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UNITARINESS OF OPERATORS

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Abstract. In this paper, we explain some sufficient conditions for unitariness of Toeplitz operators and little Hankel operators on the Bergman space.

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1. Introduction

The Bergman space is the Hilbert space of all holomorphic functions f on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, denoted as $A^2(\mathbb{D})$ for which

$$\|f\|_{A^2(\mathbb{D})} = \left(\int \left(|f(z)|^2 dA(z) \right) \right)^{\frac{1}{2}} < \infty,$$

where $dA(z)$ is the normalized Lebesgue area measure on the open unit disk $\mathbb{D}$. If $h(z) = \sum_{n=0}^{\infty} a_n z^n$ and $k(z) = \sum_{n=0}^{\infty} b_n z^n$ are two functions in $A^2(\mathbb{D})$, then the inner product of h and k is given by

$$\langle h, k \rangle = \int_{\mathbb{D}} h(z)\overline{k(z)} dA(z) = \sum_{n=0}^{\infty} \frac{a_n \overline{b_n}}{n+1} .$$

The Bergman reproducing kernel is the function $K_z \in A^2(\mathbb{D})$ for $z \in \mathbb{D}$ such that $f(z)$ = $\langle f, K_z \rangle$ for all $f \in A^2(\mathbb{D})$ and normalized reproducing kernel k_z is the function $\frac{K_z}{\|K_z\|_2}$. Here the norm $\|.\|_2$ and the inner product $\langle .,. \rangle$ are taken in the space $L^2(\mathbb{D}, dA)$. For any integer, $n \geq 0$, let $e_n(z) = \sqrt{n+1} z^n$. Then, $\{e_n\}_{n=0}^{\infty}$ forms an orthonormal basis for $A^2(\mathbb{D})$. The Toeplitz operator T_ϕ with symbol $\phi \in L^\infty(\mathbb{D})$ on $A^2(\mathbb{D})$ is defined by $T_\phi f = P(\phi f)$; here P is an orthogonal projection from $L^2(\mathbb{D}, dA)$ onto $A^2(\mathbb{D})$.

Let $\overline{A^2(\mathbb{D})}$ be the space of conjugate analytic functions in $L^2(\mathbb{D}, dA)$. Then $\overline{A^2(\mathbb{D})} = \{\overline{g} : g \in A^2(\mathbb{D})\}$ is closed in $L^2(\mathbb{D}, dA)$. Let $\phi \in L^\infty(\mathbb{D})$, the little Hankel operator $h_\phi : A^2(\mathbb{D}) \to A^2(\mathbb{D})$ be defined by $h_\phi f = \overline{P}(\phi f), f \in A^2(\mathbb{D})$ where $\overline{P}$ is the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto $\overline{A^2(\mathbb{D})}$. There are also numerous equivalent ways of defining little Hankel operators on $A^2(\mathbb{D})$. For illustration, define the map $S_\phi : A^2(\mathbb{D}) \to A^2(\mathbb{D})$ by $S_\phi f = PJ(\phi f)$, where J is selfadjoint, unitary mapping from $L^2(\mathbb{D}, dA)$ into itself given by $Jh(z) = h(\bar{z})$. Observe that, $JS_\phi = h_\phi$. So S_ϕ is unitarily equivalent to h_ϕ.

Let $Aut(\mathbb{D})$ be the Lie group of all automorphisms (biholomorphic mappings) of $\mathbb{D}$. We can define for each $a \in \mathbb{D}$, an automorphism ϕ_a in $Aut(\mathbb{D})$ such that

(i) $(\phi_a \ o \ \phi_a)(z) \equiv z$;
(ii) $\phi_a(0) = a, \phi_a(a) = 0$;
(iii) ϕ_a has a unique fixed point in $\mathbb{D}$.

In fact, $\phi_a(z) = \frac{a-z}{1-\bar{a}z}$ for all a and z in $\mathbb{D}$. It is easy to verify that the derivative of ϕ_a at z is equal to $-k_a(z)$. It implies the real Jacobian determinant of ϕ_a at z is

$$J_{\phi_a(z)} = |k_a(z)|^2 = \frac{\left(1-|a|^2\right)^2}{|1-\bar{a}z|^4}.$$

Given $a \in \mathbb{D}$ and f (any measurable function on $\mathbb{D}$), let us define a function $U_a f$ on $\mathbb{D}$ by $U_a f(w) = k_a(w) f(\phi_a(w))$. Notice that U_a is a bounded linear operator on $L^2(\mathbb{D}, dA)$ and $A^2(\mathbb{D})$ for all $a \in \mathbb{D}$. Further, it can be verified that $U_a^2 = I$, the identity operator, $U_a^* = U_a, U_a(A^2(\mathbb{D})) \subset A^2(\mathbb{D})$ and $U_a((A^2(\mathbb{D}))^\perp) \subset (A^2(\mathbb{D}))^\perp$ for all $a \in \mathbb{D}$. Thus $U_a P = P U_a$ for all $a \in \mathbb{D}$ (see [11]).

Let $H^\infty(\mathbb{D})$ denote the space of bounded analytic functions on $\mathbb{D}$. Let $\mathcal{L}(H)$ denote the algebra of bounded, linear operators from a Hilbert space H into itself and let $T \in \mathcal{L}(H)$. Then the Berezin transform of T is denoted by $\widetilde{T}$, a complex valued function on $\mathbb{D}$ defined by $\widetilde{T(z)} = \langle Tk_z, k_z \rangle$. Let $ker(T)$ denotes kernel of T and an operator T is normaloid if, $\|T\| = sup\{|\langle Tx, x\rangle|; \|x\| = 1\}$. An operator T is paranormal if, $\|Tx\|^2 \leq \|T^2 x\| \|x\|$ and p-hyponormal if $\|T\|^{2p} \geq \|T^*\|^{2p}$ for $0 < p \leq 1$.

In this article, we establish some sufficient conditions for Toeplitz operators and little Hankel operators on the Bergman space $A^2(\mathbb{D})$ to be unitary and average of unitaries.

2. RESULTS

2.1. *Preliminary considerations*

In this section we introduce the basic concepts and known results used in the course of our investigation. Our presentation is on the Bergman space. For more details, refer [1, 6, 8, 12].

Corollary 1 ([6]). *Let $T, S \in \mathbb{B}(H)$ with T invertible. If $T^{-1}S = ST^*$ with $0 \notin cl(W(S))$ and if T is hyponormal, then T is unitary.*

Proposition 1 ([12]). *For an invertible operator $T \in \mathbb{B}(H)$, the following are equivalent: (1) $T\Theta T^{-1}$ and $\|T\| \leq 1$ (2) T is unitary.*

Corollary 2 ([1]). *Let $T_1 \in \mathcal{L}(H)$ be injective $w-$ hyponormal operator and $T_2 \in \mathcal{L}(H)$ be an isometry. Assume that there exists quasiaffinities X and Y such that $T_1X = XT_2$ and $YT_1 = T_2Y$, if either X or Y is compact, then T_1 and T_2 are unitary operators and unitarily equivalent.*

Theorem 1 ([8]). *If a normal operator N_1 is a quasi-affine transform of a normal operator N_2, then N_1 is unitarily equivalent to N_2.*

Moreover, the present study has certain advanced applications which are related to the outcomes of the recent results of Padhy *et al.* [3] and Srivastava *et al.* [14].

3. Main Result

3.1. *Toeplitz operators*

In this section, we present some sufficient conditions for Toeplitz operators on $A^2(\mathbb{D})$ to be unitary.

Theorem 2. *Let ϕ be an essentially bounded Lebesgue measurable function on $\mathbb{D}$ such that $\|\phi\|_\infty \leq 1$. Then T_ψ can be represented as an average of n unitary operators, where $\psi = (1-\frac{1}{n})\phi + \frac{1}{n}$, $n > 2$ and $n \in \mathbb{N}$.*

Proof. Since $\|\phi\|_\infty \leq 1$,
$$\|I - T_{\frac{1+\phi}{2}}\| = \|I - \frac{I}{2} - \frac{T_\phi}{2}\| = \|\frac{I}{2} - \frac{T_\phi}{2}\| \leq \frac{1}{2} + \frac{1}{2}\|T_\phi\| \leq \frac{1+\|\phi_\infty\|}{2} < 1.$$
Therefore, $T_{\frac{1+\phi}{2}}$ is invertible. Let $T_{\frac{1+\phi}{2}}$ has a polar decomposition VP, where P is a positive operator and V is a unitary operator. Therefore, $T_{\frac{1+\phi}{2}} = \frac{V(P_1+P_2)}{2}$, where $P_1 = P + i\sqrt{I-P^2}$ and $P_2 = P - i\sqrt{I-P^2}$ are unitary operators. So, $T_{1+\phi} = VP_1 + VP_2$. Similarly there are unitary operators $V_1, V_2, ..., V_{n-1}$ and $W_1, W_2, ..., W_{n-1}(= V_n)$ such that
$$\begin{aligned} T_{1+\phi} &= V_1 + W_1 \\ T_{1+2\phi} &= V_1 + W_1 + T_\phi. \end{aligned}$$
Therefore,
$$\begin{aligned} T_{1+(n-1)\phi} &= V_1 + W_1 + T_{(n-2)\phi} \\ &= V_1 + V_2 + W_2 + T_{(n-3)\phi} \\ &= \ldots \\ &= V_1 + V_2 + \ldots + (V_n = W_{n-1}). \end{aligned}$$
Then,
$$T_\psi = T_{(1-\frac{1}{n})\phi+\frac{1}{n}} = \frac{1}{n} T_{1+(n-1)\phi} = \frac{1}{n}(V_1 + V_2 + \ldots + V_n).$$
Hence the result holds. □

Theorem 3. *Let $\phi \in L^\infty(\mathbb{D})$ and $\phi \geq 0$. If $T_\phi T_{\phi-1} = 0$, then T_ψ is unitary where $\psi = 2(\phi \circ \phi_a) - 1$.*

Proof. Since $\phi \geq 0, T_\phi \geq 0$. Then T_ϕ is self-adjoint. Further, Since $T_\phi T_{\phi-1} = 0$, that implies $T_\phi(T_\phi - I) = 0$ or $T_\phi^2 = T_\phi$. It is well known that $U_a T_\phi U_a = T_{\phi\circ\phi_a}$ for all $a \in \mathbb{D}$. Now

$$\begin{aligned}
T_\psi^* T_\psi &= T^*_{2(\phi\circ\phi_a)-1} T_{2(\phi\circ\phi_a)-1} \\
&= (2T^*_{\phi\circ\phi_a} - I)(2T_{\phi\circ\phi_a} - I) \\
&= (2(U_a^* T_\phi^* U_a^*) - I)(2(U_a T_\phi U_a) - I) \\
&= 4(U_a^* T_\phi^* U_a^* U_a T_\phi U_a) - 2U_a^* T_\phi^* U_a^* - 2U_a T_\phi U_a + I \\
&= 4U_a T_\phi^2 U_a - 4U_a T_\phi U_a + I \qquad (\because U_a^* = U_a\ ,\ T_\phi^* = T_\phi \text{ and } U_a^2 = I) \\
&= 4U_a T_\phi U_a - 4U_a T_\phi U_a + I \qquad (\because T_\phi^2 = T_\phi) \\
&= I.
\end{aligned}$$

Similarly, one can verify that $T_\psi T_\psi^* = I$. Hence T_ψ is unitary. □

Theorem 4. *Let ϕ be an essentially bounded Lebesgue measurable function on $\mathbb{D}$. If T_ϕ is an isometry and $T_{\bar{\phi}\circ\phi_a}$ is hyponormal for $a \in \mathbb{D}$. Then T_ϕ is unitary.*

Proof. For all $a \in \mathbb{D}$, it is known that

$$T_\phi U_a = U_a T_{\phi\circ\phi_a}. \tag{3.1}$$

Now,

$$U_a T^*_{\phi\circ\phi_a} = U_a (U_a T_\phi U_a)^* = U_a U_a^* T_\phi^* U_a^* = T_\phi^* U_a.$$

Multiplying $T_{\bar{\phi}}$ on the left of equation (3.1), we obtain

$$T_{\bar{\phi}} T_\phi U_a = T_{\bar{\phi}} U_a T_{\phi\circ\phi_a}. \tag{3.2}$$

Since, T_ϕ is an isometry and $T_{\bar{\phi}} U_a = U_a T_{\bar{\phi}\circ\phi_a}$, equation (3.2) turns out to be

$$U_a = T_{\bar{\phi}} U_a T_{\phi\circ\phi_a} = U_a T_{\bar{\phi}\circ\phi_a} T_{\phi\circ\phi_a}.$$

Thus, $T_{\bar{\phi}\circ\phi_a} T_{\phi\circ\phi_a} = I$. Further, since $T_{\bar{\phi}\circ\phi_a}$ is hyponormal and $T_{\bar{\phi}\circ\phi_a} T_{\phi\circ\phi_a} = I$. This implies, $T_{\bar{\phi}\circ\phi_a} T_{\phi\circ\phi_a} = T_{\phi\circ\phi_a} T_{\bar{\phi}\circ\phi_a} = I$. Hence, T_ϕ is unitary. □

Theorem 5. *Let $\phi \in L^\infty(\mathbb{D})$. If $T^{*^2}_{\phi\circ\phi_a} T^2_{\phi\circ\phi_a} - 2T^*_{\phi\circ\phi_a} T_{\phi\circ\phi_a} + I = 0$ and $\widetilde{T_\phi^2}(z) = 1$ then T_ϕ is unitary.*

Proof. It is well known that $U_a T_\phi U_a = T_{\phi\circ\phi_a}$. Given that $\widetilde{T_\phi^2}(z) = 1$. Then $T_\phi^2 = I$ if and only if $T_{\bar{\phi}}^2 = I$. Again $T^{*^2}_{\phi\circ\phi_a} T^2_{\phi\circ\phi_a} - 2T^*_{\phi\circ\phi_a} T_{\phi\circ\phi_a} + I = 0$ gives $U_a T_\phi^{*^2} T_\phi U_a - 2U_a T_\phi^* T_\phi U_a + I = 0$. Thus, $2 - 2U_a T_\phi^* T_\phi U_a = 0$ or $U_a T_\phi^* T_\phi U_a = I$ by using $T_\phi^2 = I$ and $T_\phi^{*^2} = I$. Hence T_ϕ is isometry and invertible. Therefore T_ϕ is unitary. □

Theorem 6. *If ϕ is a bounded harmonic function on $\mathbb{D}$ with $\|\phi\|_\infty = 1$ and $\widetilde{T_\phi^n}(z) = 1$ then T_ϕ is unitary.*

Proof. Since ϕ is a bounded harmonic function on $\mathbb{D}$ then it follows from (Theorem 5, [4]) that $\|T_\phi\| = \|\phi\|_\infty = 1$. Now $\widetilde{T_\phi^n}(z) = 1$ that implies $\langle T_\phi^n k_z, k_z\rangle = 1 = \langle k_z, k_z\rangle$, $\forall z \in \mathbb{D}$. Thus, $T_\phi^n = I$. So T_ϕ is invertible. Then T_ϕ and T_ϕ^{-1} are power bounded and therefore similar to unitary operator (Proposition 3.8 and Corollary 1.16, [10]). Hence $\|T_\phi\| = 1$ implies that T_ϕ is unitarily equivalent to unitary operator. Therefore T_ϕ is unitary. $\square$

Theorem 7. *Let $\phi, \phi_1, \phi_2 \in L^\infty(\mathbb{D})$ and $\|\phi\|_\infty \leq 1$. Then T_ϕ can be expressed as sum of three unitary operators.*

Proof. Let $T_\phi = T_{\phi_1} + iT_{\phi_2}$ be the Cartesian decomposition of T_ϕ. Since $\|\phi\|_\infty \leq 1$, $\|T_\phi\| \leq \|\phi\|_\infty \leq 1$. So T_ϕ is a contraction. Thus the real and imaginary parts of T_ϕ are also contractions. Let $T_1 = \frac{1}{2}(T_{\phi_1} - (I - T_{\phi_2}^2)^{\frac{1}{2}})$ and notice that T_1 is a contraction. Therefore one can define the operators $W_1 = T_1 + i(I - T_1^2)^{\frac{1}{2}}$ and $W_2 = (I - T_{\phi_2}^2)^{\frac{1}{2}} + iT_{\phi_2}$. Here $W_1W_1^* = W_1^*W_1 = I$ and $W_2W_2^* = W_2^*W_2 = I$. Hence $T_\phi = W_1 + W_1^* + W_2$, which is sum of three unitary operators. $\square$

Theorem 8. *Let $\phi, \phi_1, \phi_2 \in L^\infty(\mathbb{D})$ and $T_\phi = T_{\phi_1} + iT_{\phi_2}$ be the Cartesian decomposition of T_ϕ. Then $T_{1-\phi}$ is unitary if and only if $T_{\phi_1}T_{\phi_2} = T_{\phi_2}T_{\phi_1}$ and $T_{\phi_1-1}^2 = T_{1-\phi_2}T_{1+\phi_2}$.*

Proof. Suppose $T_{\phi_1}T_{\phi_2} = T_{\phi_2}T_{\phi_1}$. Then $T_\phi T_{\bar{\phi}} = T_{\bar{\phi}}T_\phi = T_{\phi_1}^2 + T_{\phi_2}^2$. Again, since $T_{\phi_1-1}^2 = T_{1-\phi_2}T_{1+\phi_2}$ then, $(T_{\phi_1} - I)^2 = (I - T_{\phi_2})(I + T_{\phi_2})$. So $T_\phi T_{\bar{\phi}} = T_{\bar{\phi}}T_\phi = T_{\phi+\bar{\phi}}$, it can be easily shown that $T_{1-\bar{\phi}}T_{1-\phi} = I = T_{1-\phi}T_{1-\bar{\phi}}$. Conversely, suppose $T_{1-\phi}$ is unitary. Then $T_\phi T_{\bar{\phi}} = T_{\bar{\phi}}T_\phi = T_{\phi+\bar{\phi}}$. Since $T_\phi T_{\bar{\phi}} = T_{\bar{\phi}}T_\phi$, we have $T_{\phi_1}T_{\phi_2} = T_{\phi_2}T_{\phi_1}$. Again, since $T_\phi T_{\bar{\phi}} = T_{\phi+\bar{\phi}}$, therefore $T_{\phi_1-1}^2 = T_{1-\phi_2}T_{1+\phi_2}$. $\square$

Corollary 3. *Let $\phi, \phi_1, \phi_2 \in L^\infty(\mathbb{D})$ where $\phi_1 \geq 0$ and $\phi_2 \geq 0$. If $T_\phi = T_{\phi_1} + iT_{\phi_2}$ be the Cartesian decomposition of T_ϕ with $T_{\phi_1}T_{\phi_2}$ is p-hyponormal and $T_{\phi_1-1}^2 = T_{1-\phi_2}T_{1+\phi_2}$, then $T_{1-\phi}$ is unitary.*

Proof. If $\phi_1 \geq 0$ and $\phi_2 \geq 0$, then T_{ϕ_1}, T_{ϕ_2} are positive. Let $A = T_{\phi_1}T_{\phi_2}$, then $AT_{\phi_1} = T_{\phi_1}A^*$. So from [15], we get $A^*T_{\phi_1} = T_{\phi_1}A$, then $(T_{\phi_1}T_{\phi_2})^*T_{\phi_1} = T_{\phi_1}T_{\phi_1}T_{\phi_2}$. Thus $T_{\overline{\phi_2}}T_{\overline{\phi_1}}T_{\phi_1} = T_{\phi_1}^2T_{\phi_2}$. Therefore $T_{\phi_2}T_{\phi_1}^2 = T_{\phi_1}^2T_{\phi_2}$. As T_{ϕ_1} is positive, so $T_{\phi_1}T_{\phi_2} = T_{\phi_2}T_{\phi_1}$. Hence by Theorem 8, the corollary holds. $\square$

Corollary 4. *Let $\phi, \phi_1 \in L^\infty(\mathbb{D})$ and $T_\phi = T_{\phi_1} + iT_{\overline{\phi_1}}$ be the Cartesian decomposition of T_ϕ. If T_{ϕ_1}, $T_{\overline{\phi_1}}$ are paranormal with $ker(T_{\phi_1}) = ker(T_{\overline{\phi_1}})$ and $T_{\phi_1-1}^2 = T_{1-\overline{\phi_1}}T_{1+\overline{\phi_1}}$. Then $T_{1-\phi}$ is unitary.*

Proof. Since T_{ϕ_1} and $T_{\overline{\phi_1}}$ are paranormal with $ker(T_{\phi_1}) = ker(T_{\overline{\phi_1}})$, then by (Theorem 5, [2]), T_{ϕ_1} is normal. That is $T_\phi T_{\bar{\phi}} = T_{\bar{\phi}}T_\phi$. Therefore by replacing $T_{\phi_2} = T_{\overline{\phi_1}}$ in Theorem 8, the assertion holds. $\square$

Theorem 9. *Let* $\|\phi\|_\infty \leq 1$ *for* $\phi \in L^\infty(\mathbb{D})$ *and* $\|U_a - T_\phi\| < 1$ *for all* $a \in \mathbb{D}$. *If* $\|T_\phi^n\|^{-1} - \|T_\phi^{-n}\| - (|1 - \|T_\phi^n\|^{-1}| + |1 - \|T_\phi^{-n}\||) = 0$ *for all* $n \in \mathbb{N}$. *Then* T_ϕ *is unitary.*

Proof. Since $\|\phi\|_\infty \leq 1$ and $\|T_\phi\| \leq 1$, supposing that $\|U_a - T_\phi\| < 1$, it follows from [13] that T_ϕ is invertible. Then, T_ϕ^n is also invertible for $n \in \mathbb{N}$. As T_ϕ is a contraction, $\|T_\phi^n\|^{-1} \geq 1$. Thus, $1 \leq \|T_\phi^{-n}\|\|T_\phi^n\|$ and therefore, $\|T_\phi^{-n}\| \geq 1$. Since, $\|T_\phi^n\|^{-1} - \|T_\phi^{-n}\| = (|1 - \|T_\phi^n\|^{-1}| + |1 - \|T_\phi^{-n}\||)$, this implies $\|T_\phi^{-n}\| = 1$. Therefore, T_ϕ^n is unitary on $A^2(\mathbb{D})$. Further, assuming that $\|T_\phi\| \leq 1$ and T_ϕ^n is an isometry, T_ϕ is unitary. In particular, if $\|T_\phi\| \leq 1$ then $\|T_\phi^{n-1}\| \leq 1$. This implies that $T_{\bar{\phi}}^{n-1} T_\phi^{n-1} \leq I$. Therefore, $I \geq T_{\bar{\phi}} T_\phi \geq T_{\bar{\phi}}(T_{\bar{\phi}}^{n-1} T_\phi^{n-1}) T_\phi = T_{\bar{\phi}}^n T_\phi^n = I$. Hence, T_ϕ is an isometry. Further, if $\|T_\phi\| \leq 1$ and $T_{\bar{\phi}}^n$ is an isometry, $T_{\bar{\phi}}$ is an isometry. So, $\|T_\phi\| \leq 1$ and T_ϕ^n is unitary, it clearly implies that T_ϕ is a unitary operator. □

Theorem 10. *Let* ϕ *be a positive essentially bounded Lebesgue measurable function with* $\|\phi\|_\infty \leq 1$. *Then* $\mp T_\phi \pm i\sqrt{I - T_\phi^2}$ *are unitary operators on* $A^2(\mathbb{D})$.

Proof. As ϕ is positive, T_ϕ is positive on $A^2(\mathbb{D})$ and since $\|\phi\|_\infty \leq 1$, $\|T_\phi\| \leq 1$. Therefore, $I - T_\phi^2$ is a positive operator on $A^2(\mathbb{D})$. Let's define $U_1 = T_\phi + i\sqrt{I - T_\phi^2}$, $U_2 = T_\phi - i\sqrt{I - T_\phi^2}, U_3 = -T_\phi + i\sqrt{I - T_\phi^2}$ and $U_4 = -T_\phi - i\sqrt{I - T_\phi^2}$. One can easily observe that, $U_1^* = U_2, U_3^* = U_4$. Hence, $U_1^* U_1 = U_1 U_1^* = I, U_2^* U_2 = U_2 U_2^* = I, U_3^* U_3 = U_3 U_3^* = I$ and $U_4^* U_4 = U_4 U_4^* = I$. Therefore, U_1, U_2, U_3 and U_4 are unitary operators on $A^2(\mathbb{D})$. □

Proposition 2. *Let* $\phi \in L^\infty(\mathbb{D})$. *If* $\widetilde{T_\phi^2}(z) = 1$ *then* T_ϕ *is unitarily equivalent with* $T_{\bar{\phi}}$.

Proof. Let $T_\phi = UP$ be the polar decomposition of T where U is partial isometry and P is the positive operator. Since positive operators are self-adjoint, so $P^* = P$. Further since $\widetilde{T_\phi^2}(z) = 1$, $\langle T_\phi^2 k_z, k_z \rangle = 1 = \langle k_z, k_z \rangle$, $\forall z \in \mathbb{D}$. That implies $T_\phi^2 = I$ and therefore T_ϕ is invertible. Hence the partial isometry U will become a unitary operator U_a(say) $\forall a \in \mathbb{D}$. Thus $T_\phi = U_a P$ where $U_a = U_a^* = U_a^{-1}$. Then

$$U_a T_\phi U_a = U_a U_a P U_a = P U_a = T_{\bar{\phi}}.$$

Hence T_ϕ is unitarily equivalent with $T_{\bar{\phi}}$. □

Theorem 11. *Let* $\phi \in L^\infty(\mathbb{D})$ *and* $\phi \geq 0$. *Then* T_ϕ *is unitary if and only if* $\widetilde{T_\phi^2}(z) = 1$.

Proof. Let $\phi \geq 0$, that implies $T_\phi \geq 0$. Since positive operators are self adjoint, $T_\phi^* = T_\phi$. Given $\widetilde{T_\phi^2}(z) = 1$, so we get $T_\phi^2 = I$. Now

$$T_\phi^* T_\phi = T_\phi^2 = I = T_\phi T_\phi^*. \tag{3.3}$$

So T_ϕ is unitary. Conversely suppose T_ϕ is unitary, then $T_\phi^2 = I$. Thus $\langle T_\phi^2 f, g\rangle = \langle f, g\rangle \ \forall f, g \in A^2(\mathbb{D})$. If $f = g = k_z, \ \forall z \in \mathbb{D}$, then $\widetilde{T_\phi^2}(z) = 1$. $\square$

3.2. *Little Hankle operators*

In this section, we discuss some sufficient conditions for little Hankle operators on $A^2(\mathbb{D})$ to be unitary.

Theorem 12. *Let $\phi \in L^\infty(\mathbb{D})$. If S_ϕ is power bounded and bounded below with $\|S_\phi^2\|^2 + 1 = 2\|S_\phi\|^2$ then S_ϕ is unitary.*

Proof. Given that

$$\|S_\phi^2\|^2 + 1 = 2\|S_\phi\|^2. \tag{3.4}$$

We have to show S_ϕ^n will satisfy the equation (3.4). We will prove this by method of induction. Now the above equation is true for $n = 1$. Let us assume that equation (3.4) is true for $n = k$, that is $\|S_\phi^{2k}\|^2 + 1 = 2\|S_\phi^k\|^2$. This implies

$$S_\phi^{*^{2k}} S_\phi^{2k} + I = 2S_\phi^{*^k} S_\phi^k. \tag{3.5}$$

Then we will prove that equation (3.4) is true for $n = k+1$. Now

$$\begin{aligned}
S_\phi^{*^{2(k+1)}} S_\phi^{2(k+1)} + I - 2S_\phi^{*^{k+1}} S_\phi^{k+1} &= S_\phi^{*^2}(S_\phi^{*^{2k}} S_\phi^{2k}) S_\phi^2 + I - 2S_\phi^{*^{k+1}} S_\phi^{k+1} \\
&= S_\phi^{*^2}(2S_\phi^{*^k} S_\phi^k - I) S_\phi^2 + I - 2S_\phi^{*^{k+1}} S_\phi^{k+1} \\
&= 2S_\phi^{*^{k+2}} S_\phi^{k+2} - S_\phi^{*^2} S_\phi^2 + I - 2S_\phi^{*^{k+1}} S_\phi^{k+1} \\
&= 2S_\phi^{*^k}(S_\phi^{*^2} S_\phi^2 - S_\phi^* S_\phi) S_\phi^k - S_\phi^{*^2} S_\phi^2 + I \\
&= 2S_\phi^{*^k}(S_\phi^* S_\phi - I) S_\phi^k - S_\phi^{*^2} S_\phi^2 + I \qquad (by\ (3.4)) \\
&= 2S_\phi^{*^{k+1}} S_\phi^{k+1} - 2S_\phi^{*^k} S_\phi^k - S_\phi^{*^2} S_\phi^2 + I \\
&= 2S_\phi^{*^{k-1}}(S_\phi^{*^2} S_\phi^2 - S_\phi^* S_\phi) S_\phi^{k-1} - S_\phi^{*^2} S_\phi^2 + I \\
&= 2S_\phi^{*^{k-1}}(S_\phi^* S_\phi - I) S_\phi^{k-1} - S_\phi^{*^2} S_\phi^2 + I \\
&= 2S_\phi^{*^k} S_\phi^k - 2S_\phi^{*^{k-1}} S_\phi^{k-1} - S_\phi^{*^2} S_\phi^2 + I \\
&\textit{continuing in this fashion} \\
&\vdots \\
&= 2S_\phi^{*^2} S_\phi^2 - 2S_\phi^* S_\phi - S_\phi^{*^2} S_\phi^2 + I \\
&= S_\phi^{*^2} S_\phi^2 - 2S_\phi^* S_\phi + I \\
&= 0,
\end{aligned}$$

where the second equality follows from (3.5). Hence S_ϕ^n is satisfying equation (3.4) that is, $\|S_\phi^{2n}\|^2+1=2\|S_\phi^n\|^2$. Therefore,

$$S_\phi^{*^{2n}}S_\phi^{2n}-2S_\phi^{*^n}S_\phi^n+I=0. \tag{3.6}$$

Again from equation (3.4),

$$\|S_\phi^2\|^2=2\|S_\phi\|^2-1.$$

So

$$\|S_\phi^4\|^2=2\|S_\phi^2\|^2-1=2(2\|S_\phi\|^2-1)-1=4\|S_\phi\|^2-3.$$

Again,

$$\|S_\phi^8\|^2=2\|S_\phi^4\|^2-1=2(4\|S_\phi\|^2-3)-1=8\|S_\phi\|^2-7.$$

In general, we get

$$\|S_\phi^{2^n}\|^2=2^n\|S_\phi\|^2-(2^n-1) \tag{3.7}$$

for every $n\in\mathbb{N}$. Further, since S_ϕ is power bounded, so $\exists$ a positive real number M such that

$$\|S_\phi^n\|\leq M \quad for \;\; n=1,2,3\dots\;. \tag{3.8}$$

Thus, equation (3.7) and (3.8) will provide $M^2\geq\|S_\phi^{2^n}\|^2=2^n\|S_\phi\|^2-(2^n-1)$. This implies, $\frac{M^2}{2^n}\geq\|S_\phi\|^2-1+2^{-n}\geq 0$. As $n\to\infty$, we get $\|S_\phi\|^2=1$. Therefore,

$$S_\phi^*S_\phi=I. \tag{3.9}$$

Since any isometry is one-to-one, $ker(S_\phi)=\{0\}$. It follows from ([5], Lemma-2.1) that $ker(S_\phi^*)=\{0\}$. It is well known that $\overline{Range(S_\phi)}=ker(S_\phi^*)^\perp=\{0\}^\perp=A^2(\mathbb{D})$. That implies $Range(S_\phi)$ is dense in $A^2(\mathbb{D})$. Since S_ϕ is bounded below and has dance range, then by [7], S_ϕ is invertible and $Range(S_\phi)=A^2(\mathbb{D})$. Let $S_\phi=UP$ be the polar decomposition of S_ϕ where U is the partial isometry and P is the positive operator. Since S_ϕ is invertible, so the partial isometry U can be extended to a unitary operator V(say). Therefore, $S_\phi=VP$. It can be easily shown that $S_\phi=VN$, where N is a normal operator. Hence,

$$S_\phi S_\phi^*=VS_\phi^*S_\phi V^*=I. \tag{3.10}$$

Therefore, S_ϕ is unitary. □

Corollary 5. *Let $\phi\in L^\infty(\mathbb{D})$ and $\|\phi\|_\infty\leq 1$. If $\widetilde{(S_\phi^*S_\phi)}(z)\geq 1$ and S_ϕ is bounded below. Then S_ϕ is unitary.*

Proof. Now for any $f\in A^2(\mathbb{D})$,

$$\|S_\phi f\|\leq\|S_\phi\|\|f\|\leq\|\phi\|_\infty\|f\|\leq\|f\|. \tag{3.11}$$

Thus,

$$\|S_\phi\|^2\leq\|f\|^2\Rightarrow\langle S_\phi f,S_\phi f\rangle\leq\langle f,f\rangle\Rightarrow S_\phi^*S_\phi\leq I.$$

Again, since $\widetilde{(S_\phi^* S_\phi)}(z) \geq 1$, $\langle S_\phi k_z, S_\phi k_z\rangle \geq 1 = \|k_z\|^2$, $\langle k_z, k_z\rangle \leq \langle S_\phi^* S_\phi k_z, k_z\rangle$ and also $I \leq S_\phi^* S_\phi$. Hence, S_ϕ is isometry. Thus, the corollary follows from Theorem 12. $\square$

Corollary 6. *Let $\phi \in L^\infty(\mathbb{D})$. If $\|S_\phi^2\|^2 + 1 = 2\|S_\phi\|^2$ and $\widetilde{S_\phi^2}(z) = 1$. Then S_ϕ is unitary.*

Proof. Given that $\widetilde{S_\phi^2}(z) = 1$. Then $\langle S_\phi^2 k_z, k_z\rangle = 1 = \|k_z\|^2$. That implies $\langle S_\phi^2 k_z, k_z\rangle = \langle k_z, k_z\rangle$. That is $S_\phi^2 = I$. It is easy to verify that $S_\phi^2 = I$ if and only if $S_\phi^{*^2} = I$. Since $\|S_\phi^2\|^2 + 1 - 2\|S_\phi\|^2 = 0$ then $S_\phi^{*^2} S_\phi^2 + I - 2S_\phi^* S_\phi = 0$ and an easy computation demonstrates $2I - 2S_\phi^* S_\phi = 0$ or $I = S_\phi^* S_\phi$. Thus S_ϕ is isometry. Further , Since $S_\phi^2 = I$ then S_ϕ is right invertible as well as left invertible and hence S_ϕ is invertible. Therefore, the corollary follows from Theorem 12. $\square$

Corollary 7. *Let $\phi \in L^\infty(\mathbb{D})$. If S_ϕ is normaloid and $\widetilde{S_\phi^n}(z) = 1$, then S_ϕ is unitary.*

Proof. It is well known that $\|S_\phi^n\| \leq \|S_\phi\|^n$, for $n \in \mathbb{N}$. Since $\widetilde{S_\phi^n}(z) = 1$, we have that $\langle S_\phi^n k_z, k_z\rangle = 1 = \|k_z\|^2 = \langle k_z, k_z\rangle$. That is $S_\phi^n = I$. Again, since S_ϕ is normaloid, then $\|S_\phi^n\| = \|S_\phi\|^n = \|S_\phi\|$. Since $S_\phi^n = I$, then it is obvious that $1 = \|S_\phi^n\| = \|S_\phi\|^n = \|S_\phi\|$. Thus S_ϕ is isometry and $S_\phi^2 = I$. Therefore, the corollary follows from Corollary 6. $\square$

Theorem 13. *Let $\phi \in L^\infty(\mathbb{D})$ and T_ϕ, S_ϕ^* are p-hyponormal operators with $T_\phi X = XS_\phi$ where $X : A^2(\mathbb{D}) \to A^2(\mathbb{D})$ such that $ker(X) = \{0\}$ and $\overline{Range(X)} = A^2(\mathbb{D})$. If $T_{1-\phi}$ is unitary then $S_{1-\phi}$ is unitary.*

Proof. Since T_ϕ, S_ϕ^* are p-hyponormal operators with $T_\phi X = XS_\phi$ where $X : A^2(\mathbb{D}) \to A^2(\mathbb{D})$ such that $ker(X) = \{0\}$ and $\overline{Range(X)} = A^2(\mathbb{D})$ then from [9], S_ϕ is unitarily equivalent to T_ϕ. That is $S_\phi = V^* T_\phi V$ where V is unitary. As $T_{1-\phi} = I - T_\phi$ is unitary, so $T_{\bar\phi} T_\phi = T_{\phi+\bar\phi} = T_\phi T_{\bar\phi}$. Thus $S_\phi + S_\phi^* = V^* T_{\phi+\bar\phi} V$. Now

$$\begin{aligned} S_{1-\phi}^* S_{1-\phi} &= (I - S_\phi)^*(I - S_\phi) \\ &= S_\phi^* S_\phi - S_\phi^* - S_\phi + I \\ &= V^* T_\phi^* T_\phi V - S_\phi^* - S_\phi + I \\ &= V^* T_\phi^* T_\phi V - V^* T_{\phi+\bar\phi} V + I = I. \end{aligned}$$

Similarly, we have $S_{1-\phi} S_{1-\phi}^* = I$. Hence the assertion holds. $\square$

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REFERENCES

[1] M. H. M. Rashid, M. S. M. Noorani, and A. S. Saari, " On the spectra of some non-normal operators," *Bull. Malays. Math. Sci. Soc*, vol. 31, pp. 135–143, 2008.

[2] T. Ando, "Operators with norm condition." *Acta Sci. Math. (Szeged)*, vol. 33, pp. 169–178, 1972.

[3] C. Padhy, P. K. Jena, and S. K. Paikray, "Aluthge transform of operators on the Bergman space," *Arab. J. Math.*, vol. 8, pp. 1–11, 2019, doi: 10.1007/s40065-019-00272-y.

[4] N. Das, " Norm of Toeplitz operators on the Bergman space," *Indian J. Pure Appl. Math*, vol. 33, pp. 255–267, 2002.

[5] N. Das and P. K. Jena, " On the range and kernel Of Toeplitz and little Hankel operators." *Methods Funct. Anal. Topology*, vol. 19, pp. 55–67, 2013.

[6] C. R. Deprima, " Remarks on "Operators with inverses similar to their adjoints"," *Porc. Amer. Math. Soc.*, vol. 43, pp. 478–480, 1974, doi: 10.1090/S0002-9939-1974-0331080-4.

[7] R. G. Douglas, *Banach Algebra Techniques in Operator Theory.* New York: Academic Press, 1972.

[8] H.L.Wang, " A note on uniform operators," *Porc. Amer. Math. Soc.*, vol. 96, pp. 643–646, 1986, doi: 10.2307/2045921.

[9] I. H. Kim, " The Fuglede-Putnam theorem for (P,K)- quasihyponormal operators," *J. Inequal Appl.*, vol. 2006, pp. 1–7, 2004, doi: 10.1155/JIA/2006/47481.

[10] C. S. Kubrusly, *An introduction to models and decompositions in operator theory*. Boston: Birkh'auser, 1997.

[11] K.Zhu, *Operator Theory in Function Spaces*. New York: Marcel Dekker, 1990.

[12] I. S. Othman, " Nearly equivalent operators," *Mathematica Bohemica*, vol. 121, pp. 133–141, 1996.

[13] C. S. Lin, " The unilateral shift and a norm equality for bounded linear operators," *Porc. Amer. Math. Soc.*, vol. 127, pp. 1693–1696, 1999.

[14] H. M. Srivastava, Q. Z. Ahmad, N. Khan, and N. Khanand B.Khan, " Hankel and Toeplitz determinant for a subclass of q starlike functions associated with a general conic domain," *Mathematics*, vol. 7, pp. 1–15, 2019.

[15] A. Uchiyama and K. Tanahashi, " Fuglede-Putnam's theorem for p-hyponormal or log-hyponormal operators," *Glasgow Math. J.*, vol. 44, pp. 397–410, 2002, doi: 10.1017/S0017089502020139.

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