Miskolc Mathematical Notes
Vol. 20 (2019), No. 2, pp. 1215–1225
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2019.3021

ON THE TRANSLATION HYPERSURFACES WITH GAUSS MAP G SATISFYING $\Delta G = AG$

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Received 11 September, 2019

Abstract. It is a known fact that a translation hypersurface is obtained by combination of any three curves in the 4-dimensional Euclidean space. We examine a special situation where the Gauss map of a translation hypersurface satisfies the condition $\Delta G = AG$ where Δ represents the Laplace operator and A is a 4×4-real matrix. Our result is that such a translation hypersurface is one of the following three hypersurfaces: the hypersurface of translation surface and a constant vector along this surface, the hyperplane, the hypersurface $\Sigma \times \mathbb{R}$ where Σ is a translation surface.

2010 *Mathematics Subject Classification:* 53B25; 53C50

Keywords: translation hypersurface, Gauss map, Euclidean space

1. Introduction

It was Chen [5], at first, who defined the notation of finite type immersions in order to study the submanifolds in Euclidean and pseudo-Euclidean spaces. Later, the notation was expanded from the finite type immersions to smooth maps. Gauss map is one of the most important smooth map and it is frequently used in the studies of surfaces, submanifolds and etc [7, 8]. The Gauss map G renders each point of a surface to the unit normal vector of surface, and we consider the special case, where the condition $\Delta G = AG$ is satisfied.

Let M be a connected surface in the Euclidean 3-space E^3 and let G be the Gauss map given by $G : M \longrightarrow S^2 \subseteq E^3$ where S^2 is the unit sphere in E^3 which is centered at the origin. The basic connection between Gauss map and Laplacian is arised from surface M with constant mean curvature. For such surfaces, ΔG is equal to $\|dG\|^2 G$ where Δ is the Laplace operator with respect to the induced metric on M and d is a differential operator [9]. This can be considered as surfaces whose Gauss map is an eigenfuction of the Laplacian; that is, $\Delta G = \lambda G, \lambda \in \mathbb{R}$. Generalizing this equation to

$$\Delta G = AG, A \in Mat(3, \mathbb{R}) \tag{1.1}$$

where $Mat(3,\mathbb{R})$ is the set of 3×3 real matrices, Dillen, Pas and Verstraelen [6] investigated surfaces of revolution in the Euclidean 3-space E^3 which satisfy condition (1.1). Then, they showed that such a surface is part of a plane, a sphere or a circular cylinder. Condition (1.1) is also used in further studies [3,11]. Thus, the surfaces are classified by considering the relation between the Gauss map and the Laplacian.

Differently from the existing studies, our motivation in this work is to examine the translation hypersurfaces of Euclidean 4-space which satisfies the condition

$$\Delta G = AG, A \in Mat(4,\mathbb{R}). \tag{1.2}$$

We prove that such a translation hypersurface is one of the following three hypersurfaces: a hypersurface of translation surface and a constant vector along this surface, a hyperplane, a hypersurface $\Sigma\times\mathbb{R}$ where Σ is a translation surface. For the proof the Gauss map of translation hypersurface and its Laplacian are obtained firstly. Then, the translation hypersurfaces are classified by using the obtained equations. Our conclusions expand the results of [1,4].

2. TRANSLATION HYPERSURFACES IN 4-DIMENSIONAL EUCLIDEAN SPACE

In this section, by using some basic concepts, we will obtain ΔG for the translation hypersurfaces in Euclidean 4-space.

We assume that M is a hypersurface of a Euclidean 4-space $\mathbb{E}^4$ and $\overline{g}$ is an induced metric tensor to M. It is a translation hypersurface in $\mathbb{E}^4$ if it is given with an immersion

$$x : C \subseteq \mathbb{E}^3 \to \mathbb{E}^4, (u,v,z) \mapsto (u,v,z,\tilde{f}(u)+\tilde{g}(v)+\tilde{h}(z)) \tag{2.1}$$

where $\tilde{f},\tilde{g},\tilde{h}$ are smooth functions [10]. We denote the Gauss map by $G : M \to S^3$ where S^3 is the unit hypersphere of $\mathbb{E}^4$ [2].

The Laplacian Δ with respect to the induced metric tensor $\overline{g}$ is

$$\Delta = \frac{-1}{\sqrt{|\omega|}}\sum_{i,j}\frac{\partial}{\partial x_i}\left(\sqrt{|\omega|}\overline{g}^{ij}\frac{\partial}{\partial x^j}\right) \tag{2.2}$$

where $\omega = \det(\overline{g})$, $\overline{g}^{ij} = (\overline{g})^{-1}$ [11].

Differentiating $x(u,v,z)$ in (2.1) we get

$$x_u = (1,0,0,f),\ x_v = (0,1,0,g),\ x_z = (0,0,1,h),$$

where $f = \frac{d\tilde{f}}{du}$, $g = \frac{d\tilde{g}}{dv}$, $h = \frac{d\tilde{h}}{dz}$. For the induced metric on M we have

$$\begin{aligned}
&\overline{g}_{11} = \langle x_u,x_u\rangle = 1+f^2, && \overline{g}_{12} = \langle x_u,x_v\rangle = fg, && \overline{g}_{13} = \langle x_u,x_z\rangle = fh,\\
&\overline{g}_{21} = \langle x_v,x_u\rangle = fg, && \overline{g}_{22} = \langle x_v,x_v\rangle = 1+g^2, && \overline{g}_{23} = \langle x_v,x_z\rangle = gh,\\
&\overline{g}_{31} = \langle x_z,x_u\rangle = fh, && \overline{g}_{32} = \langle x_z,x_v\rangle = gh, && \overline{g}_{33} = \langle x_z,x_z\rangle = 1+h^2.
\end{aligned}$$

So in a matrix form

$$\overline{g} = \begin{bmatrix} 1+f^2 & fg & fh \\ fg & 1+g^2 & gh \\ fh & gh & 1+h^2 \end{bmatrix},$$

hence

$$\det(\overline{g}) = 1+f^2+g^2+h^2.$$

For later use, we define the smooth function

$$\omega = \|x_u \wedge x_v \wedge x_z\|^2 = \langle (f,g,h,-1), (f,g,h,-1) \rangle,$$

and observe

$$\omega = 1+f^2+g^2+h^2 = \det(\overline{g}) \tag{2.3}$$

Then the Gauss map G of hypersurface is

$$G = (G_1, G_2, G_3, G_4) = \frac{x_u \wedge x_v \wedge x_z}{\|x_u \wedge x_v \wedge x_z\|} = \frac{1}{\sqrt{\omega}}(f,g,h,-1). \tag{2.4}$$

From (2.2), we get

$$\Delta = \frac{-1}{\sqrt{\omega}} \left\{ \begin{array}{l} \frac{\partial}{\partial u}\left(\frac{\omega - f^2}{\sqrt{\omega}} \frac{\partial}{\partial u} - \frac{fg}{\sqrt{\omega}} \frac{\partial}{\partial v} - \frac{fh}{\sqrt{\omega}} \frac{\partial}{\partial z} \right) + \\ + \frac{\partial}{\partial v}\left(\frac{-fg}{\sqrt{\omega}} \frac{\partial}{\partial u} + \frac{\omega - g^2}{\sqrt{\omega}} \frac{\partial}{\partial v} - \frac{gh}{\sqrt{\omega}} \frac{\partial}{\partial z} \right) + \\ + \frac{\partial}{\partial z}\left(\frac{-fh}{\sqrt{\omega}} \frac{\partial}{\partial u} - \frac{gh}{\sqrt{\omega}} \frac{\partial}{\partial v} + \frac{\omega - h^2}{\sqrt{\omega}} \frac{\partial}{\partial z} \right) \end{array} \right\}.$$

By straightforward computations we obtain

$$\Delta = \frac{1}{\omega} \left\{ \begin{array}{l} (f^2-\omega) \frac{\partial^2}{\partial u^2} + (g^2-\omega) \frac{\partial^2}{\partial v^2} + (h^2-\omega) \frac{\partial^2}{\partial z^2} + \\ + 2fg \frac{\partial^2}{\partial u \partial v} + 2fh \frac{\partial^2}{\partial u \partial z} + 2gh \frac{\partial^2}{\partial v \partial z} + \\ + \frac{1}{\omega} \left(\begin{array}{l} (\omega - f^2) f' + (\omega - g^2) g' \\ + (\omega - h^2) h' \end{array} \right) \left(f \frac{\partial}{\partial u} + g \frac{\partial}{\partial v} + h \frac{\partial}{\partial z} \right) \end{array} \right\}. \tag{2.5}$$

Now, we calculate ΔG_1, ΔG_2, ΔG_3 and ΔG_4.

Firstly, by considering (2.5) for $G_1 = \frac{f}{\sqrt{\omega}}$, we obtain

$$\Delta G_1 = \frac{1}{\omega^{\frac{7}{2}}}\left\{\begin{array}{l} \left(\omega - f^2\right)^2 \left(4ff' - \omega f''\right) + \\ + f\left(g^2 - \omega\right)\left[4g^2\left(g'\right)^2 - \omega\left(\left(g'\right)^2 + gg''\right)\right] + \\ + f\left(h^2 - \omega\right)\left[4h^2\left(h'\right)^2 - \omega\left(\left(h'\right)^2 + hh''\right)\right] + \\ + \left(2f^2 - \omega\right)4ff'\left(g^2g' + h^2h'\right) + \\ + \omega ff'\left(\omega - f^2\right)\left(g' + h'\right) + 6g^2h^2h'g'f - \\ - fg'h'\left[\left(\omega - g^2\right)h^2 + \left(\omega - h^2\right)g^2\right] \end{array}\right\}. \tag{2.6}$$

Secondly, by considering (2.5) for $G_2 = \frac{g}{\sqrt{\omega}}$ we get

$$\Delta G_2 = \frac{1}{\omega^{\frac{7}{2}}}\left\{\begin{array}{l} g\left(f^2 - \omega\right)\left[4\left(f'\right)^2 f^2 - \omega\left(\left(f'\right)^2 + f''f\right)\right] + \\ + \left(\omega^2 - g^2\right)^2\left(4g\left(g'\right)^2 - \omega g''\right) + \\ + g\left(h^2 - \omega\right)\left[4h^2\left(h'\right)^2 - \omega\left(\left(h'\right)^2 + hh''\right) + \right] \\ + \left(2g^2 - \omega\right)4gg'\left(f'f^2 + h^2h'\right) + \\ + gg'\omega\left(\omega - g^2\right)\left(f' + h'\right) + 6f^2h^2gh'f' - \\ - f'h'g\left(h^2\left(\omega - f^2\right) + f^2\left(\omega - h^2\right)\right) \end{array}\right\}. \tag{2.7}$$

Then, by considering (2.5) for $G_3 = \frac{h}{\sqrt{\omega}}$ we obtain

$$\Delta G_3 = \frac{1}{\omega^{\frac{7}{2}}}\left\{\begin{array}{l} h\left(f^2 - \omega\right)\left[4\left(f'\right)^2 f^2 - \omega\left(\left(f'\right)^2 + f''f\right)\right] + \\ + h\left(g^2 - \omega\right)\left[4g^2\left(g'\right)^2 - \omega\left(\left(g'\right)^2 + gg''\right)\right] + \\ + \left(\omega - h^2\right)^2\left(4h\left(h'\right)^2 - \omega h''\right) + \\ + 4hh'\left(2h^2 - \omega\right)\left(g'g^2 + f'f^2\right) + \\ + \omega hh'\left(\omega - h^2\right)\left(f' + g'\right) + 6f^2g^2hf'g' - \\ - hf'g'\left[\left(\omega - f^2\right)g^2 + \left(\omega - g^2\right)f^2\right] \end{array}\right\}. \tag{2.8}$$

Finally, by considering (2.5) for $G_4 = \frac{-1}{\sqrt{\omega}}$ we get

$$\Delta G_4 = \frac{1}{\omega^{\frac{7}{2}}} \left\{ \begin{array}{l} (f^2-\omega)\,[(f')^2\,(\omega-f^2)+ff''\omega-3f^2\,(f')^2- \\ -f'\,(g^2g'+h^2h')]+(g^2-\omega)\,[(g')^2\,(\omega-g^2)+ \\ +gg''\omega-3g^2\,(g')^2-g'\,(f^2f'+h^2h')]+ \\ +(h^2-\omega)\,[(h')^2\,(\omega-h^2)+hh''\omega-3h^2\,(h')^2- \\ -h'\,(f^2f'+g^2g')]- \\ -6\,(f^2g^2f'g'+f^2h^2f'h'+g^2h^2g'h') \end{array} \right\}. \tag{2.9}$$

So, the left-hand side of condition (1.2), which we are particularly interested in, is obtained for translation hypersurfaces.

3. Main result

Now, let us consider the translation hypersurface M which satisfies the condition $\Delta G = AG$, for a 4×4—real matrix A.

Theorem 1. *A translation hypersurface in Euclidean* 4*-space whose Gauss map satisfies (1.2) is one of the following hypersurfaces:*

(1) *the hypersurface which consists of translation surface and a constant vector along this surface,*
(2) *the hyperplane,*
(3) *the hypersurface $\Sigma\times\mathbb{R}$, where Σ is a translation surface.*

Proof. By (1.2), we can write ΔG in the matrix form

$$\begin{bmatrix}\Delta G_1\\ \Delta G_2\\ \Delta G_3\\ \Delta G_4\end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44}\end{bmatrix} \begin{bmatrix} G_1\\ G_2\\ G_3\\ G_4\end{bmatrix}. \tag{3.1}$$

When the component G_i of the Gauss map are substituted in (3.1), ΔG_1 is obtained as

$$\Delta G_1 = a_{11}\frac{f}{\sqrt{\omega}} + a_{12}\frac{g}{\sqrt{\omega}} + a_{13}\frac{h}{\sqrt{\omega}} - a_{14}\frac{1}{\sqrt{\omega}}. \tag{3.2}$$

If we substitute (2.6) into (3.2) and then multiply both sides by $\sqrt{\omega}$, we find that

$$\begin{aligned}
&(\omega-f^2)^2\left(4f\,(f')^2-\omega f''\right)+f\,(g^2-\omega)\left[4g^2\,(g')^2-\omega\left((g')^2+gg''\right)\right]+\\
&+f\,(h^2-\omega)\left[4h^2\,(h')^2-\omega\left((h')^2+hh''\right)\right]+4ff'\,(2f^2-\omega)\,(g^2g'+h^2h')+\\
&+\omega ff'\,(\omega-f^2)\,(g'+h')+6g^2h^2h'g'f-fg'h'\left[(\omega-g^2)\,h^2+(\omega-h^2)\,g^2\right]\\
&=\omega^3\,(a_{11}f+a_{12}g+a_{13}h-a_{14}).
\end{aligned} \tag{3.3}$$

Similarly, substitution of (2.7) into (3.1) and then multiplying both sides by $\sqrt{\omega}$, results in

$$\begin{aligned}&(\omega-g^2)^2\left(4g\left(g'\right)^2-\omega g''\right)+g\left(f^2-\omega\right)\left[4f^2\left(f'\right)^2-\omega\left(\left(f'\right)^2+ff''\right)\right]+\\&+g\left(h^2-\omega\right)\left[4h^2\left(h'\right)^2-\omega\left(\left(h'\right)^2+hh''\right)\right]+4gg'\left(2g^2-\omega\right)\left(f'f^2+h'h^2\right)+\\&+gg'\omega\left(\omega-g^2\right)\left(f'+h'\right)+6f^2h^2gh'f'-f'h'g\left[h^2\left(\omega-f^2\right)+f^2\left(\omega-h^2\right)\right]\\&=\omega^3\left(a_{21}f+a_{22}g+a_{23}h-a_{24}\right).\end{aligned}\tag{3.4}$$

According to the matrix form in (3.1), substitution of (2.8), (2.9) into (3.1) and then multiplying both sides by $\sqrt{\omega}$, we get

$$\begin{aligned}&(\omega-h^2)^2\left(4h\left(h'\right)^2-\omega h''\right)+h\left(f^2-\omega\right)\left[4f^2\left(f'\right)^2-\omega\left(\left(f'\right)^2+ff''\right)\right]+\\&+h\left(g^2-\omega\right)\left[4g^2\left(g'\right)^2-\omega\left(\left(g'\right)^2+gg''\right)\right]+4hh'\left(2h^2-\omega\right)\left(g'g^2+f'f^2\right)+\\&+\omega hh'\left(\omega-h^2\right)\left(f'+g'\right)+6f^2g^2hf'g'-hf'g'\left[\left(\omega-f^2\right)g^2+\left(\omega-g^2\right)f^2\right]\\&=\omega^3\left(a_{31}f+a_{32}g+a_{33}h-a_{34}\right),\end{aligned}\tag{3.5}$$

and

$$\begin{aligned}&\left(f^2-\omega\right)\left[\left(f'\right)^2\left(\omega-f^2\right)+ff''\omega-3f^2\left(f'\right)^2-f'\left(g'g^2+h'h^2\right)\right]+\\&+\left(g^2-\omega\right)\left[\left(g'\right)^2\left(\omega-g^2\right)+gg''\omega-3g^2\left(g'\right)^2-g'\left(f'f^2+h'h^2\right)\right]+\\&+\left(h^2-\omega\right)\left[\left(h'\right)^2\left(\omega-h^2\right)+hh''\omega-3h^2\left(h'\right)^2-h'\left(f'f^2+g'g^2\right)\right]-\\&-6\left(f^2g^2f'g'+f^2h^2f'h'+g^2h^2h'g'\right)\\&=\omega^3\left(a_{41}f+a_{42}g+a_{43}h-a_{44}\right),\end{aligned}\tag{3.6}$$

respectively.

Rearranging (3.6) so that the left-hand side of the equation is $\omega\left(f^2-\omega\right)ff''$, results in

$$\begin{aligned}\omega\left(f^2-\omega\right)ff''=&-\left(f^2-\omega\right)\left(\omega-f^2\right)\left(f'\right)^2+\left(f^2-\omega\right)3f^2\left(f'\right)^2+\\&+\left(f^2-\omega\right)f'\left(g^2g'+h^2h'\right)-\\&-\left(g^2-\omega\right)[\left(g'\right)^2\left(\omega-g^2\right)+gg''\omega-3g^2\left(g'\right)^2-\\&-g'\left(f^2f'+h^2h'\right)]-\\&-\left(h^2-\omega\right)[\left(h'\right)^2\left(\omega-h^2\right)+hh''\omega-3h^2\left(h'\right)^2-\\&-h'\left(f^2f'+g^2g'\right)]+\end{aligned}\tag{3.7}$$

$$+6\left(f^2g^2f'g'+f^2h^2f'h'+g^2h^2h'g'\right)+$$
$$+\omega^3\left(a_{41}f+a_{42}g+a_{43}h-a_{44}\right).$$

Then we arrange (3.3) so that the left-hand side of the equation is $-\omega\left(\omega-f^2\right)^2f''$. After we multiply (3.3) by f and multiply (3.7) by $\left(\omega-f^2\right)$, we get

$$A_1\left(f'\right)^2+B_1f'=\Gamma_1, \tag{3.8}$$

where

$$\begin{aligned}
&A_1=\omega\left(\omega-f^2\right)^2, \qquad (3.9)\\
&B_1=\left(g^2g'+h^2h'\right)\left[\omega\left(3f^2-1\right)+f^2\left(f^2+1\right)\right],\\
&\Gamma_1=\omega\left(\omega-g^2\right)\left[-\left(\left(g'\right)^2\omega+gg''\omega\right)+\left(4g^2\left(g'\right)^2+h^2h'g'\right)\right]+\\
&\quad+\omega\left(\omega-h^2\right)\left[-\left(\left(h'\right)^2\omega+hh''\omega\right)+\left(4h^2\left(h'\right)^2+h'g^2g'\right)\right]-\\
&\quad-6\omega g^2h^2g'h'+f\omega^3\left(a_{11}f+a_{12}g+a_{13}h-a_{14}\right)-\\
&\quad-\left(\omega-f^2\right)\omega^3\left(a_{41}f+a_{42}g+a_{43}h-a_{44}\right).
\end{aligned}$$

Similarly, using equation (3.4) and (3.7) multiplied by $-g$, we get

$$A_2\left(f'\right)^2+B_2f'=\Gamma_2, \tag{3.10}$$

where

$$\begin{aligned}
&A_2=0, \qquad (3.11)\\
&B_2=\omega gg'\left(\omega-3f^2\right),\\
&\Gamma_2=\left(\omega-g^2\right)\left(g''\omega^2-g'\omega h'-3\omega g\left(g'\right)^2\right)-3gg'h^2h'\omega+\\
&\quad+g\omega^3\left(a_{41}f+a_{42}g+a_{43}h-a_{44}\right)+\\
&\quad+\omega^3\left(a_{21}f+a_{22}g+a_{23}h-a_{24}\right).
\end{aligned}$$

Using equation (3.5) and (3.7) multiplied by $-h$, we obtain

$$A_3\left(f'\right)^2+B_3f'=\Gamma_3, \tag{3.12}$$

where

$$\begin{aligned}
&A_3=0, \qquad (3.13)\\
&B_3=\omega hh'\left(4f^2-\omega\right)+h^2f^2\left(h^2-\omega\right)+h^3h'f^2,\\
&\Gamma_3=\left(\omega-h^2\right)\left(3h\left(h'\right)^2\omega-\omega^2h''\right)+\omega hh'g'\left(\omega-3g^2\right)-\\
&\quad-\omega^3\left(a_{31}f+a_{32}g+a_{33}h-a_{34}\right)-\\
&\quad-h\omega^3\left(a_{41}f+a_{42}g+a_{43}h-a_{44}\right).
\end{aligned}$$

So we have the system of equations

$$\begin{aligned} A_1 \left(f'\right)^2 + B_1 f' &= \Gamma_1, \\ A_2 \left(f'\right)^2 + B_2 f' &= \Gamma_2, \\ A_3 \left(f'\right)^2 + B_3 f' &= \Gamma_3, \end{aligned} \tag{3.14}$$

with matrices

$$D = \begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \\ A_3 & B_3 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ 0 & B_2 \\ 0 & B_3 \end{bmatrix} \tag{3.15}$$

and

$$E = \begin{bmatrix} A_1 & B_1 & \Gamma_1 \\ A_2 & B_2 & \Gamma_2 \\ A_3 & B_3 & \Gamma_3 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 & \Gamma_1 \\ 0 & B_2 & \Gamma_2 \\ 0 & B_3 & \Gamma_3 \end{bmatrix}. \tag{3.16}$$

For the solution of system (3.14), the rank of E must be equal to the rank of D. So $rankE \neq 3$ hence $B_2\Gamma_3 = \Gamma_2 B_3$. Then, in the case of $rankE = rankD = 1$ or $rankE = rankD = 2$, the solution of (3.14) should be examined.

CASE 1: Let $rankE = rankD = 1$. Then the determinants of $(2\times 2)-$ matrices of E vanish. These determinants are

(1) $\begin{vmatrix} A_1 & B_1 \\ 0 & B_2 \end{vmatrix} = \omega^2 g g' \left(\omega - f^2\right)^2 \left(\omega - 3f^2\right);$

(2) $\begin{vmatrix} A_1 & \Gamma_1 \\ 0 & \Gamma_2 \end{vmatrix} = \omega \left(\omega - f^2\right)^2 \Big\{\left(\omega - g^2\right)\left(g''\omega^2 - g'\omega h' - 3\omega g \left(g'\right)^2\right) -$
$- 3gg'h^2h'\omega + g\omega^3 \left(a_{41} f + a_{42} g + a_{43} h - a_{44}\right) +$
$+ \omega^3 \left(a_{21} f + a_{22} g + a_{23} h - a_{24}\right)\Big\};$

(3) $\begin{vmatrix} A_1 & B_1 \\ 0 & B_3 \end{vmatrix} = \omega \left(\omega - f^2\right)^2 \left[\omega h h' \left(4f^2 - \omega\right) + h^2 f^2 \left(h^2 - \omega\right) + h^3 h' f^2\right];$

(4) $\begin{vmatrix} A_1 & \Gamma_1 \\ 0 & \Gamma_3 \end{vmatrix} = \omega \left(\omega - f^2\right)^2 \Big\{\left(\omega - h^2\right)\left(3h \left(h'\right)^2 \omega - \omega^2 h''\right) +$
$+ \omega h h' g' \left(\omega - 3g^2\right) - \omega^3 \left(a_{31} f + a_{32} g + a_{33} h - a_{34}\right) -$
$- h\omega^3 \left(a_{41} f + a_{42} g + a_{43} h - a_{44}\right)\Big\};$

(5) $\begin{vmatrix} B_1 & \Gamma_1 \\ B_2 & \Gamma_2 \end{vmatrix} = \left(g^2 g' + h^2 h'\right)\left[\omega \left(3f^2 - 1\right) + f^2 \left(f^2 + 1\right)\right] \times$

$$\times\left\{\left(\omega-g^2\right)\left(g''\omega^2-g'\omega h'-3\omega g\left(g'\right)^2\right)-3gg'h^2h'\omega+\right.$$

$$+g\omega^3\left(a_{41}f+a_{42}g+a_{43}h-a_{44}\right)+$$

$$\left.+\omega^3\left(a_{21}f+a_{22}g+a_{23}h-a_{24}\right)\right\}-$$

$$-\omega gg'\left(\omega-3f^2\right)\left\{\omega\left(\omega-g^2\right)\left[-\left(\left(g'\right)^2\omega+gg''\omega\right)+\right.\right.$$

$$+\left(4g^2(g')^2+h^2h'g'\right)]+\omega\left(\omega-h^2\right)\left[-\left(\left(h'\right)^2\omega+hh''\omega\right)+\right.$$

$$+\left(4h^2\left(h'\right)^2+h'g^2g'\right)]-6\omega g^2h^2g'h'+$$

$$+f\omega^3\left(a_{11}f+a_{12}g+a_{13}h-a_{14}\right)-$$

$$\left.-\left(\omega-f^2\right)\omega^3\left(a_{41}f+a_{42}g+a_{43}h-a_{44}\right)\right\};$$

$$(6)\quad \begin{vmatrix} B_2 & \Gamma_2 \\ B_3 & \Gamma_3 \end{vmatrix}=\omega gg'\left(\omega-3f^2\right)\left\{\left(\omega-h^2\right)\left(3h\left(h'\right)^2\omega-\omega^2h''\right)+\right.$$

$$+\omega hh'g'\left(\omega-3g^2\right)-\omega^3\left(a_{31}f+a_{32}g+a_{33}h-a_{34}\right)-$$

$$\left.-h\omega^3\left(a_{41}f+a_{42}g+a_{43}h-a_{44}\right)\right\}-$$

$$-\left\{\omega hh'\left(4f^2-\omega\right)+h^2f^2\left(h^2-\omega+hh'\right)\right\}\times$$

$$\times\left\{\left(\omega-g^2\right)\left(g''\omega^2-g'\omega h'-3\omega g\left(g'\right)^2\right)-3gg'h^2h'\omega+\right.$$

$$+g\omega^3\left(a_{41}f+a_{42}g+a_{43}h-a_{44}\right)+$$

$$\left.+\omega^3\left(a_{21}f+a_{22}g+a_{23}h-a_{24}\right)\right\}.$$

Also, we have $A_1B_2=0$ and $A_1B_3=0$ for $rankD=1$. Since $rankE=rankD=1$, all determinants of $(2\times2)-$ matrices of E are zero. When f is isolated by using the results of determinants, it is written by depending on the functions g,g',g'',h,h',h''. Since the functions g and h are independent from the parameter u of the function f, the derivative of f is zero. So, f is a constant.

CASE 2: Let $rankE=rankD=2$. Since $rankD=2$, we have

$$(1)\quad \begin{vmatrix} A_1 & B_1 \\ 0 & B_2 \end{vmatrix}=\omega^2gg'\left(\omega-f^2\right)^2\left(\omega-3f^2\right);$$

$$(2)\quad \begin{vmatrix} A_1 & B_1 \\ 0 & B_3 \end{vmatrix}=\omega\left(\omega-f^2\right)^2\left[\omega hh'\left(4f^2-\omega\right)+h^2f^2\left(h^2-\omega\right)+h^3h'f^2\right].$$

At least one of the products A_1B_2 and A_1B_3 is different from zero. Then, at least one of the determinants of $(2\times 2)-$matrices of E is different from zero and so, $rank E = 2$. For the system

$$\begin{aligned} A_1\left(f'\right)^2 + B_1 f' &= \Gamma_1, \\ 0 + B_2 f' &= \Gamma_2, \end{aligned}$$

we have

$$f' = \frac{\Gamma_2}{B_2} \text{ and } \left(f'\right)^2 = \frac{\Gamma_1 B_2 - B_1 \Gamma_2}{A_1 B_2},$$

so

$$B_2\left(\Gamma_1 B_2 - \Gamma_2 B_1\right) = A_1 \Gamma_2^2. \tag{3.17}$$

When we substitute (3.9) and (3.11) into (3.17) and isolate function f, we obtain a polynomial whose coefficients depend on the functions g, g', g'', h, h', h''. Since the functions g and h are independent from the parameter u of function f, the derivative of f is zero. So, f is a constant.

By considering Case 1 and 2, we have two cases: $f = 0$ or $f = c \neq 0$ where c is a constant.

If $f = 0$ then $\tilde{f}$ is a constant. So, this hypersurface has the form

$$x(u,v,z) = \left(u, v, z, c + \tilde{g}(v) + \tilde{h}(z)\right) \tag{3.18}$$

and it consists of a translation surface and a constant vector along it.

If $f = c \neq 0$, c is constant then $\tilde{f} = cu$ and we have

$$x(u,v,z) = \left(u, v, z, cu + \tilde{g}(v) + \tilde{h}(z)\right). \tag{3.19}$$

Thus from [2], we conclude that M is a hypersurface $\sum \times \mathbb{R}$ or a hyperplane. Here $\sum$ is a translation surface.

After that, similar conclusions, which we give for the function f, could be found for the functions g, h. So the desired result is obtained. □

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