x) \leq C$. Besides, by the conditions (2.1) and (2.2)

$$|M_n^\alpha(y, x)| \leq \frac{1}{n} \sum_{k=0}^{n-1} D_{\alpha_1(|n|, k)}(y^1, x^1) D_{\alpha_2(|n|, k)}(y^2, x^2) \leq C \frac{1}{n} \sum_{k=0}^{n-1} M_{|n|} M_{|n|} \leq C M_{|n|}^2.$$

Consequently,

$$\int_{I_{|n|}(y^1) \times I_{|n|}(y^2)} |M_n^\alpha(y, x)| d\mu(x) \leq C.$$

Hence, $||M_n^\alpha(y, \cdot)|| \leq C$.

$\qquad \square$

Now, let's check the quasi-locality of the maximal operator t_*^α is quasi-local (for the definition of the quasi-locality, see e.g. [16, page 262]).

Lemma 5. *Let $f \in L^1(G_m^2)$ such that $\operatorname{supp} f \subset I_a(u^1) \times I_a(u^2)$, $\int f d\mu(x) = 0$ for some $u \in G_m^2$ and $a \in \mathbb{N}$. Then, $\int_{G_m^2 \setminus I_a(u^1) \times (I_a(u^2))} t_*^\alpha f(x) d\mu(x) \leq C||f||_1$.*

Proof. If $|n| \leq a - c$ for some fixed constant $c > 0$ depending only on α_1 and α_2, then we have by (4) that $\alpha_1(|n|, k), \alpha_2(|n|, k) < M_a$ for every $k < n$. Consequently, the kernel $M_n^\alpha(y, x)$, which is a linear combination of the product of Vilenkin-like functions χ_k with $k < M_a$, which is $\mathcal{A}_a$ measurable. This implies that

$$t_n^\alpha f(y) = \int_{I_a^2(u)} f(x) M_n^\alpha(x, y) d\mu(x) = M_n^\alpha(y) \int_{I_a(u^1) \times I_a(u^2)} f(x) d\mu = 0.$$

That is, $|n| \geq a - c$ can be supposed. By the theorem of Fubini, by Lemma 4 and by the fact that for kernel $|M_n^\alpha(y,x)| = |M_n^\alpha(x,y)|$ we get,

$$\int_{G_m^2 \setminus I_a^2(u)} t_*^\alpha f$$

$$= \int_{G_m^2 \setminus I_a^2(u)} \sup_{|n| \geq a-c} |t_n^\alpha f| d\mu(x)$$

$$= \int_{G_m^2 \setminus I_a^2(u)} \sup_{|n| \geq a-c} \left| \int_{I_a^2(u)} f(x) M_n^\alpha(y,x) d\mu(x) \right| d\mu(y)$$

$$\leq \int_{I_a^2(u)} |f(x)| \int_{G_m^2 \setminus I_a^2(u)} \sup_{|n| \geq a-c} |M_n^\alpha(y,x)| d\mu(y) | d\mu(x) \leq C \int_{I_a^2(u)} |f(x)| d\mu(x) = C ||f||_1.$$

This completes the proof of Lemma 5.   □

Theorem 1. *The operator t_*^α is of weak type (L^1, L^1) and it is also of type (L^p, L^p) for all $1 < p \leq \infty$.*

Proof. Now, we know that the operator t_*^α is of type (L^∞, L^∞) which is given by Corollary 1 and it is quasi-local by Lemma 5. Consequently, to prove that operator t_*^α is of weak type (L^1, L^1) is nothing else but to follow the standard argument (see e.g. [16]). Finally, the interpolation lemma of the Marcinkiwicz gives that it is also of type (L^p, L^p) for all $1 < p \leq \infty$. Consequently, by the above written by standard argument, we have the following theorem.   □

Theorem 2. *Let α satisfy conditions (2.1) and (2.2). Then, we have $t_n^\alpha \to f$ for each $f \in L^1(G_m^2)$ a.e. with respect to every bounded Vilenkin-like system.*

Proof. The proof of Theorem 2 is just a standard consequence of the fact that the maximal operator t_*^α is of weak type (L^1, L^1), the fact that it holds for each two-dimensional Vilenkin-like polynomial (linear combinations of $\chi_k(x^1)\chi_n(x^2)$) and the fact that the set of two-dimensional Vilenkin-like polynomials is dense in $L^1(G_m^2)$. This density property comes from the behavior of the one dimensional kernel function D_{M_n}. That is, it is either M_n or zero.   □

Finally, we give an application of Theorem 2. Before this a corollary is given:

Corollary 2. *Let (a_n) be a lacunary sequence of positive reals, i.e. $a_{n+1} \geq a_n q$ for some $q > 1$ $(n \in \mathbb{N})$ and α satisfy conditions (3) and $\alpha_j(n,k) \leq Ca_n$ $(k < a_n, j = 1, 2)$ (modified version of condition (2.2). Then for every integrable function $f \in L^1(G_m^2)$ we have*

$$\frac{1}{a_n} \sum_{k=0}^{a_n-1} S_{\alpha_1(n,k),\alpha_2(n,k)} f(x) \to f(x)$$

for a.e. $x \in G_m^2$.

Proof. The proof of this corollary runs as follows. Let b_n be defined as $M_{b_n-1} \leq a_n < M_{b_n}$ (that is, $b_n = |a_n| + 1$) and

$$\tilde{\alpha}_j(b_n, k) = \begin{cases} \alpha(n,k), & \text{for } 0 \leq k < a_n, \\ k, & \text{if } a_n \leq k < M_{b_n} \end{cases} \quad (j = 1, 2).$$

Then, $\tilde{\alpha}$ satisfies conditions (2.1) (trivially) and (2.2) since for $k < a_n$, $\tilde{\alpha}_j(b_n, k) = \alpha_j(n,k) \leq Ca_n \leq CM_{b_n}$. By Theorem 2 it follows that for the maximal operator $t_*^{\tilde{\alpha}} f := \sup |t_n^{\tilde{\alpha}} f|$ we have $\mu\{t_*^{\tilde{\alpha}} f \geq \lambda\} \leq C\|f\|_1/\lambda$ for all $f \in L^1(G_m^2)$ and $\lambda > 0$. Since

$$\frac{1}{a_n} \sum_{k=0}^{a_n-1} S_{\alpha_1(n,k),\alpha_2(n,k)} f = \frac{M_{b_n}}{a_n} \frac{1}{M_{b_n}} \sum_{k=0}^{M_{b_n}-1} S_{\tilde{\alpha}_1(b_n,k),\tilde{\alpha}_2(b_n,k)} f - \frac{M_{b_n}}{a_n} \frac{1}{M_{b_n}} \sum_{k=a_n}^{M_{b_n}-1} S_{k,k} f,$$

and consequently, $|t_{a_n}^{\alpha} f| \leq C|t_{M_{b_n}}^{\tilde{\alpha}} f| + C|t_{M_{b_n}}^{\text{id}} f| + C|t_{a_n}^{\text{id}} f|$, then (id denotes the "identical function", i.e. $\text{id}(n,k) = (k,k)$) $t_*^{\alpha} f \leq C t_*^{\tilde{\alpha}} f + C t_* f$. The ordinary maximal Marcinkiewicz operator is of weak type (L^1, L^1) (see e.g. [6]) and this by standard argument [16] completes the proof of this corollary.   □

In the sequel we give an application of the Corollary above. The triangular partial sums of the 2-dimensional Fourier series and the triangular Dirichlet kernels (with respect to the Vilenkin-like system χ) are defined as

$$S_k^{\triangle} f(x^1, x^2) := \sum_{i=0}^{k-1} \sum_{j=0}^{k-i-1} \hat{f}(i,j) \chi_i(x^1) \chi_j(x^2), \quad D_k^{\triangle}(x^1, x^2) := \sum_{i=0}^{k-1} \sum_{j=0}^{k-i-1} \chi_i(x^1) \chi_j(x^2).$$

The Fejér means of the triangular partial sums of the two-dimensional integrable function f (see e.g. [11]) are

$$\sigma_n^{\triangle} f := \frac{1}{n} \sum_{k=0}^{n-1} S_k^{\triangle} f.$$

For the trigonometric system Herriot proved [12] the a.e. (and norm) convergence $\sigma_n^{\triangle} f \to f$ ($f \in L^1$). His method can not be adopted for the Vilenkin system, since for the time being there is no kernel formula available for these systems. The first result in this a.e. convergence issue of triangular means is due to Goginava and Weisz [11]. They proved for the Walsh-Paley system and each integrable function the a.e. convergence relation $\sigma_{2^n}^{\triangle} f \to f$. This result for the whole sequence of the triangular mean operators in the Walsh case is given by the first author [9]. In the Vilenkin situation there is nothing proved yet. By the corollary above, (Corollary 2) we prove for bounded Vilenkin-like systems:

Theorem 3. *For every lacunary sequence (a_n) (that is, $a_{n+1} \geq qa_n$, $q > 1$) we have the a.e. convergence $\sigma_{a_n}^{\triangle} f \to f$ for each $f \in L^1(G_m^2)$.*

To demonstrate the proof of this, see some calculations below [9] between the triangle and the one dimensional Dirichlet kernels.

$$K_n^{\triangle}(y^1,y^2,x^1,x^2) = \frac{1}{n}\sum_{k=0}^{n-1} D_k^{\triangle}(y^1,y^2,x^1,x^2) = \frac{1}{n}\sum_{k=1}^{n-1}\sum_{i=0}^{k-1}\sum_{j=0}^{k-i-1} \chi_i(y^1,x^1)\chi_j(y^2,x^2)$$

$$= \frac{1}{n}\sum_{k=1}^{n-1}\sum_{i=0}^{k-1} \chi_i(y^1,x^1)D_{k-i}(y^2,x^2) = \frac{1}{n}\sum_{k=1}^{n-1}\sum_{i=1}^{k} \chi_{k-i}(y^1,x^1)D_i(y^2,x^2)$$

$$= \frac{1}{n}\sum_{i=1}^{n-1}\sum_{k=i}^{n-1} \chi_{k-i}(y^1,x^1)D_i(y^2,x^2) = \frac{1}{n}\sum_{i=1}^{n-1} D_{n-i}(y^1,x^1)D_i(y^2,x^2)$$

$$= \frac{1}{n}\sum_{i=1}^{n-1} D_i(y^1,x^1)D_{n-i}(y^2,x^2)$$

which is a generalized Marcienkiwicz kernel discussed in this paper (see Corollary 8:
$\alpha_1(|n|,k) = k$, $\alpha_2(|n|,k) = a_n - k$.

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