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HERMITE-HADAMARD INEQUALITIES FOR UNIFORMLY CONVEX FUNCTIONS AND ITS APPLICATIONS IN MEANS

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This paper is dedicated to Professor Hossien Mohebi on his 60^{th} birthday.

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Abstract. In this paper, we prove Hermite-Hadamard inequality for uniformly convex, uniformly s-convex functions. Also, we obtain Hermite Hadamard inequality for fractional integral by using these functions. Finally, some applications of these inequalities are given.

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1. Introduction and Preliminaries

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$, then the following inequality holds:

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

The above inequality is well known in the literature as the Hermite-Hadamard inequality. Recently, the generalizations, improvements, variations and applications for convexity and the Hermite-Hadamard inequality have attracted the attention of many researchers, see [4–8, 11] and the references therein.

The following definitions can be found in [2, 12] and [1].

Definition 1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then f is called uniformly convex with modulus $\psi : [0,+\infty) \to [0,+\infty]$ if ψ is increasing, ψ vanishes only at 0, and

$$f(tx+(1-t)y)+t(1-t)\psi(|x-y|) \leq tf(x)+(1-t)f(y), \tag{1.1}$$

for each $x, y \in \mathbb{R}$ and $t \in (0,1)$.
If (1.1) holds with $\psi = \frac{\beta}{2}|.|^2$ for some $\beta > 0$, then f is called strongly convex with constant β.

In the following we give a simple example of a uniformly convex function (see [2], Corollary 2.14).

Example 1. In view of the following equality,

$$(\alpha x+(1-\alpha)y)^2+\alpha(1-\alpha)(x-y)^2=\alpha x^2+(1-\alpha)y^2,$$

for all $\alpha \in (0,1)$ and $x,y \in \mathbb{R}$, the function $f(t)=t^2$ for $t \in \mathbb{R}$ is uniformly convex with modulus $\psi(t)=t^2$ for all $t \geq 0$.

In the following proposition, the relation between convex functions and strongly convex functions is expressed. For more details about uniformly and strongly convex functions see [2].

Proposition 1. *Let $f:\mathbb{R}\to\mathbb{R}$ be a function and $\beta>0$. Then f is a strongly convex function with constant β if and only if $f-\dfrac{\beta}{2}|.|^2$ is a convex function.*

Clearly, strong convexity implies uniformly convexity, uniformly convexity implies strict convexity, and strict convexity implies convexity.
We can define the concept of uniformly s-convexity as follows:

Definition 2. Let $f:\mathbb{R}\to\mathbb{R}$ be a function. Then f is called s-uniformly convex function with modulus $\psi:[0,+\infty)\to[0,+\infty]$ if ψ is increasing, ψ vanishes only at 0, and

$$f(tx+(1-t)y)+t^s(1-t)\psi(|x-y|)\leq t^s f(x)+(1-t)^s f(y), \tag{1.2}$$

for each $x,y\in\mathbb{R}$, $t\in(0,1)$ and $s\in(0,1)$.
If Definition (1.2) holds with $\psi=\frac{\beta}{2}|.|^2$ for some $\beta>0$, then f is called strongly s-convex with constant β.

Definition 3. Let $f\in L[a,b]$. The left-sided and right-sided Riemann-Liouville fractional integrals $J^{\alpha}_{a^+}f$ and $J^{\alpha}_{b^-}f$ of order $\alpha>0$ with $a\geq 0$ are defined by

$$J^{\alpha}_{a^+}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt \quad \text{with} \quad x>a$$

$$J^{\alpha}_{b^-}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt \quad \text{with} \quad x<b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function and its defnition is

$$\Gamma(\alpha)=\int_0^{+\infty} e^{-t}t^{\alpha-1}dt.$$

It is to be noted that $J^0_{a^+}f(x)=J^0_{b^-}f(x)=f(x)$. In the case of $\alpha=1$, the fractional integral reduces to the classical integral.

In [12], M. Z. Sarikaya et al. presented the following Hermite-Hadamard's inequalities for fractional integrals.

Theorem 1 ([12]). *Let $f : I \to \mathbb{R}$ be a positive function with $0 \le a < b$ and $f \in L[a,b]$. If f is a convex function on $[a,b]$, then the following inequality for fractional integrals holds:*

$$f(\frac{a+b}{2}) \le \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J^{\alpha}_{a^+}f(b)+J^{\alpha}_{b^-}f(a)] \le \frac{f(a)+f(b)}{2}.$$

2. Main Results

In this section, we shall state our main results. At the first, we obtain Hermite-Hadamard type inequalities for the class of uniformly convex, uniformly s-convex and strongly convex functions.

Theorem 2. *Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly convex function. Then, the following inquality holds:*

$$f(\frac{a+b}{2}) + \frac{1}{8(b-a)}\int_{a-b}^{b-a}\psi(|t|)dt \le \frac{1}{b-a}\int_a^b f(t)dt \le \frac{f(a)+f(b)}{2} - \frac{1}{6}\psi(|a-b|).$$

Proof. In (1.1), set $t = \frac{1}{2}$, then one has

$$f(\frac{x+y}{2}) + \frac{1}{4}\psi(|x-y|) \le \frac{f(x)+f(y)}{2}. \tag{2.1}$$

Now in (2.1), set $x = ta+(1-t)b$ and $y = (1-t)a+tb$, and integrate inequality (2.1) on $[0,1]$ with respect to t. We conclude

$$f(\frac{a+b}{2}) + \frac{1}{4}\int_0^1 \psi(|(2t-1)(a-b)|)dt$$
$$\le \frac{1}{2}\int_0^1 f(ta+(1-t)b)dt + \frac{1}{2}\int_0^1 f((1-t)a+tb)dt.$$

Also, the following equalities holds

$$\frac{1}{4}\int_0^1 \psi(|(2t-1)(a-b)|)dt = \frac{1}{4}\int_{b-a}^{a-b}\psi(|u|)\frac{du}{2(a-b)}$$
$$= \frac{1}{8(b-a)}\int_{a-b}^{b-a}\psi(|t|)dt$$

and

$$\int_0^1 f((1-t)a+tb)dt = \int_0^1 f(ta+(1-t)b)dt = \frac{1}{b-a}\int_a^b f(t)dt.$$

Therefore,

$$f(\frac{a+b}{2}) + \frac{1}{8(b-a)}\int_{a-b}^{b-a}\psi(|t|)dt \le \frac{1}{b-a}\int_a^b f(t)dt.$$

On the other hand, in (1.1) put $x = a$, $y = b$ and integrate on $[0,1]$ with respect to t. Hence

$$\int_0^1 f(ta+(1-t)b)dt + \int_0^1 t(1-t)\psi(|a-b|)dt \le \int_0^1 \frac{f(a)+f(b)}{2}dt,$$

and so

$$\frac{1}{b-a}\int_a^b f(t)dt + \psi(|a-b|)\frac{\Gamma(2)\Gamma(2)}{\Gamma(4)} \le \frac{f(a)+f(b)}{2}.$$

Therefore,

$$\frac{1}{b-a}\int_a^b f(t)dt \le \frac{f(a)+f(b)}{2} - \frac{1}{6}\psi(|a-b|),$$

which completes the proof. It is worth noting that we used the following fact:

$$\int_0^1 t(1-t)dt = B(2,2) = \frac{\Gamma(2)\Gamma(2)}{\Gamma(4)} = \frac{1}{6},$$

where

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \ \ \Gamma(x) = \int_0^{+\infty} e^{-t}t^{x-1}dt, \ x>0, \ y>0,$$

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

□

In order to prove the main theorems, we need the following lemma that has been proved in [3].

Lemma 1. *Let $f : I^o \to \mathbb{R}$ be a differentiable function on I^o, $a,b \in I^o$ with $a<b$. If $f' \in L[a,b]$, then the following equality holds:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt = \frac{b-a}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)dt.$$

Theorem 3. *Let $f : I^o \to \mathbb{R}$ be a differentiable function on I^o, $a,b \in I^o$ with $a<b$. If $|f'|$ is uniformly convex function on I^o, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt\right| \le \frac{b-a}{8}(|f'(a)|+|f'(b)|) - \frac{b-a}{32}\psi(|a-b|).$$

Proof. In view of Lemma 1 and uniformly convexity of $|f'|$, one has

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt\right| \le \frac{b-a}{2}\int_0^1 |(1-2t)||f'(ta+(1-t)b)|dt$$

$$\le \frac{b-a}{2}\int_0^1 |1-2t|(t|f'(a)|+(1-t)|f'(b)|+t(t-1)\psi(|a-b|))dt$$

$$\le \frac{b-a}{2}\int_0^1 t|1-2t||f'(a)|dt + \int_0^1 |1-2t|(1-t)|f'(b)|dt$$

$$+\int_0^1 |1-2t|t(t-1)\psi(|a-b|))dt$$
$$\leq \frac{b-a}{8}(|f'(a)|+(f'(b)))-\frac{b-a}{32}\psi(|a-b|),$$

which completes the proof. Also, note that

$$\int_0^1 t|1-2t|dt=\int_0^1 (1-t)|1-2t|dt=\frac{1}{4},$$
$$\int_0^1 |1-2t|t(t-1)\psi(|a-b|)dt=-\frac{1}{16}\psi(|a-b|).$$

□

Theorem 4. *Let $f: I^o \to \mathbb{R}$ be a differentiable mapping on I^o, $a,b \in I^o$ with $a<b$ and $p>1$. If $|f'|^q$ is uniformly convex on I^o, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{b-a}{2(p+1)^{\frac{1}{p}}}(\frac{|f'(a)|^q+|f'(b)|^q}{2}-\frac{1}{6}\psi(|a-b|))^{\frac{1}{q}},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By Lemma 1 and Hölder's inequality, we conclude

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{b-a}{2}\int_0^1 |(1-2t)||f'(ta+(1-t)b)|dt$$
$$\leq \frac{b-a}{2}\left(\int_0^1 |1-2t|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}$$
$$\leq \frac{b-a}{2}\frac{1}{(p+1)^{\frac{1}{p}}}\left(|f(a)|^q\int_0^1 tdt+|f'(b)|^q\int_0^1 (1-t)dt+\psi(|a-b|)\int_0^1 t(t-1)dt\right)^{\frac{1}{q}}$$
$$\leq \frac{b-a}{2(p+1)^{\frac{1}{p}}}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}-\frac{1}{6}\psi(|a-b|)\right)^{\frac{1}{q}}.$$

Hence, the proof is complete. □

Theorem 5. *Let $f:\mathbb{R}\to\mathbb{R}$ be strongly convex function. Then*

$$f(\frac{a+b}{2})+\frac{\beta}{24}(b-a)^2 \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}-\frac{\beta}{12}(b-a)^2.$$

Proof. From Hermite-Hadamard inequality for convex functions, we have

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}. \tag{2.2}$$

Since from Proposition 1 f is a strongly convex function, we have $f-\frac{\beta}{2}|.|^2$ is convex. Hence in (2.2) replace f by $f-\frac{\beta}{2}|.|^2$ and after some calculations the result is obtained.

□

Theorem 6. *Let $f:\mathbb{R}\to\mathbb{R}$ be uniformly s-convex function. Then*

$$2^{s-1}f(\frac{a+b}{2})+\frac{1}{8(b-a)}\int_{a-b}^{b-a}\psi(|t|)dt \leq \frac{1}{b-a}\int_a^b f(t)dt$$
$$\leq \frac{f(a)+f(b)}{s+1}-\frac{1}{(s+1)(s+2)}\psi(|a-b|).$$

Proof. In (1.2), set $t=\frac{1}{2}$, then we have

$$f(\frac{x+y}{2})+\frac{1}{2^{s+1}}\psi(|x-y|)\leq \frac{f(x)+f(y)}{2^s}. \tag{2.3}$$

Now, set $x=ta+(1-t)b$ and $y=(1-t)a+tb$ in (2.5) and integrate on $[0,1]$ with respect to t. We get

$$f(\frac{a+b}{2})+\frac{1}{2^{s+1}}\int_0^1 \psi(|(2t-1)(a-b)|)dt$$
$$\leq \frac{1}{2^s}\int_0^1 f(ta+(1-t)b)dt+\frac{1}{2^s}\int_0^1 f((1-t)a+tb)dt.$$

Now,

$$\frac{1}{2^{s+1}}\int_0^1 \psi(|(2t-1)(a-b)|)dt = \frac{1}{2^{s+1}}\int_{b-a}^{a-b}\psi(|u|)\frac{du}{2(a-b)}$$
$$=\frac{1}{2^{s+2}(b-a)}\int_{a-b}^{b-a}\psi(|t|)dt.$$

Also, we have $\int_0^1 f((1-t)a+tb)dt=\int_0^1 f((1-t)b+ta)dt=\frac{1}{b-a}\int_a^b f(t)dt$. Therefore

$$f(\frac{a+b}{2})+\frac{1}{2^{s+2}(b-a)}\int_{a-b}^{b-a}\psi(|t|)dt\leq \frac{1}{2^{s-1}(b-a)}\int_a^b f(t)dt.$$

On the other hand, in (1.1) put $x=a$, $y=b$ and integrate on $[0,1]$ with respect to t. Then we obtain

$$\int_0^1 f(ta+(1-t)b)dt+\int_0^1 t^s(1-t)\psi(|a-b|)dt\leq \int_0^1 t^s f(a)+(1-t)^s f(b)dt$$

so,

$$\frac{1}{b-a}\int_a^b f(t)dt+\psi(|a-b|)\frac{\Gamma(s+1)\Gamma(2)}{\Gamma(s+3)}\leq \frac{f(a)+f(b)}{s+1},$$

finally,

$$\frac{1}{b-a}\int_a^b f(t)dt\leq \frac{f(a)+f(b)}{s+1}-\frac{1}{(s+1)(s+2)}\psi(|a-b|),$$

which completes the proof. □

Theorem 7. *Let $p \in [2, +\infty)$, then the following inequality holds:*

$$|\frac{a+b}{2}|^p + \frac{1}{8(b-a)} 2^{1-p} \min\{p2^{-\frac{p}{2}}, 1-2^{-\frac{p}{2}}\} \int_{a-b}^{b-a} |t|^p dt \leq \frac{1}{b-a}\int_a^b |t|^p dt$$
$$\leq \frac{|a|^p + |b|^p}{2} - \frac{1}{6}\min\{p2^{-\frac{p}{2}}, 1-2^{-\frac{p}{2}}\}|a-b|^p.$$

Proof. According to ([2], Proposition 10.13), since $|.|^2$ is uniformly convex with modules of convexity $|.|^2$. Hence for $p \in [2, +\infty)$ is uniformly convex with modules of convexity ψ such that ψ satisfing

$$\psi \geq 2^{1-p} \min\{p2^{-\frac{p}{2}}, 1-2^{-\frac{p}{2}}\}|.|^p, \tag{2.4}$$

Hence, in view of Theorem 2 for function $f(t) = |t|^p$ and (2.4), one has

$$|\frac{a+b}{2}|^p + \frac{1}{8(b-a)} 2^{1-p} \min\{p2^{-\frac{p}{2}}, 1-2^{-\frac{p}{2}}\} \int_{a-b}^{b-a} |t|^p dt$$
$$\leq |\frac{a+b}{2}|^p + \frac{1}{8(b-a)} \int_{a-b}^{b-a} \psi(t) dt$$
$$\leq \frac{1}{b-a}\int_a^b |t|^p dt$$
$$\leq \frac{|a|^p + |b|^p}{2} - \frac{1}{6}\psi(|a-b|)$$
$$\leq \frac{|a|^p + |b|^p}{2} - \frac{1}{6}\min\{p2^{-\frac{p}{2}}, 1-2^{-\frac{p}{2}}\}|a-b|^p.$$

□

Proposition 2. *Let p be an even number and let $a, b \in \mathbb{R}$ with $0 < a < b$, then the following inequality holds:*

$$(p+1)(\frac{a+b}{2})^p + \frac{(b-a)^{p+1}}{2^{p+2}(b-a)} \min\{p2^{-\frac{p}{2}}, 1-2^{-\frac{p}{2}}\}$$
$$\leq \frac{b^{p+1} - a^{p+1}}{b-a}$$
$$\leq \left(\frac{a^p + b^p}{2} - \frac{(b-a)^p}{6} \min\{p2^{-\frac{p}{2}}, 1-2^{-\frac{p}{2}}\}\right)(p+1).$$

Proof. The proof is immediate consequence of Theorem 7. □

2.1. *Hermite-Hadamard's inequalities for fractional integrals*

Theorem 8. *Let $f : [a, b] \to \mathbb{R}$ be a uniformly convex function. Then, for $\alpha > 0$ the following inquality for fractional integrals holds:*

$$f(\frac{a+b}{2}) + \frac{\Gamma(\alpha+1)}{2^{\alpha+2}(b-a)^\alpha} J^\alpha_{(a-b)^+}\psi(|a-b|) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a^+} f(b) + J^\alpha_{b^-} f(a)]$$

$$\leq \frac{f(a)+f(b)}{2} - \alpha\beta(\alpha+1,2)\psi(|a-b|).$$

Proof. In (1.1), set $t=\frac{1}{2}$, then we have

$$f(\frac{x+y}{2}) + \frac{1}{4}\psi(|x-y|) \leq \frac{f(x)+f(y)}{2}. \tag{2.5}$$

Now, set $x = ta+(1-t)b$ and $y = (1-t)a+tb$ in (2.5). Multiplying both sides of (2.5) by $t^{\alpha-1}$ and then integrating the resulting inequality with respect to t over [0,1], we obtain

$$\int_0^1 t^{\alpha-1} f(\frac{a+b}{2})dt + \frac{1}{4}\int_0^1 t^{\alpha-1}\psi(|(2t-1)(a-b)|)dt$$
$$\leq \frac{1}{2}\int_0^1 t^{\alpha-1} f(ta+(1-t)b)dt + \frac{1}{2}\int_0^1 t^{\alpha-1} f((1-t)a+tb)dt.$$

Let $ta+(1-t)b = r$, $(1-t)a+tb = s$ and $(2t-1)(a-b) = x$, then

$$\frac{f(\frac{a+b}{2})}{\alpha} + \frac{1}{4}\int_{b-a}^{a-b} (\frac{b-a-x}{2(b-a)})^{\alpha-1}\psi(|x|)\frac{dx}{2(a-b)} \leq$$
$$\frac{1}{2}\int_b^a (\frac{b-r}{b-a})^{\alpha-1} f(r)\frac{dr}{a-b} + \frac{1}{2}\int_a^b (\frac{s-a}{b-a})^{\alpha-1} f(s)\frac{ds}{b-a}.$$

So, we have

$$\frac{f(\frac{a+b}{2})}{\alpha} + \frac{1}{2^{\alpha+2}(b-a)^{\alpha}} J^{\alpha}_{(a-b)^+}\psi(|a-b|) \leq \frac{\Gamma(\alpha)}{2(b-a)^{\alpha}}[J^{\alpha}_{a^+}f(b) + J^{\alpha}_{b^-}f(a)].$$

Conversely, since f is uniformly convex one has

$$f(tx+(1-t)y) + t(1-t)\psi(|x-y|) \leq tf(x) + (1-t)f(y). \tag{2.6}$$

Now, replacing x by y we have

$$f(ty+(1-t)x) + t(1-t)\psi(|x-y|) \leq tf(y) + (1-t)f(x). \tag{2.7}$$

Adding the two equations (2.6) and (2.7) we obtain

$$f(tx+(1-t)y) + f((1-t)x+ty) + 2t(1-t)\psi(|x-y|) \leq f(x) + f(y). \tag{2.8}$$

Set $x=a$ and $y=b$ in (2.8) and also multiplying both sides of (2.8) by $t^{\alpha-1}$ and then integrating the resulting inequality with respect to t over [0,1], we obtain

$$\int_0^1 t^{\alpha-1} f(ta+(1-t)b)dt + \int_0^1 t^{\alpha-1} f((1-t)a+tb)dt + \int_0^1 2t^{\alpha}(1-t)\psi(|a-b|)dt$$
$$\leq \int_0^1 t^{\alpha-1} f(a)dt + \int_0^1 t^{\alpha-1} f(b)dt.$$

So,

$$\frac{\Gamma(\alpha)}{2(b-a)^{\alpha}}[J^{\alpha}_{a^+}f(b) + J^{\alpha}_{b^-}f(a)] \leq \frac{f(a)+f(b)}{2\alpha} - \beta(\alpha+1,2)\psi(|a-b|),$$

which completes the proof. □

3. Applications to special means

Consider the following special means for two nonnegative real numbers α,β with $\alpha \neq \beta$ as follows (see [3, 5, 9, 10]):

(1) The arithmetic mean:

$$A = A(\alpha,\beta) = \frac{\alpha+\beta}{2},\ \alpha,\beta \in \mathbb{R},$$

with $\alpha,\beta > 0$.

(2) The logarithmic mean:

$$\overline{L} = \overline{L}(\alpha,\beta) = \frac{\beta-\alpha}{\ln\beta - \ln\alpha},\quad \alpha \neq \beta, \alpha,\beta \in \mathbb{R},$$

with $\alpha,\beta > 0$.

(3) The generalized logarithmic mean:

$$L_n(\alpha,\beta) = [\frac{\beta^{n+1}-\alpha^{n+1}}{(n+1)(\beta-\alpha)}]^{\frac{1}{n}},\ n \in \mathbb{R}\backslash\{-1,0\}, \alpha \neq \beta, \alpha,\beta \in \mathbb{R},$$

with $\alpha,\beta > 0$.

Proposition 3. *Let $a,b \in \mathbb{R}$ with $0 < a < b$ and let p be an even number. Then the following inequality holds:*

$$\left((\frac{a+b}{2})^p + \frac{(b-a)^{p+1}}{2^{p+2}(p+1)(b-a)} \min\{p2^{-\frac{p}{2}}, 1-2^{-\frac{p}{2}}\} \right)^{\frac{1}{p}}$$
$$\leq L_p(a,b) \leq \left((\frac{a^p+b^p}{2} - \frac{(b-a)^p}{6} \min\{p2^{-\frac{p}{2}}, 1-2^{-\frac{p}{2}}\}) \right)^{\frac{1}{p}}$$

Proof. Since the function $g(t) = t^{\frac{1}{p}}$ is increasing for $t \geq 0$ and $p > 0$, in view of Proposition 2, the proof is complete. □

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