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A STUDY ON THE UNIFORM CONVERGENCE OF SPECTRAL EXPANSIONS FOR CONTINUOUS FUNCTIONS ON A STURM-LIOUVILLE PROBLEM

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Abstract. The paper is about investigating the uniform convergence conditions of spectral expansions of continuous functions in terms of root functions of a spectral problem with the same eigenparameter in the second-order differential equation and depending on quadratically in one of the boundary conditions on a closed interval.

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1. Introduction

In this paper, the spectral problem

$$-y'' + q(x)y = \lambda y, \qquad 0 < x < 1, \tag{1.1}$$

$$y'(0)\sin\beta = y(0)\cos\beta, \qquad 0 \leq \beta < \pi, \tag{1.2}$$

$$y'(1) = (a\lambda^2 + b\lambda + c)y(1) \tag{1.3}$$

is considered and studied the uniform convergence of Fourier series expansions in terms of root functions of this problem for continuous functions. Here, λ is a spectral parameter; $q(x)$ is real-valued continuous function on the interval $[0,1]$; $a \neq 0$, b and c are real constants.

Lineer differantial operators with the second order differential equation can be addressed by some authors [19–21] and their significant properties have been developed using novel techniques in some functional spaces. Sturm-Liouville operators with the boundary condition (or conditions) depending on the eigenparameter, one of a concrete classes of this linear differential operators, have been studied to research their various properties by many authors. For example, problems of proving existence of eigenvalues, obtaining oscillation of eigenfunctions, giving asymptotic formulae of

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eigenvalues and eigenfunctions have been studied in papers [3,5]; problems of investigating the basis properties of the system of root functions in $L_p(0,1), 1 < p < \infty$ have been investigated in papers [1, 13]; problems of obtaining the convergence conditions of the spectral expansions in terms of root functions in given some linear spaces has been proved in papers [6, 9, 10, 13–16] for one boundary value problem.

In applications, the spectral problems [11, 12], which are investigated the uniform convergence conditions of the spectral expansions, that underline an important class of mathematical physics problems appears on a model of a transrelaxation heat process, torsional vibrations of a rod with a pulley at one end, the current in a cable grounded at one end through a concentrated capacitance or inductance and vibrations of a homogeneous loaded string.

Firstly, in 2005, Code and Browne [5] studied existence and asymptotics of eigenvalues of this problem (see also [4, Section 4.1]) for the problem (1.1)-(1.3). They proved that the eigenvalues of the boundary value problem (1.1)-(1.3) form an infinite sequence λ_n $(n = 0,1,2,\dots)$ without finite limit points and only following cases are possible:

- all the eigenvalues are real and simple,
- all the eigenvalues are real and all, except one double, are simple,
- all the eigenvalues are real and all, except one triple, are simple,
- all the eigenvalues are simple and all, except a conjugate pair of non-real, are real.

Note that the eigenvalues λ_n $(n = 0,1,2,\dots)$ were considered to be listed according to non-decreasing real part and repeated according to algebraic multiplicity. Therefore, the results of the article [5] cannot be directly applied to the problem (1.1)-(1.3).

Secondly, in 2008, Aliyev and Kerimov [1] studied basisness of root functions of the problem (1.1)-(1.3). Namely, they determined the explicit form of the biorthogonal system and they proved that the root functions system is a basis of $L_p(0,1), 1 < p < \infty$; moreover, if $p = 2$, then this basis is unconditional.

Finally, solutions that obtained by using the Fourier method of partial differential equations are represented by a series. Therefore, the investigation of the properties (such as convergence or divergence) of these series is of great importance [7, 21]. We aim to investigate the uniform convergence conditions of spectral expansions of continuous functions in terms of root functions of the problem (1.1)-(1.3)

Let us give a brief description of the structure of our study. In Section 2, we express some fundamental notations and some auxiliary results to prove our hypotheses. In Section 3, we give three theorems: Theorem 1 relates to the sharpening the asymptotic formulae for eigenvalues and eigenfunctions of the problem (1.1)-(1.3). Moreover, Theorem 2, the main theorem of the paper, gives the uniform convergence conditions of the spectral expansions in terms of root functions for this problem and Theorem 3 gives the uniform convergence conditions of the spectral expansions in terms of the systems that is biorhogonally conjugate to root functions for this problem. In Section 4, an example is given in accordance with the main theorem.

2. Preliminaries

Some properties of eigenvalues, eigenfunctions and associated functions of the problem (1.1)-(1.3) should be needed and explicit forms of the root functions system of this problem should be expressed to use in the hypothesis of main theorem to reach the desired results in this article. As follows:

Let $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ denote the solutions of the equation (1.1) which satisfy the initial conditions

$$\varphi(0,\lambda)=1,\,\varphi'(0,\lambda)=h, \tag{2.1}$$

$$\psi(0,\lambda)=0,\,\psi'(0,\lambda)=1, \tag{2.2}$$

where $h=\cot\beta$ $(0<\beta<\pi)$.

Let λ_n $(n=0,1,2,\dots)$ be eigenvalues and $y_n(x)$ $(n=0,1,2,\dots)$ be eigenfunctions corresponding to λ_n $(n=0,1,2,\dots)$ of the problem (1.1)-(1.3).

If λ_k is the multiple eigenvalue $(\lambda_k=\lambda_{k+1})$, then the following relations hold for the first order associated function y_{k+1} corresponding to eigenfunction y_k [18, p28]:

$$-y''_{k+1}+q(x)y_{k+1}=\lambda_k y_{k+1}+y_k,$$
$$y'_{k+1}(0)\sin\beta=y_{k+1}(0)\cos\beta,$$
$$y'_{k+1}(1)=(a\lambda_k^2+b\lambda_k+c)y_{k+1}(1)+(2a\lambda_k+b)y_k(1).$$

If λ_k is the triple eigenvalue $(\lambda_k=\lambda_{k+1}=\lambda_{k+2})$, then there exist the second order associated function y_{k+2} for which following relations hold together with the first order associated function y_{k+1} [18, p28]:

$$-y''_{k+2}+q(x)y_{k+2}=\lambda_k y_{k+2}+y_{k+1},$$
$$y'_{k+2}(0)\sin\beta=y_{k+2}(0)\cos\beta,$$
$$y'_{k+2}(1)=(a\lambda_k^2+b\lambda_k+c)y_{k+2}(1)+(2a\lambda_k+b)y_{k+1}(1)+2ay_k.$$

Note that the functions $y_{k+1}+dy_k$ and $y_{k+2}+ey_k$, d and e are arbitrary constants, are also associated functions of the first and second order respectively.

We shall denote the solution of the equation (1.1) which satisfies the initial condition (2.1), if $0<\beta<\pi$ or (2.2), if $\beta=0$ by $y(x,\lambda)$. Then, the eigenvalues of the problem (1.1)-(1.3) are the roots of the characteristic function

$$\omega(\lambda)=y'(1,\lambda)-(a\lambda_k^2+b\lambda_k+c)y(1,\lambda). \tag{2.3}$$

In [1], it has been proven that if λ_k is a multiple (double or triple) eigenvalue of the problem (1.1)-(1.3), then

$$\begin{gathered} y(x,\lambda)\to y_k(x),\quad y'(x,\lambda)\to y'_k(x),\\ y_\lambda(x,\lambda)\to\tilde{y}_{k+1}(x),\quad y'_\lambda(x,\lambda)\to\tilde{y}'_{k+1}(x)\end{gathered} \tag{2.4}$$

uniformly according to $x \in [0,1]$, as $\lambda \to \lambda_k$, where $\tilde{y}_{k+1}$ is one of the associated functions of the first order. It is clear that $\tilde{y}_{k+1} = y_{k+1} + \tilde{c} y_k$. Furthermore, if λ_k is a triple eigenvalue of the problem (1.1)-(1.3), then

$$y_{\lambda\lambda}(x,\lambda) \to 2\tilde{y}_{k+2}(x), \qquad y'_{\lambda\lambda}(x,\lambda) \to 2\tilde{y}'_{k+2}(x) \tag{2.5}$$

uniformly according to $x \in [0,1]$, as $\lambda \to \lambda_k$, where $\tilde{y}_{k+2}$ is one of the associated functions of the second order corresponding to the first associated function $\tilde{y}_{k+1}$. It is obvious that $\tilde{y}_{k+2} = y_{k+2} + \tilde{c} y_{k+1} + \tilde{d} y_k$ (see also [8]).

By (2.4) and (2.5), it is easily seen that

$$\tilde{c} = \begin{cases} -y'_{k+1}(0), & \text{if } \beta = 0\,, \\ -y_{k+1}(0), & \text{if } 0 < \beta < \pi\,, \end{cases}$$

$$\tilde{d} = \begin{cases} \left(y'_{k+1}(0)\right)^2 - y'_{k+2}(0), & \text{if } \beta = 0\,, \\ y_{k+1}^2(0) - y_{k+2}(0), & \text{if } 0 < \beta < \pi\,. \end{cases}$$

Additionally, the following auxiliary associated functions were considered in [1]:

$$y^*_{k+1} = y_{k+1} + c_1 y_k, \tag{2.6}$$

$$y^{**}_{k+1} = y_{k+1} + c_2 y_k, \tag{2.7}$$

$$y^{\#\#}_{k+2} = y_{k+2} + c_2 y_{k+1} + c_3 y_k, \tag{2.8}$$

where

$$c_1 = -\frac{\omega'''(\lambda_k)}{3\omega''(\lambda_k)} - \frac{y_{k+1}(1)}{y_k(1)} + \tilde{c},$$

$$c_2 = -\frac{\omega^{IV}(\lambda_k)}{4\omega'''(\lambda_k)} - \frac{y_{k+1}(1)}{y_k(1)} + \tilde{c},$$

$$c_3 = -\frac{\omega^{V}(\lambda_k)}{20\omega'''(\lambda_k)} - \frac{\omega^{IV}(\lambda_k)}{4y_k(1)\omega'''(\lambda_k)}\left(y_{k+1}(1) - \tilde{c} y_k(1)\right) - \frac{y_{k+2}(1)}{y_k(1)} + \frac{\tilde{c} y_{k+1}(1)}{y_k(1)} - \tilde{c}^2 + \widetilde{d} + c_2^2,$$

Let i and j be arbitrary non-negative different integers. The following systems were investigated in [1]:*

(i) $y_n(x)$ $(n = 0,1,...; n \neq i,j)$, if all of eigenvalues of the problem (1.1)-(1.3) are real and simple or all eigenvalues of the problem (1.1)-(1.3), except a conjugate pair of non-real, are real and simple.

(ii) $y_n(x)$ $(n = 0,1,...; n \neq k,k+1)$, if λ_k is double eigenvalue $(\lambda_k = \lambda_{k+1})$ of the problem (1.1)-(1.3).

(iii) $y_n(x)$ $(n = 0,1,...; n \neq k+1,j)$, if λ_k is double eigenvalue $(\lambda_k = \lambda_{k+1})$ of the problem (1.1)-(1.3), where $j \neq k,k+1$.

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*(i) - (xii) will be used also the numbering of systems in these cases.

(iv) $y_n(x)$ $(n = 0,1,...;n \neq k, j)$, if λ_k is double eigenvalue $(\lambda_k = \lambda_{k+1})$ of the problem (1.1)-(1.3) and $y^*_{k+1}(1)\left(\lambda_j - \lambda_k\right) \neq y_k(1)$, where $j \neq k, k+1$.
(v) $y_n(x)$ $(n = 0,1,...;n \neq i, j)$, if λ_k is double eigenvalue $(\lambda_k = \lambda_{k+1})$ of the problem (1.1)-(1.3), where $i, j \neq k, k+1$.
(vi) $y_n(x)$ $(n = 0,1,...;n \neq k+1, k+2)$, if λ_k is triple eigenvalues $(\lambda_k = \lambda_{k+1} = \lambda_{k+2})$ of the problem (1.1)-(1.3).
(vii) $y_n(x)$ $(n = 0,1,...;n \neq k, k+2)$, if λ_k is triple eigenvalues $(\lambda_k = \lambda_{k+1} = \lambda_{k+2})$ of the problem (1.1)-(1.3) and $y^{\#}_{k+1}(1) \neq 0$.
(viii) $y_n(x)$ $(n = 0,1,...;n \neq k, k+1)$, if λ_k is triple eigenvalues $(\lambda_k = \lambda_{k+1} = \lambda_{k+2})$ of the problem (1.1)-(1.3) and $\left(y^{\#}_{k+1}(1)\right)^2 \neq y_k(1)\, y^{\#}_{k+2}(1)$.
(ix) $y_n(x)$ $(n = 0,1,...;n \neq k+2, j)$, if λ_k is triple eigenvalues $(\lambda_k = \lambda_{k+1} = \lambda_{k+2})$ of the problem (1.1)-(1.3), where $j \neq k, k+1, k+2$.
(x) $y_n(x)$ $(n = 0,1,...;n \neq k+1, j)$, if λ_k is triple eigenvalue $(\lambda_k = \lambda_{k+1} = \lambda_{k+2})$ of the problem (1.1)-(1.3) and $y^{\#}_{k+1}(1)\left(\lambda_j - \lambda_k\right) \neq y_k(1)$, where $j \neq k, k+1, k+2$.
(xi) $y_n(x)$ $(n = 0,1,...;n \neq k, j)$, if λ_k is triple eigenvalue $(\lambda_k = \lambda_{k+1} = \lambda_{k+2})$ of the problem (1.1)-(1.3) and $y^{\#}_{k+2}(1)\left(\lambda_j - \lambda_k\right) \neq y^{\#}_{k+1}(1)$, where $j \neq k, k+1, k+2$.
(xii) $y_n(x)$ $(n = 0,1,...;n \neq i, j)$, if λ_k is triple eigenvalues $(\lambda_k = \lambda_{k+1} = \lambda_{k+2})$ of the problem (1.1)-(1.3), where $i, j \neq k, k+1, k+2$.

The given conditions $y^*_{k+1}(1)(\lambda_j - \lambda_k) \neq y_k(1)$, $y^{\#}_{k+1}(1) \neq 0$, $(y^{\#}_{k+1}(1))^2 \neq y_k(1)\, y^{\#}_{k+2}(1)$, $y^{\#}_{k+1}(1)\left(\lambda_j - \lambda_k\right) \neq y_k(1)$, $y^{\#}_{k+1}(1)\left(\lambda_j - \lambda_k\right) \neq y_k(1)$ and $y^{\#}_{k+2}(1)\left(\lambda_j - \lambda_k\right) \neq y^{\#}_{k+1}(1)$ for the above-mentioned system are sufficient and necessary conditions for the basisness of the systems (iv), (vii), (viii), (x) and (xi) respectively. Moreover, for example, if $y^*_{k+1}(1)(\lambda_j - \lambda_k) = y_k(1)$, the system (iv) is neither complete nor minimal in $L_p(0,1), 1 < p < \infty$.

Denote the systems that is biorhogonally conjugate to each of the systems (i)-(xii) by the system $\{u_n(x)\}$. For example, the system $u_n(x)$ $(n = 0,1,...;n \neq k, j)$ is biorhogonally conjugate to the system (iv) or the system (xi) if λ_k is double or triple respectively.

3. Main results

Firstly, we shall give the sharpened asymptotics for eigenvalues and eigenfunctions, since such asymptotics are very important for the prove of main theorem.

We here note that the following asymptotic formulae are valid for sufficiently large n [5, Section 2, Theorem 1]

$$\lambda_n = \begin{cases} (n-1)^2\pi^2 + O(1), & \text{if } \beta = 0, \\ \left(n-\frac{3}{2}\right)^2 \pi^2 + O(1), & \text{if } 0<\beta<\pi. \end{cases} \tag{3.1}$$

The following theorem aims to sharp the asymptotic formula (3.1).

Theorem 1. *Let* $\lambda_n = s_n^2$ $(\operatorname{Re} s_n \geq 0)$. *The following asymptotic formulae are valid for sufficiently large* n*:*

(a) *If* $\beta = 0$, *then*

$$s_n = (n-1)\pi + \frac{A_1}{n\pi} + O\left(\frac{\delta_{n,1}}{n}\right), \tag{3.2}$$

$$\begin{aligned} y_n(x) = \psi(x,\lambda_n) = {} & \frac{\sin(n-1)\pi x}{(n-1)\pi} + \frac{\alpha_1(x)\cos(n-1)\pi x}{(n\pi)^2} \\ & + \frac{\alpha_{n,1}(x)\cos(n-1)\pi x}{2(n\pi)^2} + \frac{\beta_{n,1}(x)\sin(n-1)\pi x}{2(n\pi)^2} \\ & + O\left(\frac{\delta_{n,1}}{n^2}\right), \end{aligned} \tag{3.3}$$

where

$$A_1 = \frac{1}{2}\int_0^1 q(\tau)d\tau, \quad \alpha_1(x) = A_1 x - \frac{1}{2}\int_0^x q(\tau)d\tau,$$

$$\alpha_{n,1}(x) = \int_0^x q(\tau)\cos 2(n-1)\pi\tau d\tau,$$

$$\beta_{n,1}(x) = \int_0^x q(\tau)\sin 2(n-1)\pi\tau d\tau \quad an \quad \delta_{n,1} = \left|\int_0^1 q(\tau)\cos(2n-1)\pi\tau d\tau\right| + \frac{1}{n}.$$

(b) *If* $0<\beta<\pi$, *then*

$$s_n = \left(n-\frac{3}{2}\right)\pi + \frac{A_2}{n\pi} + O\left(\frac{\delta_{n,2}}{n}\right), \tag{3.4}$$

$$
\begin{aligned}
y_n(x) = \varphi(x,\lambda_n) &= \cos\left(n-\frac{3}{2}\right)\pi x + \frac{\alpha_2(x)\sin\left(n-\frac{3}{2}\right)\pi x}{n\pi} \\
&+ \frac{\alpha_{n,2}(x)\sin\left(n-\frac{3}{2}\right)\pi x}{2n\pi} - \frac{\beta_{n,2}(x)\cos\left(n-\frac{3}{2}\right)\pi x}{2n\pi} \\
&+ O\left(\frac{\delta_{n,2}}{n}\right),
\end{aligned}
\tag{3.5}
$$

where $A_2 = h + \frac{1}{2}\int_0^1 q(\tau)d\tau$, $h = \cot\beta$, $\alpha_2(x) = h - A_2 x + \frac{1}{2}\int_0^x q(\tau)d\tau$, $\alpha_{n,2}(x) = \int_0^x q(\tau)\cos(2n-3)\pi\tau d\tau$, $\beta_{n,2}(x) = \int_0^x q(\tau)\sin(2n-3)\pi\tau d\tau$ *and* $\delta_{n,2} = \left|\int_0^1 q(\tau)\cos(2n-3)\pi\tau d\tau\right| + \frac{1}{n}$.

Proof. We will only give the proof of case (b). The other case is proven similarly. Suppose that, $\lambda = s^2$. By virtue of (2.1), the equality

$$
\varphi(x,\lambda) = \cos sx + \frac{h}{s}\sin sx + \frac{1}{s}\int_0^x \sin(s(x-\tau))q(\tau)\varphi(\tau,\lambda)d\tau \tag{3.6}
$$

holds [17, Chapter I, Section 1.2, Lemma 1.2.1].

Let $\lambda = \sigma + it$. There exists $s_0 > 0$ such that for $|s| > s_0$, the estimate

$$
\varphi(x,\lambda) = \cos sx + O(e^{|t|x}|s|^{-1}) \tag{3.7}
$$

is valid [17, Chapter I, Section 1.2, Lemma 1.2.2], where the function $O(e^{|t|x}|s|^{-1})$ is the entire function of s for any fixed x in $[0,1]$. Moreover, the function (3.7) is uniform with respect to x for $0 \le x \le 1$.

Because of $0 < \beta < \pi$ in assumption of case (b), the formulae

$$
s_n = \sqrt{\lambda_n} = \left(n - \frac{3}{2}\right)\pi + O(n^{-1}) \tag{3.8}
$$

is satisfied by (3.1). The formulae (3.6)-(3.8) yield the following:

$$\varphi(x,\lambda_n) = \cos s_n x + \frac{h}{s_n}\sin s_n x + \frac{\sin s_n x}{2s_n}\int_0^x q(\tau)d\tau$$
$$+\frac{\sin s_n x}{2s_n}\int_0^x q(\tau)\cos 2s_n\tau d\tau \tag{3.9}$$
$$-\frac{\cos s_n x}{2s_n}\int_0^x q(\tau)\sin 2s_n\tau d\tau + O(n^{-2}).$$

In addition, by differentiating the equality (3.6) with respect to x and substituting the formulae (3.8), we obtain

$$\varphi'(x,\lambda_n) = -s_n\sin s_n x + O(1). \tag{3.10}$$

Consequently, the roots of the equation

$$\varphi'(1,\lambda_n) = \left(a\lambda_n^2 + b\lambda_n + c\right)\varphi(1,\lambda_n) \tag{3.11}$$

are the eigenvalues of the problem (1.1)-(1.3).

Let $s_n = \left(n - \frac{3}{2}\right)\pi + \varepsilon_n$. Since the formulas

$$\sin s_n = (-1)^n + O(n^{-2}),$$
$$\cos s_n = (-1)^{n-1}\varepsilon_n + O(n^{-3})$$

by using (3.9) and (3.10), we obtain the estimates

$$\varphi(1,\lambda_n) = (-1)^{n-1}\varepsilon_n + \frac{(-1)^n h}{n\pi} + \frac{(-1)^n}{2n\pi}\int_0^1 q(\tau)d\tau + O\left(\frac{\delta_{n,2}}{n}\right), \tag{3.12}$$

$$\varphi'(1,\lambda_n) = (-1)^{n-1}n\pi + O(1). \tag{3.13}$$

By substituting (3.12) and (3.13) in equation (3.11), the equality

$$(-1)^{n-1}\varepsilon_n + \frac{(-1)^n h}{n\pi} + \frac{(-1)^n}{2n\pi}\int_0^1 q(\tau)d\tau + O\left(\frac{\delta_{n,2}}{n}\right) = 0$$

is obtained, where $\delta_{n,2} = \left|\int_0^1 q(\tau)\cos(2n-3)\pi\tau d\tau\right| + \frac{1}{n}$. The last equation shows that the asymptotic formulae (3.4) is valid.

By using (3.4), we obtain

$$\cos s_n x = \cos\left(n - \frac{3}{2}\right)\pi x - \frac{A_2 x}{n\pi}\sin\left(n - \frac{3}{2}\right)\pi x + O\left(\frac{\delta_{n,2}}{n}\right). \tag{3.14}$$

The asymptotic formulae (3.5) follows from (3.9) and (3.14) and the proof of theorem 1 is completed. □

The trigonometric system $\{\theta_n(x)\}_{n=1}^{\infty}$ are defined as follows:

$$\theta_n(x) = \begin{cases} \sqrt{2}\sin n\pi x, & \text{if } \beta = 0, \\ \sqrt{2}\cos\left(n - \frac{1}{2}\right)\pi x, & \text{if } 0 < \beta < \pi. \end{cases}$$

The following theorems are related to uniformly convergent spectral expansions in terms of root functions and the functions which are biorthogonally conjugate to root functions of the problem (1.1)-(1.3), respectively.

Now, we are ready to give main theorem, as follows:

Theorem 2. *Suppose that* $f \in C[0,1]$ *and* $f(x)$ *has a uniformly convergent Fourier expansion in the system* $\{\theta_n(x)\}_{n=1}^{\infty}$ *on the interval* $[0,1]$*. Then, this function can be expanded in Fourier series in each of the systems* (i)-(xii) *and these expansions are uniformly convergent on every interval* $[0,r]$ $(0 < r < 1)$*. Moreover, the Fourier series of* $f(x)$ *in the systems* (i)-(xii) *are uniformly convergent on* $[0,1]$ *if and only if*

$(f, y_i)\, y_j(1) = 0$ *for the systems* (i), (v) *and* (xii)*;*

$(f, y_k)\, y_{k+1}(1) = (f, y_{k+1})\, y_k(1)$ *for the systems* (ii) *and* (vi)*;*

$(f, y_k)\, y_j(1) = (f, y_i)\, y_k(1)$ *for the systems* (iii) *and* (ix)*;*

$\left(f, y_{k+1}^{*}\right) y_j(1) = \left(f, y_j\right) y_{k+1}^{*}(1)$ *for the system* (iv)*;*

$(f, y_k)\, y_{k+2}^{\#}(1) = \left(f, y_{k+2}^{\#}\right) y_k(1)$ *for the system* (vii)*;*

$\left(f, y_{k+1}^{\#}\right) y_{k+2}^{\#}(1) = \left(f, y_{k+2}^{\#}\right) y_{k+1}^{\#}(1)$ *for the system* (viii)*;*

$\left(f, y_{k+1}^{\#}\right) y_j(1) = \left(f, y_j\right) y_{k+1}^{\#}(1)$ *for the system* (x) *and*

$\left(f, y_{k+2}^{\#}\right) y_j(1) = \left(f, y_j\right) y_{k+2}^{\#}(1)$ *for the system* (xi).

Proof. The theorem will only be proven for the system (iv). The theorem for other systems can be proved similarly.

Consider the Fourier series of $f(x)$ in the system $y_n(x)$ $(n = 0, 1, \ldots; n \neq k, j)$ on the interval $[0,1]$:

$$F(x) = \sum_{n=0; n \neq k, j}^{\infty} (f, u_n) y_n(x). \tag{3.15}$$

The elements of the $u_n(x)$ $(n = 0, 1, \ldots; n \neq k, j)$ biorthogonal system defined as biorthogonally conjugate to the system (iv) can be represented in the following form [1]:

$$u_n(x)=\frac{1}{B_n\Delta^*_{k_j}}\begin{vmatrix} y_n(x) & y_n(1) & \lambda_n y_n(1)\\ y^*_{k+1}(x) & y^*_{k+1}(1) & \lambda_k y^*_{k+1}(1)+y_k(1)\\ y_j(x) & y_j(1) & \lambda_j y_j(1)\end{vmatrix}\quad (n\neq k,k+1,j), \tag{3.16}$$

$$u_{k+1}(x)=\frac{1}{B_{k+1}\Delta^*_{k_j}}\begin{vmatrix} y_k(x) & y_k(1) & \lambda_k y_k(1)\\ y^*_{k+1}(x) & y^*_{k+1}(1) & \lambda_k y^*_{k+1}(1)+y_k(1)\\ y_j(x) & y_j(1) & \lambda_j y_j(1)\end{vmatrix},$$

where

$$B_n=\|y_n\|^2+(2a\lambda_n+b)|y_n(1)|^2\quad (n\neq k,k+1,j), \tag{3.17}$$

$$B_{k+1}=-\frac{y_k(1)\omega''(\lambda_k)}{2},$$

$$\Delta^*_{k_j}=y_j(1)\left[(\lambda_j-\lambda_k)y^*_{k+1}(1)-y_k(1)\right]$$

and y^*_{k+1} is defined by (2.6).

Note that, the eigenvalues λ_n are real and simple because of $n\neq k,k+1$. From this reason, $B_n\neq 0$ $(n\neq k,k+1)$ [1, Corollary 1.3]. In addition, $\Delta^*_{k_j}\neq 0$ from the condition $y^*_{k+1}(1)(\lambda_j-\lambda_k)\neq y_k(1)$ for the system (iv).

Let $r=1+\max\{k,j\}$. Examining the uniform convergence of the series (3.15) on the interval $[0,1]$ is equivalent to that of the series

$$F_1(x)=\sum_{n=r}^{\infty}(f,u_n)y_n(x). \tag{3.18}$$

Therefore, let's investigate the uniform convergence of the series (3.18):

Firstly, suppose that $\beta=0$ and the sequence $S_{N,1}(x)$ is defined by the partial sum of the series (3.18):

$$S_{N,1}(x)=\sum_{n=r}^{N}(f,u_n)y_n(x). \tag{3.19}$$

From the equality (3.16), we obtain

$$u_n(x)=\frac{y_n(x)}{B_n}+\frac{y_n(1)}{B_n}C^{(1)}_{k_j}(x)+\frac{\lambda_n y_n(1)}{B_n}C^{(2)}_{k_j}(x), \tag{3.20}$$

where

$$C^{(1)}_{k_j}(x)=-\frac{1}{\Delta^*_{k_j}}\begin{vmatrix} y^*_{k+1}(x) & \lambda_k y^*_{k+1}(1)+y_k(1)\\ y_j(x) & \lambda_j y_j(1)\end{vmatrix},$$

$$C^{(2)}_{k_j}(x)=\frac{1}{\Delta^*_{k_j}}\begin{vmatrix} y^*_{k+1}(x) & y^*_{k+1}(1)\\ y_j(x) & y_j(1)\end{vmatrix}. \tag{3.21}$$

The inequality

$$\max_{0\le x\le 1}\left|y'_n(x)\right| \le const.\left(1+\sqrt{\lambda_n}\right)\max_{0\le x\le 1}|y_n(x)|. \tag{3.22}$$

is valid for sufficiently large numbers of n [22, Theorem 2].

From the expressions (1.3), (3.1), (3.3) and (3.22), the inequality

$$\left|(a\lambda_n^2+b\lambda_n+c)y_n(1)\right| = \left|y_n'(1)\right| \le const.\left(1+\sqrt{\lambda_n}\right)\max_{0\le x\le 1}|y_n(x)| = O(1)$$

holds. Also,

$$y_n(1) = O(n^{-4}), \tag{3.23}$$

is obtained from the last inequality. By virtue of (3.2), (3.3) and (3.23) in the equality (3.17), we give

$$B_n = \frac{1}{2(n-1)^2\pi^2} + O(n^{-3}) \tag{3.24}$$

and from this,

$$\frac{1}{B_n} = 2(n-1)^2\pi^2 + O(n) \tag{3.25}$$

hods. By using the equalities (3.23) and (3.24) in the equality (3.20), we have

$$u_n(x) = \frac{y_n(x)}{B_n} + \frac{\lambda_n y_n(1)}{B_n}C_{k_j}^{(2)}(x) + O(n^{-2}). \tag{3.26}$$

By considering the equality (3.26),

$$\begin{aligned} S_{N,1}(x) = {} & \sum_{n=r}^{N}\frac{1}{B_n}(f,y_n)y_n(x) + \left(f, C_{k_j}^{(2)}\right)\sum_{n=r}^{N}\frac{\lambda_n y_n(1)}{B_n}y_n(x) \\ & + \sum_{n=r}^{N}O(n^{-2}) \end{aligned} \tag{3.27}$$

is obtained from the sequence (3.19). Consequently, the equality (3.27) by considering the equality (3.3) and (3.25) shows that the formula

$$\begin{aligned} S_{N,1}(x) = {} & \sum_{n=r}^{N}\left(f,\sqrt{2}\sin(n-1)\pi x\right)\sqrt{2}\sin(n-1)\pi x \\ & + \sum_{n=1}^{N}K_n(x) + \left(f, C_{k_j}^{(2)}\right)\sum_{n=r}^{N}\frac{\lambda_n y_n(1)}{B_n}y_n(x) \\ & + \sum_{n=r}^{N}O(n^{-2}) \end{aligned} \tag{3.28}$$

is also valid. Here $K_n(x)$ is given by

$$\begin{aligned} K_n(x) &= (f, \sin(n-1)\pi x)\, O(n^{-1}) + (f\alpha_1, \cos(n-1)\pi x)\, O(n^{-1}) \\ &\quad + \left(f, \alpha_{n,1}(x)\cos(n-1)\pi x\right) O(n^{-1}) \\ &\quad + \left(f, \beta_{n,1}(x)\sin(n-1)\pi x\right) O(n^{-1}) \\ &\quad + O\left(\frac{\delta_{n,1}}{n}\right). \end{aligned} \tag{3.29}$$

The first series to the right of the equality (3.28) is uniformly convergent in the interval $[0,1]$ due to assumption of the theorem while the last series is absolutely and uniformly convergent in same interval. On the other hand, from the equality (3.29) the following inequality is obtained for sufficiently large n:

$$\begin{aligned} |K_n(x)| &\le \frac{const.}{n}\{|(f, \sin(n-1)\pi x)| + |(f\alpha_1, \cos(n-1)\pi x)| \\ &\quad + \left|\left(f, \alpha_{n,1}(x)\cos(n-1)\pi x\right)\right| + \left|\left(f, \beta_{n,1}(x)\sin(n-1)\pi x\right)\right| \\ &\quad + O\left(\delta_{n,1}\right)\} \\ &\le const. \left\{|(f, \sin(n-1)\pi x)|^2 + |(f\alpha_1, \cos(n-1)\pi x)|^2 \right. \\ &\quad \left. + \left|\left(f, \alpha_{n,1}(x)\cos(n-1)\pi x\right)\right|^2 + \left|\left(f, \beta_{n,1}(x)\sin(n-1)\pi x\right)\right|^2 \right\} \\ &\quad + O\left(\frac{\delta_{n,1}}{n}\right). \end{aligned}$$

It is easily seen that,

$$\sum_{n=r}^{\infty} |(f, \sin(n-1)\pi x)|^2 < +\infty, \qquad \sum_{n=r}^{\infty} |(f\alpha_1, \cos(n-1)\pi x)|^2 < +\infty,$$

$$\sum_{n=r}^{\infty} \left| O\left(\frac{\delta_{n,1}}{n}\right) \right| < +\infty.$$

And, by virtue of Bessel inequality, we obtain

$$\begin{aligned}\sum_{n=r}^{\infty}\left|\left(f,\alpha_{n,1}(x)\cos(n-1)\pi x\right)\right|^2 &= \sum_{n=r}^{\infty}\left|\int_0^1 f(x)\alpha_{n,1}(x)\cos(n-1)\pi x dx\right|^2 \\ &\leq \sum_{n=r}^{\infty}\left[\int_0^1 |f(x)\alpha_{n,1}(x)|\, dx\right]^2 \\ &\leq \|f\|^2\int_0^1\left[\sum_{n=r}^{\infty}\left|\int_0^x q(\tau)\cos(n-1)\pi\tau d\tau\right|^2\right]dx \\ &\leq const.\|f\|^2\int_0^1\int_0^x |q(\tau)|^2 d\tau dx \\ &\leq const.\|f\|^2\|q\|^2.\end{aligned}$$

Similarly, we also obtain

$$\sum_{n=r}^{\infty}\left|\left(f,\beta_{n,1}(x)\sin(n-1)\pi x\right)\right|^2 \leq const.\|f\|^2\|q\|^2.$$

Therefore, the series $\sum\limits_{n=1}^{N} K_n(x)$ in the equality (3.28) is absolutely and uniformly convergent on the interval $[0,1]$.

At now, let us investigate the uniform convergence of the third series to the right of the equality (3.28):

By (3.21), we have

$$\left(f,C_{k_j}^{(2)}\right) = \frac{1}{\Delta_{k_j}^*}\left[y_j(1)(f,y_{k+1}^*) - y_{k+1}^*(1)(f,y_j)\right].$$

If $(f,C_{k_j}^{(2)}) = 0$, $y_j(1)(f,y_{k+1}^*) = y_{k+1}^*(1)(f,y_j)$. Hence, this completes the proof of the second part of the theorem.

Suppose that, $(f,C_{k_j}^{(2)}) \neq 0$. For the sufficiently large of n, the zeros of the functions $y_{n+1}(x)$ and $y_n(x)$ in the whole of $(0,1)$ are ranked one after the other. Namely, the k^{th} zero of $y_{n+1}(x)$ is less than the k^{th} zero of $y_n(x)$ [17, p.14, Comparison Theorem]. Furthermore, $y_n(x) > 0$ with the initial condition (2.2) in $(0,x_n^{(1)})$. Here, $x_n^{(1)} > 0$ is the zero nearest to $x=0$ of the function $y_n(x)$. According to these discussions, the inequality

$$y_{n+1}(1)y_n(1) < 0 \tag{3.30}$$

can be verified. From (3.23) and (3.30)

$$y_n(1) = \frac{(-1)^n k_n}{n^4}, \tag{3.31}$$

holds, and using (3.1), (3.3) and (3.25) together with (3.31)

$$\frac{\lambda_n y_n(1)}{B_n} y_n(x) = \frac{\widetilde{k}_n}{n} \sin((n-1)\pi(1+x)) + O(n^{-2})$$

is obtained, where k_n and $\widetilde{k}_n$ are a positive (or negative) and bounded sequence of all terms. On the other hand, the third series to the right of the equality (3.28) is uniformly convergent in the interval $[0,r]$ $(0<r<1)$ since $\frac{\widetilde{k}_n}{n} \to 0$ and the relation

$$\begin{aligned}
\left|\sum_{n=r}^{N} \sin((n-1)\pi(1+x))\right| &= \frac{1}{2\left|\sin\frac{\pi(1+x)}{2}\right|}\left|\sum_{n=r}^{N} \sin((n-1)\pi(1+x))\sin\frac{\pi(1+x)}{2}\right| \\
&= \frac{\left|\cos\frac{(2r-3)\pi(1+x)}{2} - \cos\frac{(2N-1)\pi(1+x)}{2}\right|}{2\left|\sin\frac{\pi(1+x)}{2}\right|} \\
&\le \frac{1}{\sin\frac{\pi(1+r)}{2}} \qquad (0 \le x \le r < 1)
\end{aligned}$$

is verify [2, Abel's Lemma].

Finally, suppose that $\beta \neq 0$. Then, if we consider the expressions (3.4), (3.5), (3.17), (3.22) and the partial sum of the series (3.27), we have the following equalities

$$y_n(1) = O(n^{-3}), \tag{3.32}$$

$$\frac{1}{B_n} = 2 + O(n^{-1}), \tag{3.33}$$

$$\begin{aligned}
S_{N,1}(x) = &\sum_{n=r}^{N}\left(f, \sqrt{2}\cos\left(n-\frac{3}{2}\right)\pi x\right)\sqrt{2}\cos\left(n-\frac{3}{2}\right)\pi x \\
&+ \sum_{n=1}^{N} R_n(x) + \left(f, C_{k_j}^{(2)}\right)\sum_{n=r}^{N}\frac{\lambda_n y_n(1)}{B_n} y_n(x) \\
&+ \sum_{n=r}^{N} O(n^{-2}),
\end{aligned} \tag{3.34}$$

where

$$
\begin{aligned}
R_n(x) = &\left(f, \cos\left(n-\frac{3}{2}\right)\pi x\right) O(n^{-1}) + \left(f\alpha_2, \sin\left(n-\frac{3}{2}\right)\pi x\right) O(n^{-1}) \\
&+ \left(f, \alpha_{n,2}(x) \sin\left(n-\frac{3}{2}\right)\pi x\right) O(n^{-1}) \\
&+ \left(f, \beta_{n,2}(x) \cos\left(n-\frac{3}{2}\right)\pi x\right) O(n^{-1}) \\
&+ O\left(\frac{\delta_{n,2}}{n}\right).
\end{aligned}
$$

The first series to the right of the equality (3.34) is uniformly convergent in the interval $[0,1]$ due to assumption of the theorem while the last series is absolutely and uniformly convergent in same interval. On the other hand, showing that the series $\sum_{n=1}^{N} R_n(x)$ in (3.34) is absolutely and uniformly convergent in the interval $[0,1]$ is completely similar to that of the series $\sum_{n=1}^{N} K_n(x)$.

If $(f, C_{k_j}^{(2)}) = 0$, then the second part of the theorem was proven.

Suppose that, $(f, C_{k_j}^{(2)}) \neq 0$. From (3.1), (3.5), (3.30), (3.32) and (3.33), the equality

$$
\frac{\lambda_n y_n(1)}{B_n} y_n(x) = \frac{\widetilde{r}_n}{n} \sin\left(\left(n-\frac{3}{2}\right)\pi(1+x)\right) + O(n^{-2})
$$

is obtained, where $\widetilde{r}_n$ is a positive (or negative) and bounded sequence of all terms.

Together with this last equality, we obtain that the third series to the right of the equality (3.34) is uniformly convergent in the interval $[0,r]$ $(0<r<1)$ with similar calculations as in the above discussion.

So, the proof of the theorem 2 is completed. □

Theorem 3. *Suppose that* $f \in C[0,1]$ *and* $f(x)$ *has a uniformly convergent Fourier expansion in the system* $\{\theta_n(x)\}_{n=1}^{\infty}$ *on the interval* $[0,1]$, *then this function can be expanded in Fourier series in each of the systems* $\{u_n(x)\}$ *which are biorthogonally conjugates to the systems* (i)-(xii) *and these expansions are uniformly convergent on the interval* $[0,1]$.

Proof. The theorem will only be proven for the system (3.16) which is biorthogonally conjugate to the system (iv). The proof of the theorem is similar for other systems.

Consider the Fourier series of $f(x)$ in the system (3.16) on the interval $[0,1]$:

$$G(x) = \sum_{n=0;n\neq k,j}^{\infty} (f, y_n) u_n(x). \tag{3.35}$$

The series (3.35) is uniformly convergent on the interval $[0,1]$ if and only if the series

$$G_1(x) = \sum_{n=r}^{\infty} (f, y_n) u_n(x) \tag{3.36}$$

is uniformly convergent on the interval $[0,1]$, where $r = 1 + \max\{k, j\}$.

Let $\beta = 0$ and the sequence $S_{N,2}(x)$ is defined by the partial sum of the series (3.36):

$$S_{N,2}(x) = \sum_{n=r}^{N} (f, y_n) u_n(x). \tag{3.37}$$

By using the equality (3.26), we obtain

$$\begin{aligned} S_{N,2}(x) = &\sum_{n=r}^{N} \frac{1}{B_n} (f, y_n) y_n(x) + C_{k_j}^{(2)}(x) \sum_{n=r}^{N} \frac{\lambda_n y_n(1)}{B_n} (f, y_n) \\ &+ \sum_{n=r}^{N} O(n^{-3}). \end{aligned} \tag{3.38}$$

The first sequences to the right of the equality (3.28) and (3.38) are the same. Therefore, the first sequence to the right of the equality (3.38) is uniformly convergent on the interval $[0,1]$. On the other hand, the equality

$$\sum_{n=r}^{N} \frac{\lambda_n y_n(1)}{B_n} (f, y_n) = \sum_{n=r}^{N} (f, \sin(n-1)\pi x)\, O(n^{-1})$$

is valid from (3.1), (3.3), (3.23) and (3.26). From here, the inequality

$$|(f, \sin(n-1)\pi x) O(n^{-1})| \leq const. \left\{ |(f, \sin(n-1)\pi x)|^2 + \frac{1}{n^2} \right\}$$

holds. The numerical series $\sum_{n=r}^{\infty} |(f, \sin(n-1)\pi x)|^2$ is convergent. Namely, the second sequences to the right of the equality (3.38) is absolutely and uniformly convergent on $[0,1]$.

On the other hand, the proof of theorem can be also proven similar in the case $0 < \beta < \pi$.

So, the proof of the theorem 3 is completed. □

4. EXAMPLE

In this section, an example is given to make the theorem 2 more understandable. This is an example of the root functions systems (iv) in the case of $0 < \beta < \pi$.

Example 1. *Consider the spectral problem*

$$-y'' = \lambda y, \quad 0 < x < 1, \tag{4.1}$$

$$y'(0) = 0, \quad y'(1) = \left(\frac{\lambda^2}{\pi^2} - \lambda\right) y(1), \tag{4.2}$$

where λ is a spectral parameter.

For this problem, $\lambda_0 = \lambda_1 = 0$ is double eigenvalue corresponding eigenfunction $y_0(x) \equiv 1$ and associated function $y_1(x) \equiv -\frac{x^2}{2} + c$ respectively, $\lambda_2 = \pi^2$ is a simple eigenvalue corresponding eigenfunction $y_2(x) \equiv \cos \pi x$ and all other simple eigenvalues $\lambda_3 < \lambda_4 < ...$ are the solutions of the equation

$$\tan \sqrt{\lambda} = \sqrt{\lambda} \left(1 - \frac{\lambda}{\pi^2}\right)$$

corresponding to eigenfunctions $y_n(x) = \cos \sqrt{\lambda_n} x$ $(n \geq 3)$, where c is an arbitrary constant (see also [1]).

Since $y(x, \lambda) = \cos \sqrt{\lambda} x$, then $\widetilde{y_1}(x) = \lim\limits_{\lambda \to 0} y_\lambda(x, \lambda) = -\frac{x^2}{2}$. Note that $\widetilde{c} = -y_1(0) = -c$, $\beta = -\frac{\pi}{2} \neq 0$. Therefore, by (2.6), $y_1^*(x) = -\frac{x^2}{2} + \frac{3\pi^2+15}{5\pi^2+15} - c$.

Consider the system

$$\left\{ \frac{x^2}{2} + c, \cos \sqrt{\lambda_n} x \ (n \geq 3) \right\},$$

that is the system of root functions (2.3) without removed functions $y_0(x) \equiv 1$ and $y_2(x) = \cos \pi x$. This system is a basis in $L_p(0,1), 1 < p < \infty$ if and if only $y_1^*(1)(\lambda_2 - \lambda_0) \neq y_0(1)$ or $c \neq \frac{\pi^2+15}{10(\pi^2+3)} - \frac{1}{\pi^2}$.

Let $f(x) = x - 1$. Since $\left(f, \cos\left(n - \frac{1}{2}\right)\pi x\right) = O\left(n^{-2}\right)$, then the function $f(x)$ can be expanded in Fourier series of the system $\left\{\sqrt{2}\cos(n - \frac{1}{2})\pi x\right\}_{n=1}^{\infty}$. It is easily calculated that $(f, y_1^*) = \frac{1}{24} - \frac{c}{2}$ and $(f, y_2) = -\frac{2}{\pi^2}$. From here, if $c = -\frac{2\pi^2+30}{(5\pi^2+15)(\pi^2-4)}$, then $(f, y_1^*)y_2(1) = (f, y_2)y_1^*(1)$.

Consequently, from theorem 2, if $c \neq \frac{\pi^2+15}{10(\pi^2+3)} - \frac{1}{\pi^2}, -\frac{2\pi^2+30}{(5\pi^2+15)(\pi^2-4)}$, then Fourier series of $f(x)$ is uniformly convergent on every interval $[0, r]$, $0 < r < 1$; $c = -\frac{2\pi^2+30}{(5\pi^2+15)(\pi^2-4)}$, then Fourier series of $f(x)$ is uniformly convergent on $[0, 1]$.

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