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A NONLINEAR ULTRA-PARABOLIC DIFFERENTIAL INEQUALITY WITH BESSEL OPERATOR

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Abstract. In this paper, we investigate the existence and nonexistence of weak solutions to an ultra-parabolic differential inequality posed on the interval $(0,1)$, under an inhomogeneous Dirichlet boundary condition. The considered problem involves the singular Bessel operator

$$\frac{d^2}{dx^2}+\frac{\alpha}{x}\frac{d}{dx}$$

and a nonlinearity of the form $x^{-\sigma}|u|^p$. Our approach is based on nonlinear capacity estimates and an appropriate choice of test functions.

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1. Introduction

In this work, we are concerned with the study of existence and nonexistence of weak solutions to the two-time nonlinear differential inequality governed by the singular Bessel operator

$$u_t+(|u|^{m-1}u)_s-\left(u_{xx}+\frac{\alpha}{x}u_x\right)\geq x^{-\sigma}|u|^p,\quad t,s>0,\ 0<x<1, \tag{1.1}$$

where $u=u(t,s,x)$, $\alpha\in\mathbb{R}$, $p>m\geq 1$ and $\sigma\geq 0$. Problem (1.1) is studied under the inhomogeneous Dirichlet boundary condition

$$u(t,s,1)=\delta,\quad t,s>0, \tag{1.2}$$

where $\delta>0$ is a constant.

Problem (1.1), involving two-time variables, is a differential inequality of ultra-parabolic-type. Ultra-parabolic problems arise in many applications such as mathematical finance (see, e.g., [3]), the theory of the Brownian motion (see, e.g., [15]) and the kinetic theory of gases (see, e.g., [2, 10]). In the linear case, ultra-parabolic

equations were first introduced by Kolmogoroff [7] in order to model the probability density of a system with 2D degrees of freedom. Since then, a lot of generalizations have been made by several authors. Nonlinear ultra-parabolic equations were first studied by Ugowski in [14], where differential inequalities of parabolic-type involving multidimensional time were investigated. For instance, a maximum principle which has several applications was established. A class of inverse problems for higher order ultraparabolic equations with unknown minor coefficients and a right-hand side function have been studied under specific constraints [12]. Recently, a class of ultra-parabolic Kolmogorov-type operators satisfying the Hörmander condition for the approximation and interpolation techniques was introduced by Pesce [11]. On the other hand, the Bessel operator arises as a result of solving the Helmholtz equation in spherical and cylindrical coordinates by means of separable variables. Therefore, Bessel operators are of great importance for many problems related to wave propagation and static potentials, see, eg., [8, 9].

The study of nonexistence of solutions to evolution problems involving multidimensional time was initiated by Kirane and his collaborators (see, e.g., [1, 5, 6]). Other contributions in this topic can be found in [4, 13]. In particular, Kerbal and Kirane [5] considered the two-time nonlinear equation

$$u_t + u_s - \Delta(|u|^m) = |u|^p, \quad t,s > 0, x \in \mathbb{R}^N, \tag{1.3}$$

where $u = u(t,s,x)$ and $p > m \geq 1$, subject to the initial conditions

$$u(t,0,x) = \varphi_1(t,x),\ u(0,s,x) = \varphi_2(s,x), \quad t,s > 0, x \in \mathbb{R}^N. \tag{1.4}$$

Namely, it was proved that, if

$$\int_0^\infty \int_{\mathbb{R}^N} \varphi_1(t,x)\,dx\,dt + \int_0^\infty \int_{\mathbb{R}^N} \varphi_2(s,x)\,dx\,ds > 0,$$

then (1.3)-(1.4) admits no weak solution, provided

$$1 \leq m < p \leq m + \frac{2m}{N+2}.$$

In the same reference [5], the authors extended the above result to systems of the form

$$\begin{cases} u_t + u_s - \Delta(|u|^m) &= |v|^p, \quad t,s > 0, x \in \mathbb{R}^N, \\ v_t + v_s - \Delta(|v|^n) &= |u|^q, \quad t,s > 0, x \in \mathbb{R}^N, \end{cases}$$

where $p > \max\{n,1\}$ and $q > \max\{m,1\}$.

To the best of our knowledge, the study of nonexistence of solutions to nonlinear ultra-parabolic problems in a bounded domain was not previously considered in the literature. Motivated by this fact, and the contribution [5], problem (1.1) under the Dirichlet boundary condition (1.2) is here investigated. Namely, using nonlinear capacity estimates, specifically adapted to the multidimensional time, the boundedness of the domain, the properties of the Bessel operator $\frac{\partial^2}{\partial x^2} + \frac{\alpha}{x}\frac{\partial}{\partial x}$ and the boundary

condition (1.2), we obtain optimal criteria for the nonexistence of solutions. The cases $\alpha < 1$ and $\alpha \geq 1$ are studied separately. In the case $\alpha < 1$, we show that, if $\sigma \leq 2 + (1-\alpha)(m-1)$, then for all $p > m$, (1.1)-(1.2) admits solutions for some $\delta > 0$. However, if $\sigma > 2+(1-\alpha)(m-1)$, then the dividing line with respect to existence and nonexistence is given by a critical exponent $p = p^*(\sigma, \alpha)$. In the case $\alpha \geq 1$, we show that the considered problem admits a different critical behavior. Namely, we show that $\sigma^* = 2$ is critical in the following sense: if $\sigma > \sigma^*$, then (1.1)-(1.2) admits no weak solution for all $p > m$; if $\sigma < \sigma^*$, then, for all $p > m$, (1.1)-(1.2) admits solutions for some $\delta > 0$.

The rest of the paper is organized as follows. In Section 2, we define weak solutions to the considered problem and state our main results. Some useful lemmas are established in Section 3. Finally, the proofs of our main results are given in Section 4.

Throughout the text, we shall use the following notations. For a positive real number ℓ, the notation $\ell \gg 1$ means that ℓ is sufficiently large. By C (or C_i), we mean a positive constant that is independent of the scaling parameters T, S, R and the solution u. The value of this constant is not important and is not necessarily the same from one line to another.

2. Main results

Before stating our main results, we need to define the notion of weak solution to (1.1)-(1.2). Let $Q = (0,\infty) \times (0,\infty) \times (0,1]$. We introduce the set of test functions

$$\Phi = \left\{ \varphi = \varphi(t,s,x) \in C^2(Q) : \sup(\varphi) \subset\subset Q, \varphi \geq 0, \varphi(\cdot,\cdot,1) \equiv 0 \right\}.$$

Here, by $\sup(\varphi) \subset\subset Q$, we mean that φ is compactly supported on Q.

Definition 1. We say that $u \in L^p_{loc}(Q)$ is a weak solution to (1.1)-(1.2) if

$$\begin{aligned} &\int_Q x^{-\sigma} |u|^p \varphi \, dx\, ds\, dt - \delta \int_{(0,\infty)\times(0,\infty)} \varphi_x(t,s,1)\, ds\, dt \\ &\qquad \leq -\int_Q u\varphi_t \, dx\, ds\, dt - \int_Q |u|^{m-1} u \varphi_s \, dx\, ds\, dt - \int_Q u \left(\varphi_{xx} - \alpha \left(\frac{\varphi}{x} \right)_x \right) dx\, ds\, dt \end{aligned} \tag{2.1}$$

for all $\varphi \in \Phi$.

Remark 1. Note that, if $u \in C^2((0,\infty) \times (0,\infty) \times (0,1])$ is a classical solution to (1.1)-(1.2) (i.e., u satisfies (1.1) for every $t, s > 0$, $0 < x < 1$, and u satisfies (1.2) for every $t, s > 0$), then, multiplying (1.1) by $\varphi \in \Phi$, integrating by parts over Q, and using (1.2), we obtain (2.1).

We first consider the case $\alpha < 1$.

Theorem 1. *Let* $\alpha < 1$ *and* $m \geq 1$.

(i) *If* $\delta > 0$ *and*

$$\sigma > 2 + (1-\alpha)(m-1), \ m < p < m + \frac{(\sigma - 2) - (1-\alpha)(m-1)}{1-\alpha}, \tag{2.2}$$

then (1.1)-(1.2) *admits no weak solution.*

(ii) *If*

$$p > \max\left\{m, m + \frac{(\sigma-2)-(1-\alpha)(m-1)}{1-\alpha}\right\},$$

then (1.1)-(1.2) *admits (stationary) solutions for some* $\delta > 0$.

Remark 2. Let $\alpha < 1$ and $m \geq 1$. From Theorem 1, we deduce the following facts:

(i) If $\sigma \leq 2 + (1-\alpha)(m-1)$, then for all $p > m$, (1.1)-(1.2) admits (stationary) solutions for some $\delta > 0$.

(ii) If $\sigma > 2 + (1-\alpha)(m-1)$, then

$$p^* := m + \frac{(\sigma-2)-(1-\alpha)(m-1)}{1-\alpha}$$

is a critical exponent for (1.1)-(1.2) in the following sense:

- If $\delta > 0$ and $m < p < p^*$, then (1.1)-(1.2) admits no weak solution.
- If $p > p^*$, then (1.1)-(1.2) admits (stationary) solutions for some $\delta > 0$.

Remark 3. When $\alpha < 1$ and $\sigma > 2 + (1-\alpha)(m-1)$, the critical case $p = p^*$ is left as an open question.

We next consider the case $\alpha \geq 1$.

Theorem 2. *Let* $\alpha \geq 1$ *and* $m \geq 1$.

(i) *If* $\delta > 0$ *and* $\sigma > 2$, *then, for all* $p > m$, (1.1)-(1.2) *admits no weak solution.*

(ii) *If* $\sigma < 2$, *then, for all* $p > m$, (1.1)-(1.2) *admits (stationary) solutions for some* $\delta > 0$.

Remark 4. If $\alpha \geq 1$ and $m \geq 1$, then it follows from Theorem 2 that (1.1)-(1.2) admits a new critical value $\sigma^* = 2$ in the following sense:

(i) If $\delta > 0$ and $\sigma > \sigma^*$, then, for all $p > m$, (1.1)-(1.2) admits no weak solution.

(ii) If $\sigma < \sigma^*$, then, for all $p > m$, (1.1)-(1.2) admits (stationary) solutions for some $\delta > 0$.

Remark 5. When $\alpha \geq 1$, the critical case $\sigma = \sigma^*$ is left as an open question.

3. Auxiliary results

Some useful lemmas for the proof of our main results are established in this section.

Let us consider a cut-off function $\zeta \in C^\infty([0,\infty))$ satisfying

$$0 \leq \zeta \leq 1, \ \sup(\zeta) \subset (0,1).$$

For $\ell \gg 1$ and $T, S > 0$, let

$$a_T(t) = \zeta^\ell\left(\frac{t}{T}\right), \quad t > 0, \tag{3.1}$$

and

$$a_S(s) = \zeta^{\ell}\left(\frac{s}{S}\right), \quad s > 0. \tag{3.2}$$

Lemma 1. *The following estimates hold:*

$$\int_0^{\infty} a_T(t)\,dt \le T, \tag{3.3}$$

and

$$\int_0^{\infty} a_S(s)\,ds \le S. \tag{3.4}$$

Proof. The inequalities (3.3) and (3.4) follow immediately from the properties of ζ, (3.1), and (3.2). □

Lemma 2. *The following inequality holds:*

$$\int_0^{\infty} a_T^{\frac{-1}{p-1}} \left|\frac{da_T}{dt}\right|^{\frac{p}{p-1}} dt \le CT^{1-\frac{p}{p-1}}. \tag{3.5}$$

Proof. For all $t > 0$, by (3.1) we have

$$\frac{da_T}{dt}(t) = \ell T^{-1}\zeta^{\ell-1}\left(\frac{t}{T}\right)\zeta'\left(\frac{t}{T}\right),$$

which implies that

$$a_T^{\frac{-1}{p-1}}(t)\left|\frac{da_T}{dt}(t)\right|^{\frac{p}{p-1}} = CT^{\frac{-p}{p-1}}\zeta^{\ell-\frac{p}{p-1}}\left(\frac{t}{T}\right)\left|\zeta'\left(\frac{t}{T}\right)\right|^{\frac{p}{p-1}}.$$

Then, by the properties of ζ, we get

$$\begin{aligned}\int_0^{\infty} a_T^{\frac{-1}{p-1}} \left|\frac{da_T}{dt}\right|^{\frac{p}{p-1}} dt &= CT^{\frac{-p}{p-1}} \int_0^{T} \zeta^{\ell-\frac{p}{p-1}}\left(\frac{t}{T}\right)\left|\zeta'\left(\frac{t}{T}\right)\right|^{\frac{p}{p-1}} dt \\ &\le CT^{\frac{-p}{p-1}} \int_0^{T} dt = CT^{1-\frac{p}{p-1}},\end{aligned}$$

which proves (3.5). □

Lemma 3. *The following inequality holds:*

$$\int_0^{\infty} a_S^{\frac{-m}{p-m}} \left|\frac{da_S}{ds}\right|^{\frac{p}{p-m}} ds \le CS^{1-\frac{p}{p-m}}. \tag{3.6}$$

Proof. For all $s > 0$, by (3.2) we have

$$\frac{da_S}{ds}(s) = \ell S^{-1}\zeta^{\ell-1}\left(\frac{s}{S}\right)\zeta'\left(\frac{s}{S}\right),$$

which implies that

$$a_S^{\frac{-m}{p-m}}(s)\left|\frac{da_S}{ds}(s)\right|^{\frac{p}{p-m}} = CS^{\frac{-p}{p-m}}\zeta^{\ell-\frac{p}{p-m}}\left(\frac{s}{S}\right)\left|\zeta'\left(\frac{s}{S}\right)\right|^{\frac{p}{p-m}}.$$

Then, by the properties of ζ, we get

$$\int_0^{\infty} a_S^{\frac{-m}{p-m}} \left|\frac{da_S}{ds}\right|^{\frac{p}{p-m}} ds = CS^{\frac{-p}{p-m}} \int_0^S \zeta^{\ell-\frac{p}{p-m}}\left(\frac{s}{S}\right)\left|\zeta'\left(\frac{s}{S}\right)\right|^{\frac{p}{p-m}} ds$$
$$\leq CS^{\frac{-p}{p-m}} \int_0^S ds = CS^{1-\frac{p}{p-m}},$$

which proves (3.6). □

For all $\alpha \in \mathbb{R}$, let us now introduce the function H_α defined in $(0,1]$ by

$$H_\alpha(x) = \begin{cases} x\left(1-x^{\alpha-1}\right) & \text{if} \quad \alpha > 1, \\ x\left(x^{\alpha-1}-1\right) & \text{if} \quad \alpha < 1, \\ -x\ln x & \text{if} \quad \alpha = 1. \end{cases} \tag{3.7}$$

It can be easily seen that

$$H_\alpha \geq 0, \ \frac{d^2H_\alpha}{dx^2} - \alpha\frac{d}{dx}\left(\frac{H_\alpha}{x}\right) = 0 \ \text{ in } (0,1) \tag{3.8}$$

and

$$\frac{dH_\alpha}{dx}(1) = \begin{cases} -|1-\alpha| & \text{if} \quad \alpha \neq 1, \\ -1 & \text{if} \quad \alpha = 1. \end{cases} \tag{3.9}$$

We also need a cut-off function $\xi \in C^\infty([0,\infty))$ satisfying

$$0 \leq \xi \leq 1, \ \xi \equiv 0 \text{ in } \left[0,\frac{1}{2}\right], \ \xi \equiv 1 \text{ in } [1,\infty).$$

For $\ell, R \gg 1$, let

$$\xi_R(x) = H_\alpha(x)\xi^\ell(Rx), \quad x \in (0,1],$$

that is,

$$\xi_R(x) = \begin{cases} 0 & \text{if} \quad 0 < x \leq \frac{1}{2R}, \\ H_\alpha(x)\xi^\ell(Rx) & \text{if} \quad \frac{1}{2R} \leq x \leq \frac{1}{R}, \\ H_\alpha(x) & \text{if} \quad \frac{1}{R} \leq x \leq 1. \end{cases} \tag{3.10}$$

Lemma 4. *The following inequality holds:*

$$\int_0^1 x^{\frac{\sigma}{p-1}} \xi_R(x)\,dx \leq C\left(\ln R + R^{-\left(\frac{\sigma}{p-1}+\alpha+1\right)}\right). \tag{3.11}$$

Proof. By the properties of ξ and (3.10), we have

$$\int_0^1 x^{\frac{\sigma}{p-1}} \xi_R(x)\,dx = \int_{\frac{1}{2R}}^1 x^{\frac{\sigma}{p-1}} H_\alpha(x)\xi^\ell(Rx)\,dx$$
$$\leq \int_{\frac{1}{2R}}^1 x^{\frac{\sigma}{p-1}} H_\alpha(x)\,dx. \tag{3.12}$$

We discuss three possible cases.

Case 1: $\alpha > 1$. In this case, by (3.7), we have

$$x^{\frac{\sigma}{p-1}} H_\alpha(x) \le x^{\frac{\sigma}{p-1}+1},$$

which implies by (3.12) that

$$\int_0^1 x^{\frac{\sigma}{p-1}} \xi_R(x)\,dx \le C. \tag{3.13}$$

Case 2: $\alpha < 1$. In this case, by (3.7), we have

$$x^{\frac{\sigma}{p-1}} H_\alpha(x) \le x^{\frac{\sigma}{p-1}+\alpha},$$

which implies by (3.12) that

$$\int_0^1 x^{\frac{\sigma}{p-1}} \xi_R(x)\,dx \le C\left(\ln R + R^{-\left(\frac{\sigma}{p-1}+\alpha+1\right)}\right). \tag{3.14}$$

Case 3: $\alpha = 1$. In this case, by (3.7), we have

$$x^{\frac{\sigma}{p-1}} H_\alpha(x) = x^{\frac{\sigma}{p-1}+1} \ln\left(\frac{1}{x}\right),$$

which implies by (3.12) that

$$\int_0^1 x^{\frac{\sigma}{p-1}} \xi_R(x)\,dx \le C \ln R. \tag{3.15}$$

Finally, (3.11) follows from (3.13), (3.14) and (3.15). □

Two more estimates will be needed.

Lemma 5. *The following inequality holds:*

$$\int_0^1 x^{\frac{\sigma m}{p-m}} \xi_R(x)\,dx \le C\left(\ln R + R^{-\left(\frac{\sigma m}{p-m}+\alpha+1\right)}\right).$$

Proof. Similar to that of Lemma 4. □

Lemma 6. *The following inequality holds:*

$$\int_0^1 x^{\frac{\sigma}{p-1}} \xi_R^{\frac{-1}{p-1}} \left|\frac{d^2\xi_R}{dx^2} - \alpha\frac{d}{dx}\left(\frac{\xi_R}{x}\right)\right|^{\frac{p}{p-1}} dx \le C R^{\frac{2p-\sigma}{p-1}-\min\{\alpha,1\}-1} \ln R. \tag{3.16}$$

Proof. By the definition of ξ_R, for all $x \in (0,1)$ we have

$$\begin{aligned}
\frac{d^2\xi_R}{dx^2} - \alpha\frac{d}{dx}\left(\frac{\xi_R}{x}\right) &= \frac{d^2}{dx^2}\left[H_\alpha(x)\xi^\ell(Rx)\right] - \alpha\frac{d}{dx}\left(\frac{H_\alpha}{x}\xi^\ell(Rx)\right) \\
&= \frac{d^2H_\alpha}{dx^2}\xi^\ell(Rx) + H_\alpha(x)\frac{d^2}{dx^2}\left[\xi^\ell(Rx)\right] + 2\frac{dH_\alpha}{dx}\frac{d}{dx}\left[\xi^\ell(Rx)\right] \\
&\quad - \alpha\frac{d}{dx}\left(\frac{H_\alpha}{x}\right)\xi^\ell(Rx) - \alpha\frac{H_\alpha}{x}\frac{d}{dx}\left[\xi^\ell(Rx)\right]
\end{aligned}$$

$$= \xi^{\ell}(Rx)\left(\frac{d^2H_\alpha}{dx^2} - \alpha\frac{d}{dx}\left(\frac{H_\alpha}{x}\right)\right) + H_\alpha(x)\frac{d^2}{dx^2}\left[\xi^{\ell}(Rx)\right] + \frac{d}{dx}\left[\xi^{\ell}(Rx)\right]\left(2\frac{dH_\alpha}{dx} - \alpha\frac{H_\alpha}{x}\right),$$

which implies, by (3.8) and the properties of ξ, that

$$\frac{d^2\xi_R}{dx^2} - \alpha\frac{d}{dx}\left(\frac{\xi_R}{x}\right) = H_\alpha(x)\frac{d^2}{dx^2}\left[\xi^{\ell}(Rx)\right] + \frac{d}{dx}\left[\xi^{\ell}(Rx)\right]\left(2\frac{dH_\alpha}{dx} - \alpha\frac{H_\alpha}{x}\right) \tag{3.17}$$

and

$$\sup\left(\frac{d^2\xi_R}{dx^2} - \alpha\frac{d}{dx}\left(\frac{\xi_R}{x}\right)\right) \subset \left[\frac{1}{2R}, \frac{1}{R}\right].$$

Consequently, we have

$$\begin{aligned}&\int_0^1 x^{\frac{\sigma}{p-1}}\xi_R^{\frac{-1}{p-1}}\left|\frac{d^2\xi_R}{dx^2} - \alpha\frac{d}{dx}\left(\frac{\xi_R}{x}\right)\right|^{\frac{p}{p-1}}dx\\ &= \int_{\frac{1}{2R}}^{\frac{1}{R}} x^{\frac{\sigma}{p-1}}\xi_R^{\frac{-1}{p-1}}\left|\frac{d^2\xi_R}{dx^2} - \alpha\frac{d}{dx}\left(\frac{\xi_R}{x}\right)\right|^{\frac{p}{p-1}}dx.\end{aligned} \tag{3.18}$$

On the other hand, from the properties of ξ, we have

$$\left|\frac{d^2}{dx^2}\left[\xi^{\ell}(Rx)\right]\right| \le CR^2\xi^{\ell-2}(Rx), \ \left|\frac{d}{dx}\left[\xi^{\ell}(Rx)\right]\right| \le CR\xi^{\ell-2}(Rx), \tag{3.19}$$

$x \in \left(\frac{1}{2R}, \frac{1}{R}\right)$. Furthermore, by (3.7), for all $x \in \left(\frac{1}{2R}, \frac{1}{R}\right)$, we have

$$\frac{2\frac{dH_\alpha}{dx} - \alpha\frac{H_\alpha}{x}}{H_\alpha(x)} = \begin{cases} \dfrac{2-\alpha-\alpha x^{\alpha-1}}{x-x^\alpha} & \text{if} \quad \alpha > 1,\\ \dfrac{\alpha x^{\alpha-1}+\alpha-2}{x^\alpha - x} & \text{if} \quad \alpha < 1,\\ \dfrac{-\ln x - 2}{-x\ln x} & \text{if} \quad \alpha = 1,\end{cases}$$

which implies that

$$\frac{\left|2\frac{dH_\alpha}{dx} - \alpha\frac{H_\alpha}{x}\right|}{H_\alpha(x)} \le CR, \quad x \in \left(\frac{1}{2R}, \frac{1}{R}\right). \tag{3.20}$$

We also have by (3.7) that

$$H_\alpha(x) \le CR^{-\min\{\alpha,1\}}\ln R, x \in \left(\frac{1}{2R}, \frac{1}{R}\right). \tag{3.21}$$

Hence, it follows from (3.17), (3.19) and (3.20) that

$$\frac{\left|\frac{d^2\xi_R}{dx^2} - \alpha\frac{d}{dx}\left(\frac{\xi_R}{x}\right)\right|}{H_\alpha(x)} \le \left|\frac{d^2}{dx^2}\left[\xi^{\ell}(Rx)\right]\right| + \left|\frac{d}{dx}\left[\xi^{\ell}(Rx)\right]\right|\frac{\left|2\frac{dH_\alpha}{dx} - \alpha\frac{H_\alpha}{x}\right|}{H_\alpha(x)}$$

$$\leq CR^2\xi^{\ell-2}(Rx), \quad x\in\left(\frac{1}{2R},\frac{1}{R}\right),$$

which implies by the definition of ξ_R and (3.21) that

$$\begin{aligned} x^{\frac{\sigma}{p-1}}\xi_R^{\frac{-1}{p-1}}\left|\frac{d^2\xi_R}{dx^2}-\alpha\frac{d}{dx}\left(\frac{\xi_R}{x}\right)\right|^{\frac{p}{p-1}} &= x^{\frac{\sigma}{p-1}}\xi^{\frac{-\ell}{p-1}}(Rx)H_\alpha(x)\left(\frac{\left|\frac{d^2\xi_R}{dx^2}-\alpha\frac{d}{dx}\left(\frac{\xi_R}{x}\right)\right|}{H_\alpha(x)}\right)^{\frac{p}{p-1}} \\ &\leq Cx^{\frac{\sigma}{p-1}}\xi^{\frac{-\ell}{p-1}}(Rx)\xi^{\frac{(\ell-2)p}{p-1}}(Rx)R^{\frac{2p}{p-1}}R^{-\min\{\alpha,1\}}\ln R \\ &= Cx^{\frac{\sigma}{p-1}}R^{\frac{2p}{p-1}-\min\{\alpha,1\}}\ln R\,\xi^{\ell-\frac{2p}{p-1}}(Rx), \quad x\in\left(\frac{1}{2R},\frac{1}{R}\right). \end{aligned}$$

Integrating now over $\left(\frac{1}{2R},\frac{1}{R}\right)$ and using (3.18), we get

$$\begin{aligned} &\int_0^1 x^{\frac{\sigma}{p-1}}\xi_R^{\frac{-1}{p-1}}\left|\frac{d^2\xi_R}{dx^2}-\alpha\frac{d}{dx}\left(\frac{\xi_R}{x}\right)\right|^{\frac{p}{p-1}}dx \\ &\qquad\leq CR^{\frac{2p}{p-1}-\min\{\alpha,1\}}\ln R\int_{\frac{1}{2r}}^{\frac{1}{R}} x^{\frac{\sigma}{p-1}}\xi^{\ell-\frac{2p}{p-1}}(Rx)\,dx \\ &\qquad\leq CR^{\frac{2p}{p-1}-\min\{\alpha,1\}}\ln R\int_{\frac{1}{2r}}^{\frac{1}{R}} x^{\frac{\sigma}{p-1}}\,dx \leq CR^{\frac{2p}{p-1}-\min\{\alpha,1\}-\frac{\sigma}{p-1}-1}\ln R, \end{aligned}$$

which proves (3.16). □

For all $\ell,T,S,R\gg 1$, let

$$\varphi(t,s,x)=a_T(t)a_S(s)\xi_R(x), \quad (t,s,x)\in Q. \tag{3.22}$$

Lemma 7. *The function* φ *belongs to* Φ.

Proof. Follows immediately from the definitions of a_T, a_S, and ξ_R. □

We now introduce the integral terms

$$\omega_1(\varphi)=\int_Q x^{\frac{\sigma}{p-1}}\varphi^{\frac{-1}{p-1}}|\varphi_t|^{\frac{p}{p-1}}\,dx\,ds\,dt, \tag{3.23}$$

$$\omega_2(\varphi)=\int_Q x^{\frac{\sigma m}{p-m}}\varphi^{\frac{-m}{p-m}}|\varphi_s|^{\frac{p}{p-m}}\,dx\,ds\,dt, \tag{3.24}$$

$$\omega_3(\varphi)=\int_Q x^{\frac{\sigma}{p-1}}\varphi^{\frac{-1}{p-1}}\left|\varphi_{xx}-\alpha\left(\frac{\varphi}{x}\right)_x\right|^{\frac{p}{p-1}}dx\,ds\,dt. \tag{3.25}$$

We obtain the following estimates for $\omega_i(\varphi)$, $i=1,2,3$.

Lemma 8. *The following estimate of* $\omega_1(\varphi)$ *holds:*

$$\omega_1(\varphi)\leq CT^{1-\frac{p}{p-1}}S\left(\ln R+R^{-\left(\frac{\sigma}{p-1}+\alpha+1\right)}\right). \tag{3.26}$$

Proof. By (3.22) and (3.23), we have

$$\omega_1(\varphi) = \left(\int_0^\infty a_T^{\frac{-1}{p-1}} \left| \frac{da_T}{dt} \right|^{\frac{p}{p-1}} dt \right) \left(\int_0^\infty a_S(s)\, ds \right) \left(\int_0^1 x^{\frac{\sigma}{p-1}} \xi_R(x)\, dx \right).$$

We obtain (3.26) by using (3.4), and Lemma 2 and 4. □

Lemma 9. *The following estimate of* $\omega_2(\varphi)$ *holds:*

$$\omega_2(\varphi) \leq CTS^{1-\frac{p}{p-m}} \left(\ln R + R^{-\left(\frac{\sigma m}{p-m}+\alpha+1\right)} \right).$$

Proof. Similar to the proof of Lemma 8, using (3.3) and Lemma 3 and 5. □

Lemma 10. *The following estimate of* $\omega_3(\varphi)$ *holds:*

$$\omega_3(\varphi) \leq CTSR^{\frac{2p-\sigma}{p-1}-\min\{\alpha,1\}-1} \ln R.$$

Proof. Similar to the proof of Lemma 8, using (3.3), (3.4), and Lemma 6. □

4. Proofs of the main results

This section is devoted to the proofs of Theorems 1 and 2.

Proof of Theorem 1. (i) We argue by contradiction, supposing that $u \in L^p_{loc}(Q)$ is a weak solution to (1.1)-(1.2). For $\ell, T, S, R \gg 1$, let φ be the function defined by (3.22). By Lemma 7, we know that $\varphi \in \Phi$. Then, by (2.1), it follows that

$$\begin{aligned} &\int_Q x^{-\sigma} |u|^p \varphi\, dx\, ds\, dt - \delta \int_{(0,\infty)\times(0,\infty)} \varphi_x(t,s,1)\, ds\, dt \qquad (4.1)\\ &\quad \leq \int_Q |u||\varphi_t|\, dx\, ds\, dt + \int_Q |u|^m |\varphi_s|\, dx\, ds\, dt + \int_Q |u| \left| \varphi_{xx} - \alpha \left(\frac{\varphi}{x} \right)_x \right| dx\, ds\, dt. \end{aligned}$$

Using Young's inequality, we get

$$\begin{aligned} \int_Q |u||\varphi_t|\, dx\, ds\, dt &= \int_Q \left(x^{\frac{-\sigma}{p}} |u| \varphi^{\frac{1}{p}} \right) \left(x^{\frac{\sigma}{p}} \varphi^{\frac{-1}{p}} |\varphi_t| \right) dx\, ds\, dt \qquad (4.2)\\ &\leq \frac{1}{3} \int_Q x^{-\sigma} |u|^p \varphi\, dx\, ds\, dt + C\omega_1(\varphi), \end{aligned}$$

where $\omega_1(\varphi)$ is the integral term given by (3.23). Similarly, we obtain

$$\int_Q |u|^m |\varphi_s|\, dx\, ds\, dt \leq \frac{1}{3} \int_Q x^{-\sigma} |u|^p \varphi\, dx\, ds\, dt + C\omega_2(\varphi) \qquad (4.3)$$

and

$$\int_Q |u| \left| \varphi_{xx} - \alpha \left(\frac{\varphi}{x} \right)_x \right| dx\, ds\, dt \leq \frac{1}{3} \int_Q x^{-\sigma} |u|^p \varphi\, dx\, ds\, dt + C\omega_3(\varphi), \qquad (4.4)$$

where $\omega_2(\varphi)$ and $\omega_3(\varphi)$ are given respectively by (3.24) and (3.25). Then, it follows from (4.1), (4.2), (4.3), and (4.4), that

$$-\delta \int_{(0,\infty)\times(0,\infty)} \varphi_x(t,s,1)\,ds\,dt \le C\sum_{j=1}^{3}\omega_j(\varphi). \tag{4.5}$$

On the other hand, by (3.1), (3.2), (3.10), and (3.22), we have for all $t,s>0$ that

$$\varphi_x(t,s,1) = a_T(t)a_S(s)\frac{d\xi_R}{dx}(1) = \zeta^\ell\left(\frac{t}{T}\right)\zeta^\ell\left(\frac{s}{S}\right)\frac{dH_\alpha}{dx}(1).$$

Furthermore, by (3.7), we have

$$\frac{dH_\alpha}{dx}(1) = -C_\alpha,$$

where $C_\alpha>0$. Consequently, we get

$$\varphi_x(t,s,1) = -C_\alpha\zeta^\ell\left(\frac{t}{T}\right)\zeta^\ell\left(\frac{s}{S}\right),$$

which implies, by the properties of ζ, that

$$-\delta\int_{(0,\infty)\times(0,\infty)}\varphi_x(t,s,1)\,ds\,dt = C\delta TS. \tag{4.6}$$

Now, we use (4.5), (4.6), and Lemmas 8, 9, and 10, to obtain

$$\begin{aligned} C\delta \le{}& T^{\frac{-p}{p-1}}\ln R + T^{\frac{-p}{p-1}}R^{-\left(\frac{\sigma}{p-1}+\alpha+1\right)} + S^{\frac{-p}{p-m}}\ln R + S^{\frac{-p}{p-m}}R^{-\left(\frac{\sigma m}{p-m}+\alpha+1\right)} \\ &+ R^{\frac{2p-\sigma}{p-1}-\min\{\alpha,1\}-1}\ln R. \end{aligned} \tag{4.7}$$

Taking $S = T^{\frac{p-m}{p-1}}$, (4.7) reduces to

$$\begin{aligned} C\delta \le{}& T^{\frac{-p}{p-1}}\ln R + T^{\frac{-p}{p-1}}R^{-\left(\frac{\sigma}{p-1}+\alpha+1\right)} + T^{\frac{-p}{p-1}}\ln R + T^{\frac{-p}{p-1}}R^{-\left(\frac{\sigma m}{p-m}+\alpha+1\right)} \\ &+ R^{\frac{2p-\sigma}{p-1}-\min\{\alpha,1\}-1}\ln R. \end{aligned} \tag{4.8}$$

Taking $T = R^\theta$ and $\theta>0$, (4.8) reduces to

$$\delta \le R^{\mu_1(\theta)}\ln R + R^{\mu_2(\theta)} + R^{\mu_3(\theta)} + R^{\mu}\ln R, \tag{4.9}$$

where

$$\begin{aligned} \mu_1(\theta) &= -\frac{\theta p}{p-1}, \\ \mu_2(\theta) &= -\frac{\theta p}{p-1} - \left(\frac{\sigma}{p-1}+\alpha+1\right), \\ \mu_3(\theta) &= -\frac{\theta p}{p-1} - \left(\frac{\sigma m}{p-m}+\alpha+1\right), \\ \mu &= \frac{2p-\sigma}{p-1} - \min\{\alpha,1\} - 1. \end{aligned}$$

Since $\alpha < 1$, then

$$\mu = \frac{2p-\sigma}{p-1} - \alpha - 1.$$

Remark that $\mu_1(\theta) < 0$ for all $\theta > 0$. Furthermore, taking

$$\theta > \frac{p-1}{p} \max\left\{ -\left(\frac{\sigma}{p-1} + \alpha + 1\right), -\left(\frac{\sigma m}{p-m} + \alpha + 1\right), 0 \right\}, \tag{4.10}$$

we get $\mu_2(\theta) < 0$ and $\mu_3(\theta) < 0$. On the other hand, from (2.2), we have

$$p < m + \frac{(\sigma-2) - (1-\alpha)(m-1)}{1-\alpha} = \frac{\sigma - 1 - \alpha}{1-\alpha},$$

which is equivalent to $\mu < 0$. Finally, since $\mu_i(\theta) < 0$ for all $i = 1,2,3$ and $\mu < 0$, passing to the limit as $R \to \infty$ in (4.9), we obtain $\delta \leq 0$, which contradicts the positivity of δ. This proves part (i) of Theorem 1.

(ii) Let

$$p > \max\left\{ m, m + \frac{(\sigma-2) - (1-\alpha)(m-1)}{1-\alpha} \right\} = \max\left\{ m, \frac{\sigma - 1 - \alpha}{1-\alpha} \right\}. \tag{4.11}$$

For

$$\max\left\{ \frac{\sigma-2}{p-1}, 0 \right\} < \gamma < 1 - \alpha \tag{4.12}$$

and

$$0 < \varepsilon \leq \left[\gamma(1-\alpha-\gamma)\right]^{\frac{1}{p-1}}, \tag{4.13}$$

let

$$u(x) = \varepsilon x^{\gamma}, \quad 0 < x < 1. \tag{4.14}$$

Note that, due to (4.11), one has

$$\frac{\sigma-2}{p-1} < 1 - \alpha,$$

which implies, together with $\alpha < 1$, that the set of γ satisfying (4.12) is nonempty. Now, using (4.12), (4.13), and (4.14), we obtain

$$-u_{xx} - \frac{\alpha}{x} u_x = x^{-\sigma} |u(x)|^p \left[\varepsilon^{1-p} \gamma (1-\gamma-\alpha) x^{\sigma - 2 - \gamma(p-1)} \right] \geq x^{-\sigma} |u(x)|^p,$$

for all $x \in (0,1)$, which shows that u is a stationary solution to (1.1). Furthermore, u satisfies (1.2) with $\delta = \varepsilon > 0$. This completes the proof of part (ii) of Theorem 1. □

Proof of Theorem 2. (i) We also argue here by contradiction, supposing that $u \in L^p_{loc}(Q)$ is a weak solution to (1.1)-(1.2). Following exactly the same steps of the proof of part (i) of Theorem 1, for $R \gg 1$, we get (4.9). In this case, since $\alpha \geq 1$ and $\sigma > 2$, we have

$$\mu = \frac{2-\sigma}{p-1} < 0.$$

Then, for θ satisfying (4.10), we have $\mu_i(\theta) < 0$ for all $i = 1,2,3$ and $\mu < 0$. Hence, passing to the limit as $R \to \infty$ in (4.9), we reach a contradiction with $\delta > 0$. This proves part (i) of Theorem 2.

(ii) We consider separately the cases $\alpha > 1$ and $\alpha = 1$.

• Case 1: $\alpha > 1$. For

$$\max\left\{1-\alpha, \frac{\sigma-2}{p-1}\right\} < \beta < 0 \tag{4.15}$$

and

$$0 < \tau \le [\beta(1-\beta-\alpha)]^{\frac{1}{p-1}}, \tag{4.16}$$

let

$$u(x) = \tau x^{\beta}, \quad 0 < x < 1. \tag{4.17}$$

Note that since $\alpha > 1$ and $\sigma < 2$, the set of β satisfying (4.15) is nonempty. Using (4.15), (4.16), and (4.17), we obtain

$$-u_{xx} - \frac{\alpha}{x}u_x = x^{-\sigma}|u(x)|^p \left[\tau^{1-p}\beta(1-\beta-\alpha)x^{\sigma-2-\beta(p-1)}\right] \ge x^{-\sigma}|u(x)|^p,$$

for all $x \in (0,1)$, which shows that u is a stationary solution to (1.1). Furthermore, u satisfies (1.2) with $\delta = \tau > 0$.

• Case 2: $\alpha = 1$. For $0 < \eta < 1$, let us consider the function

$$f_\eta(x) = x^{\sigma-2}(1-\ln x)^{\eta(1-p)-2}, \quad 0 < x \le 1.$$

Observe that $f_\eta(x) > 0$ for all $x \in (0,1]$ and, since $\sigma < 2$,

$$\lim_{x\to 0^+} f_\eta(x) = +\infty,$$

which implies that

$$0 < m_\eta := \inf_{0<x\le 1} f_\eta(x) < +\infty. \tag{4.18}$$

For

$$0 < \eta < 1,\ 0 < \varepsilon \le [\eta(1-\eta)m_\eta]^{\frac{1}{p-1}}, \tag{4.19}$$

let

$$u(x) = \varepsilon(1-\ln x)^{\eta}, \quad 0 < x < 1. \tag{4.20}$$

By (4.18), (4.19), and (4.20), we obtain

$$\begin{aligned} -u_{xx} - \frac{\alpha}{x}u_x &= x^{-\sigma}|u(x)|^p \left[\varepsilon^{1-p}\eta(1-\eta)f_\eta(x)\right] \\ &\ge x^{-\sigma}|u(x)|^p \left[\varepsilon^{1-p}\eta(1-\eta)m_\eta\right] \ge x^{-\sigma}|u(x)|^p, \end{aligned}$$

for all $x \in (0,1)$, which shows that u is a stationary solution to (1.1). Furthermore, u satisfies (1.2) with $\delta = \varepsilon > 0$. The proof of part (ii) of Theorem 2 is complete. □

5. Conclusion

We investigated the existence and nonexistence of weak solutions to the two-time nonlinear differential inequality (1.1) under the inhomogeneous Dirichlet boundary condition (1.2). Using nonlinear capacity estimates, we obtained sharp criteria for the nonexistence of weak solutions. Namely, if $\alpha < 1$ and $\sigma > 2 + (1-\alpha)(m-1)$, we established that the dividing line with respect to existence and nonexistence is given by a critical exponent $p = p*$ that depends on σ and α. However, if $\alpha \geq 1$, then the problem admits a new critical value $\sigma^* = 2$ in the following sense: if $\sigma > \sigma^*$, then for all $p > m$, there is no weak solution; if $\sigma < \sigma^*$, then for all $p > m$, solutions exist for some $\delta > 0$.

In this work, when $\alpha < 1$ and $\sigma > 2 + (1-\alpha)(m-1)$, the critical case $p = p^*$ was not investigated. It would be interesting to study this case in order to decide whether a weak solution exists or not.

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