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ON PELL HYBRINOMIALS

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Abstract. Hybrid numbers generalize complex, hyperbolic and dual numbers, simultaneously. Special kinds of hybrid numbers, related to numbers of Fibonacci type, among others Pell numbers, were introduced quite recently. In this paper we introduce and study polynomials, which are a generalization of Pell hybrid numbers and so called Pell hybrinomials.

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1. INTRODUCTION

Pell numbers are well-known numbers in the number theory and they belong to the wide class of numbers of the Fibonacci type. The nth Pell number P_n is defined recursively by the second order linear recurrence relation $P_n = 2P_{n-1} + P_{n-2}$, for $n \geq 2$ with initial conditions $P_0 = 0$, $P_1 = 1$. A special version of Pell numbers is Pell-Lucas numbers Q_n (also named as companion Pell numbers). Then $Q_n = 2Q_{n-1} + Q_{n-2}$, for $n \geq 2$ with $Q_0 = Q_1 = 2$.

Distinct properties of Pell and Pell-Lucas numbers can be found for example in [1, 2, 5]. In [3] Horadam and Mahon introduced Pell and Pell-Lucas polynomials as follows.

For any variable quantity x, the Pell polynomial $P_n(x)$ is defined as $P_n(x) = 2x \cdot P_{n-1}(x) + P_{n-2}(x)$ for $n \geq 2$ with $P_0(x) = 0$, $P_1(x) = 1$.

The Pell-Lucas polynomial $Q_n(x)$ is defined as $Q_n(x) = 2x \cdot Q_{n-1}(x) + Q_{n-2}(x)$ for $n \geq 2$ with initial terms $Q_0(x) = 2$, $Q_1(x) = 2x$.

For $x = 1$ we obtain Pell and Pell-Lucas numbers, respectively.

For any x let $\alpha(x) = x + \sqrt{x^2+1}$ and $\beta(x) = x - \sqrt{x^2+1}$. Then solving second–order linear recurrence relations, for $P_n(x)$ and $Q_n(x)$, respectively, we have

$$P_n(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)} \tag{1.1}$$

and

$$Q_n(x) = \alpha^n(x) + \beta^n(x). \tag{1.2}$$

One of the generalizations of the Pell polynomial is the Horadam polynomial, whose properties can be found in [4].

Hybrid numbers were introduced by Özdemir in [6] as a new generalization of complex, hyperbolic and dual numbers.

Let $\mathbb{K}$ be the set of hybrid numbers $\mathbf{Z}$ of the form

$$\mathbf{Z} = a + b\mathbf{i} + c\varepsilon + d\mathbf{h},$$

where $a, b, c, d \in \mathbb{R}$ and $\mathbf{i}$, ε, $\mathbf{h}$ are operators such that

$$\mathbf{i}^2 = -1,\ \varepsilon^2 = 0,\ \mathbf{h}^2 = 1 \tag{1.3}$$

and

$$\mathbf{ih} = -\mathbf{hi} = \varepsilon + \mathbf{i}. \tag{1.4}$$

If $\mathbf{Z}_1 = a_1 + b_1\mathbf{i} + c_1\varepsilon + d_1\mathbf{h}$, and $\mathbf{Z}_2 = a_2 + b_2\mathbf{i} + c_2\varepsilon + d_2\mathbf{h}$, are any two hybrid numbers then equality, addition, substraction and multiplication by scalar are defined.

Equality: $\mathbf{Z}_1 = \mathbf{Z}_2$ only if $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$, $d_1 = d_2$,
addition: $\mathbf{Z}_1 + \mathbf{Z}_2 = (a_1 + a_2) + (b_1 + b_2)\mathbf{i} + (c_1 + c_2)\varepsilon + (d_1 + d_2)\mathbf{h}$,
substraction: $\mathbf{Z}_1 - \mathbf{Z}_2 = (a_1 - a_2) + (b_1 - b_2)\mathbf{i} + (c_1 - c_2)\varepsilon + (d_1 - d_2)\mathbf{h}$,
multiplication by scalar $s \in \mathbb{R}$: $s\mathbf{Z}_1 = sa_1 + sb_1\mathbf{i} + sc_1\varepsilon + sd_1\mathbf{h}$.

The hybrid numbers multiplication is defined using (1.3) and (1.4). Note that using formulas (1.3) and (1.4) we can find the product of any two hybrid units. The following Table presents products of $\mathbf{i}$, ε, and $\mathbf{h}$ Using rules given in Table 1. the

TABLE 1. The hybrid number multiplication.

$\cdot$	$\mathbf{i}$	ε	$\mathbf{h}$
$\mathbf{i}$	-1	$1-\mathbf{h}$	$\varepsilon+\mathbf{i}$
ε	$\mathbf{h}+1$	0	$-\varepsilon$
$\mathbf{h}$	$-\varepsilon-\mathbf{i}$	ε	1

multiplication of hybrid numbers can be made analogously as multiplications of algebraic expressions. For hybrid numbers details, see [6].

A special kind of hybrid numbers, namely Pell hybrid numbers and Pell-Lucas hybrid numbers, were introduced in [7] as follows.

The nth Pell hybrid number PH_n and the nth Pell-Lucas hybrid number QH_n are defined as

$$PH_n = P_n + \mathbf{i}P_{n+1} + \varepsilon P_{n+2} + \mathbf{h}P_{n+3}, \tag{1.5}$$

$$QH_n = Q_n + \mathbf{i}Q_{n+1} + \varepsilon Q_{n+2} + \mathbf{h}Q_{n+3}, \tag{1.6}$$

respectively.

Interesting results of Pell and Pell-Lucas hybrid numbers obtained recently can be found in [8].

In this paper we introduce Pell and Pell-Lucas hybrinomials, i.e. polynomials, which are a generalization of Pell hybrid numbers and Pell-Lucas hybrid numbers, respectively.

For $n \geq 0$ Pell and Pell-Lucas hybrinomials are defined by

$$PH_n(x) = P_n(x) + \mathbf{i}P_{n+1}(x) + \varepsilon P_{n+2}(x) + \mathbf{h}P_{n+3}(x) \tag{1.7}$$

and

$$QH_n(x) = Q_n(x) + \mathbf{i}Q_{n+1}(x) + \varepsilon Q_{n+2}(x) + \mathbf{h}Q_{n+3}(x), \tag{1.8}$$

where $P_n(x)$ is the nth Pell polynomial, $Q_n(x)$ is the the n-th Pell-Lucas polynomial and $\mathbf{i}$, ε, $\mathbf{h}$ are hybrid units satisfy (1.3) and (1.4).

For $x = 1$ we obtain Pell hybrid numbers and Pell-Lucas hybrid numbers, respectively.

2. Properties of Pell and Pell-Lucas hybrinomials

Theorem 1. *Let $n \geq 0$ be an integer. For any variable quantity x, we have*

$$PH_n(x) = 2x \cdot PH_{n-1}(x) + PH_{n-2}(x) \text{ for } n \geq 2 \tag{2.1}$$

with $PH_0(x) = \mathbf{i} + \varepsilon \cdot (2x) + \mathbf{h} \cdot (4x^2 + 1)$
and $PH_1(x) = 1 + \mathbf{i} \cdot (2x) + \varepsilon \cdot (4x^2 + 1) + \mathbf{h} \cdot (8x^3 + 4x)$.

Proof. If $n = 2$ we have

$$\begin{aligned}
PH_2(x) &= 2x \cdot PH_1(x) + PH_0(x) \\
&= 2x \cdot (1 + \mathbf{i} \cdot (2x) + \varepsilon \cdot (4x^2 + 1) + \mathbf{h} \cdot (8x^3 + 4x)) \\
&\quad + \mathbf{i} + \varepsilon \cdot (2x) + \mathbf{h} \cdot (4x^2 + 1) \\
&= 2x + \mathbf{i} \cdot (4x^2 + 1) + \varepsilon \cdot (8x^3 + 4x) + \mathbf{h} \cdot (16x^4 + 12x^2 + 1) \\
&= P_2(x) + \mathbf{i}P_3(x) + \varepsilon P_4(x) + \mathbf{h}P_5(x).
\end{aligned}$$

If $n \geq 3$ then using the definition of Pell polynomials we have

$$\begin{aligned}
PH_n(x) &= P_n(x) + \mathbf{i}P_{n+1}(x) + \varepsilon P_{n+2}(x) + \mathbf{h}P_{n+3}(x) \\
&= (2x \cdot P_{n-1}(x) + P_{n-2}(x)) + \mathbf{i}(2x \cdot P_n(x) + P_{n-1}(x)) \\
&\quad + \varepsilon(2x \cdot P_{n+1}(x) + P_n(x)) + \mathbf{h}(2x \cdot P_{n+2}(x) + P_{n+1}(x)) \\
&= 2x\,(P_{n-1}(x) + \mathbf{i} \cdot P_n(x) + \varepsilon \cdot P_{n+1}(x) + \mathbf{h} \cdot P_{n+2}(x)) \\
&\quad + P_{n-2}(x) + \mathbf{i} \cdot P_{n-1}(x) + \varepsilon \cdot P_n(x) + \mathbf{h} \cdot P_{n+1}(x) \\
&= 2x \cdot PH_{n-1}(x) + PH_{n-2}(x),
\end{aligned}$$

which ends the proof. □

In the same way one can easily prove the next theorem.

Theorem 2. *Let $n \geq 0$ be an integer. For any variable quantity x, we have*

$$QH_n(x) = 2x \cdot QH_{n-1}(x) + QH_{n-2}(x) \text{ for } n \geq 2 \tag{2.2}$$

with $QH_0(x) = 2 + \mathbf{i} \cdot (2x) + \varepsilon \cdot (4x^2 + 2) + \mathbf{h} \cdot (8x^3 + 6x)$ *and* $QH_1(x) = 2x + \mathbf{i} \cdot (4x^2 + 2) + \varepsilon \cdot (8x^3 + 6x) + \mathbf{h} \cdot (16x^4 + 16x^2 + 2)$.

Now we give so called Binet formulas for Pell and Pell-Lucas hybrinomials.

Theorem 3. *Let $n \geq 0$ be an integer. Then*

$$\begin{aligned} PH_n(x) = & \frac{\alpha^n(x)}{\alpha(x) - \beta(x)} \left(1 + \mathbf{i}\alpha(x) + \varepsilon\alpha^2(x) + \mathbf{h}\alpha^3(x)\right) \\ & - \frac{\beta^n(x)}{\alpha(x) - \beta(x)} \left(1 + \mathbf{i}\beta(x) + \varepsilon\beta^2(x) + \mathbf{h}\beta^3(x)\right), \end{aligned} \tag{2.3}$$

where $\alpha(x) = x + \sqrt{x^2 + 1}$ *and* $\beta(x) = x - \sqrt{x^2 + 1}$.

Proof. Using (1.1), (1.5) and (1.7) we have

$$\begin{aligned} PH_n(x) &= P_n(x) + \mathbf{i} P_{n+1}(x) + \varepsilon P_{n+2}(x) + \mathbf{h} P_{n+3}(x) \\ &= \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)} + \mathbf{i}\frac{\alpha^{n+1}(x) - \beta^{n+1}(x)}{\alpha(x) - \beta(x)} \\ &\quad + \varepsilon\frac{\alpha^{n+2}(x) - \beta^{n+2}(x)}{\alpha(x) - \beta(x)} + \mathbf{h}\frac{\alpha^{n+3}(x) - \beta^{n+3}(x)}{\alpha(x) - \beta(x)} \end{aligned}$$

and after calculations the result follows. □

In the same way, using (1.2), (1.6) and (1.8), one can easily prove the next theorem.

Theorem 4. *Let $n \geq 0$ be an integer. Then*

$$\begin{aligned} QH_n(x) = & \alpha^n(x) \left(1 + \mathbf{i}\alpha(x) + \varepsilon\alpha^2(x) + \mathbf{h}\alpha^3(x)\right) \\ & + \beta^n(x) \left(1 + \mathbf{i}\beta(x) + \varepsilon\beta^2(x) + \mathbf{h}\beta^3(x)\right), \end{aligned} \tag{2.4}$$

where $\alpha(x) = x + \sqrt{x^2 + 1}$ *and* $\beta(x) = x - \sqrt{x^2 + 1}$.

Now we will give some identities related to the well-known identities for classical Pell numbers

$$\text{(Catalan identity)}\, P_{n-r} \cdot P_{n+r} - (P_n)^2 = (-1)^{n-r+1} P_r^2,$$

$$\text{(Cassini identity)}\, P_{n-1} \cdot P_{n+1} - (P_n)^2 = (-1)^n,$$

$$\text{(d'Ocagne identity)}\, P_m \cdot P_{n+1} - P_{m+1} \cdot P_n = (-1)^n P_{m-n}.$$

We give their versions for Pell and Pell-Lucas hybrinomials. These identities can be proved using Binet formulas.

For simplicity of notation let
$\Delta(x) = \alpha(x) - \beta(x)$,

$\hat{\alpha}(x) = 1 + \mathbf{i}\alpha(x) + \varepsilon\alpha^2(x) + \mathbf{h}\alpha^3(x)$,
$\hat{\beta}(x) = 1 + \mathbf{i}\beta(x) + \varepsilon\beta^2(x) + \mathbf{h}\beta^3(x)$.

Then we can write (2.3) and (2.4) as

$PH_n(x) = \frac{\alpha^n(x)}{\Delta(x)}\hat{\alpha}(x) - \frac{\beta^n(x)}{\Delta(x)}\hat{\beta}(x)$

and

$QH_n(x) = \alpha^n(x)\hat{\alpha}(x) + \beta^n(x)\hat{\beta}(x)$, respectively.

Moreover, $\alpha(x) \cdot \beta(x) = -1$ and $\Delta^2(x) = 4x^2 + 4$.

Theorem 5 (Catalan identity for Pell hybrinomials)**.** *Let $n \geq 0$, $r \geq 0$ be integers such that $n \geq r$. Then*

$$\begin{aligned}
&PH_{n-r}(x) \cdot PH_{n+r}(x) - (PH_n(x))^2 \\
&= \frac{(-1)^n}{4x^2+4}\hat{\alpha}(x)\hat{\beta}(x)\left(1 - \frac{\beta^r(x)}{\alpha^r(x)}\right) + \frac{(-1)^n}{4x^2+4}\hat{\beta}(x)\hat{\alpha}(x)\left(1 - \frac{\alpha^r(x)}{\beta^r(x)}\right).
\end{aligned}$$

Proof. For integers $n \geq 0$, $r \geq 0$ and $n \geq r$ we have

$$\begin{aligned}
&PH_{n-r}(x) \cdot PH_{n+r}(x) - (PH_n(x))^2 \\
&= \left(\frac{\alpha^{n-r}(x)}{\Delta(x)}\hat{\alpha}(x) - \frac{\beta^{n-r}(x)}{\Delta(x)}\hat{\beta}(x)\right) \cdot \left(\frac{\alpha^{n+r}(x)}{\Delta(x)}\hat{\alpha}(x) - \frac{\beta^{n+r}(x)}{\Delta(x)}\hat{\beta}(x)\right) \\
&\quad - \left(\frac{\alpha^n(x)}{\Delta(x)}\hat{\alpha}(x) - \frac{\beta^n(x)}{\Delta(x)}\hat{\beta}(x)\right) \cdot \left(\frac{\alpha^n(x)}{\Delta(x)}\hat{\alpha}(x) - \frac{\beta^n(x)}{\Delta(x)}\hat{\beta}(x)\right) \\
&= -\frac{\alpha^{n-r}(x)}{\Delta(x)}\hat{\alpha}(x)\frac{\beta^{n+r}(x)}{\Delta(x)}\hat{\beta}(x) - \frac{\beta^{n-r}(x)}{\Delta(x)}\hat{\beta}(x)\frac{\alpha^{n+r}(x)}{\Delta(x)}\hat{\alpha}(x) \\
&\quad + \frac{\alpha^n(x)}{\Delta(x)}\hat{\alpha}(x)\frac{\beta^n(x)}{\Delta(x)}\hat{\beta}(x) + \frac{\beta^n(x)}{\Delta(x)}\hat{\beta}(x)\frac{\alpha^n(x)}{\Delta(x)}\hat{\alpha}(x) \\
&= -\frac{\alpha^{n-r}(x)\beta^{n+r}(x)}{\Delta^2(x)}\hat{\alpha}(x)\hat{\beta}(x) - \frac{\beta^{n-r}(x)\alpha^{n+r}(x)}{\Delta^2(x)}\hat{\beta}(x)\hat{\alpha}(x) \\
&\quad + \frac{\alpha^n(x)\beta^n(x)}{\Delta^2(x)}\hat{\alpha}(x)\hat{\beta}(x) + \frac{\beta^n(x)\alpha^n(x)}{\Delta^2(x)}\hat{\beta}(x)\hat{\alpha}(x) \\
&= \frac{\alpha^n(x)\beta^n(x)}{\Delta^2(x)}\hat{\alpha}(x)\hat{\beta}(x)\left(1 - \frac{\beta^r(x)}{\alpha^r(x)}\right) + \frac{\alpha^n(x)\beta^n(x)}{\Delta^2(x)}\hat{\beta}(x)\hat{\alpha}(x)\left(1 - \frac{\alpha^r(x)}{\beta^r(x)}\right) \\
&= \frac{(-1)^n}{4x^2+4}\hat{\alpha}(x)\hat{\beta}(x)\left(1 - \frac{\beta^r(x)}{\alpha^r(x)}\right) + \frac{(-1)^n}{4x^2+4}\hat{\beta}(x)\hat{\alpha}(x)\left(1 - \frac{\alpha^r(x)}{\beta^r(x)}\right),
\end{aligned}$$

which ends the proof. □

In the same way one can easily prove the next theorem, which gives Catalan identity for Pell-Lucas hybrinomials.

Theorem 6 (Catalan identity for Pell-Lucas hybrinomials)**.** *Let $n \geq 0$, $r \geq 0$ be integers such that $n \geq r$. Then*

$$QH_{n-r}(x) \cdot QH_{n+r}(x) - (QH_n(x))^2$$
$$= (-1)^n \hat{\alpha}(x)\hat{\beta}(x)\left(\frac{\beta^r(x)}{\alpha^r(x)} - 1\right) + (-1)^n \hat{\beta}(x)\hat{\alpha}(x)\left(\frac{\alpha^r(x)}{\beta^r(x)} - 1\right).$$

Note that for $r = 1$ we get Cassini identities for Pell and Pell-Lucas hybrinomials. Moreover, for $r = 1$ we have
$1 - \frac{\beta(x)}{\alpha(x)} = \frac{\alpha(x)-\beta(x)}{\alpha(x)} = \frac{\Delta(x)}{\alpha(x)}$ and $1 - \frac{\alpha(x)}{\beta(x)} = \frac{\beta(x)-\alpha(x)}{\beta(x)} = -\frac{\Delta(x)}{\beta(x)}$.

Corollary 1 (Cassini identities for Pell and Pell-Lucas hybrinomials)**.** *Let $n \geq 0$ be an integer. Then*

$$PH_{n-1}(x) \cdot PH_{n+1}(x) - (PH_n(x))^2$$
$$= \frac{(-1)^{n-1}\beta(x)}{\Delta(x)}\hat{\alpha}(x)\hat{\beta}(x) - \frac{(-1)^{n-1}\alpha(x)}{\Delta(x)}\hat{\beta}(x)\hat{\alpha}(x).$$

$$QH_{n-1}(x) \cdot QH_{n+1}(x) - (QH_n(x))^2$$
$$= (-1)^n \hat{\alpha}(x)\hat{\beta}(x)\left(\frac{\beta(x)}{\alpha(x)} - 1\right) + (-1)^n \hat{\beta}(x)\hat{\alpha}(x)\left(\frac{\alpha(x)}{\beta(x)} - 1\right).$$

Theorem 7 (d'Ocagne identity for Pell hybrinomials)**.** *Let $m \geq 0$, $n \geq 0$ be integers such that $m \geq n$. Then*

$$PH_m(x) \cdot PH_{n+1}(x) - PH_{m+1}(x) \cdot PH_n(x)$$
$$= \frac{(-1)^n \alpha^{m-n}(x)}{\Delta(x)}\hat{\alpha}(x)\hat{\beta}(x) - \frac{(-1)^n \beta^{m-n}(x)}{\Delta(x)}\hat{\beta}(x)\hat{\alpha}(x).$$

Proof. Let m, n be as in the statement of the Theorem. Then

$$PH_m(x) \cdot PH_{n+1}(x) - PH_{m+1}(x) \cdot PH_n(x)$$
$$= \left(\frac{\alpha^m(x)}{\Delta(x)}\hat{\alpha}(x) - \frac{\beta^m(x)}{\Delta(x)}\hat{\beta}(x)\right) \cdot \left(\frac{\alpha^{n+1}(x)}{\Delta(x)}\hat{\alpha}(x) - \frac{\beta^{n+1}(x)}{\Delta(x)}\hat{\beta}(x)\right)$$
$$- \left(\frac{\alpha^{m+1}(x)}{\Delta(x)}\hat{\alpha}(x) - \frac{\beta^{m+1}(x)}{\Delta(x)}\hat{\beta}(x)\right) \cdot \left(\frac{\alpha^n(x)}{\Delta(x)}\hat{\alpha}(x) - \frac{\beta^n(x)}{\Delta(x)}\hat{\beta}(x)\right)$$
$$= \frac{\alpha^{m+n+1}(x)}{\Delta^2(x)}\hat{\alpha}^2(x) - \frac{\alpha^m(x)\beta^{n+1}(x)}{\Delta^2(x)}\hat{\alpha}(x)\hat{\beta}(x) - \frac{\alpha^{n+1}(x)\beta^m(x)}{\Delta^2(x)}\hat{\beta}(x)\hat{\alpha}(x)$$
$$+ \frac{\beta^{m+n+1}(x)}{\Delta^2(x)}\hat{\beta}^2(x) - \frac{\alpha^{m+1+n}(x)}{\Delta^2(x)}\hat{\alpha}^2(x) + \frac{\alpha^{m+1}(x)\beta^n(x)}{\Delta^2(x)}\hat{\alpha}(x)\hat{\beta}(x)$$
$$+ \frac{\alpha^n(x)\beta^{m+1}(x)}{\Delta^2(x)}\hat{\beta}(x)\hat{\alpha}(x) - \frac{\beta^{m+1+n}(x)}{\Delta^2(x)}\hat{\beta}^2(x)$$

$$\begin{aligned}
&= \frac{\alpha^{m+1}(x)\beta^{n}(x)-\alpha^{m}(x)\beta^{n+1}(x)}{\Delta^{2}(x)}\hat{\alpha}(x)\hat{\beta}(x) \\
&\quad + \frac{\alpha^{n}(x)\beta^{m+1}(x)-\alpha^{n+1}(x)\beta^{m}(x)}{\Delta^{2}(x)}\hat{\beta}(x)\hat{\alpha}(x) \\
&= \frac{\alpha^{m}(x)\beta^{n}(x)(\alpha(x)-\beta(x))}{\Delta^{2}(x)}\hat{\alpha}(x)\hat{\beta}(x) \\
&\quad + \frac{\alpha^{n}(x)\beta^{m}(x)(\beta(x)-\alpha(x))}{\Delta^{2}(x)}\hat{\beta}(x)\hat{\alpha}(x) \\
&= \frac{\alpha^{m}(x)\beta^{n}(x)}{\Delta(x)}\hat{\alpha}(x)\hat{\beta}(x) - \frac{\alpha^{n}(x)\beta^{m}(x)}{\Delta(x)}\hat{\beta}(x)\hat{\alpha}(x) \\
&= \frac{(-1)^{n}\alpha^{m-n}(x)}{\Delta(x)}\hat{\alpha}(x)\hat{\beta}(x) - \frac{(-1)^{n}\beta^{m-n}(x)}{\Delta(x)}\hat{\beta}(x)\hat{\alpha}(x).
\end{aligned}$$

Thus the Theorem is proved. □

In the same way we can prove next theorems.

Theorem 8 (d'Ocagne identity for Pell-Lucas hybrinomials)**.** *Let* $m \geq 0$, $n \geq 0$ *be integers such that* $m \geq n$. *Then*

$$\begin{aligned}
&QH_{m}(x)\cdot QH_{n+1}(x) - QH_{m+1}(x)\cdot QH_{n}(x) \\
&= (-1)^{n}\beta^{m-n}(x)\Delta(x)\hat{\beta}(x)\hat{\alpha}(x) - (-1)^{n}\alpha^{m-n}(x)\Delta(x)\hat{\alpha}(x)\hat{\beta}(x).
\end{aligned}$$

Theorem 9. *Let* $m \geq 0$, $n \geq 0$ *be integers. Then*

$$\begin{aligned}
&PH_{m}(x)\cdot QH_{n}(x) - QH_{m}(x)\cdot PH_{n}(x) \\
&= \frac{2(-1)^{n}\alpha^{m-n}(x)}{\Delta(x)}\hat{\alpha}(x)\hat{\beta}(x) - \frac{2(-1)^{n}\beta^{m-n}(x)}{\Delta(x)}\hat{\beta}(x)\hat{\alpha}(x).
\end{aligned}$$

Some identities for Pell and Pell-Lucas hybrinomials can be found by analogy to well–known identities for the Pell and Pell-Lucas polynomials. In the next part of this paper we indicate such identities.

Theorem 10 ([3])**.** *Let* $n \geq 1$ *be an integer. Then*

$$P_{n+1}(x) + P_{n-1}(x) = Q_{n}(x) = 2x\cdot P_{n}(x) + 2P_{n-1}(x). \tag{2.5}$$

Theorem 11. *Let* $n \geq 1$ *be an integer. Then*

$$PH_{n+1}(x) + PH_{n-1}(x) = QH_{n}(x) = 2x\cdot PH_{n}(x) + 2PH_{n-1}(x).$$

Proof. Using (2.5) we have

$$\begin{aligned}
&PH_{n+1}(x)+PH_{n-1}(x)\\
&=P_{n+1}(x)+\mathbf{i}P_{n+2}(x)+\varepsilon P_{n+3}(x)+\mathbf{h}P_{n+4}(x)\\
&\quad+P_{n-1}(x)+\mathbf{i}P_{n}(x)+\varepsilon P_{n+1}(x)+\mathbf{h}P_{n+2}(x)\\
&=(P_{n+1}(x)+P_{n-1}(x))+\mathbf{i}(P_{n+2}(x)+P_{n}(x))\\
&\quad+\varepsilon(P_{n+3}(x)+P_{n+1}(x))+\mathbf{h}(P_{n+4}(x)+P_{n+2}(x))\\
&=Q_n(x)+\mathbf{i}Q_{n+1}(x)+\varepsilon Q_{n+2}(x)+\mathbf{h}Q_{n+3}(x)\\
&=QH_n(x).
\end{aligned}$$

On the other hand

$$\begin{aligned}
&2x\cdot PH_n(x)+2PH_{n-1}(x)\\
&=2x\cdot(P_n(x)+\mathbf{i}P_{n+1}(x)+\varepsilon P_{n+2}(x)+\mathbf{h}P_{n+3}(x))\\
&\quad+2(P_{n-1}(x)+\mathbf{i}P_{n}(x)+\varepsilon P_{n+1}(x)+\mathbf{h}P_{n+2}(x))\\
&=(2x\cdot P_n(x)+P_{n-1}(x))+\mathbf{i}(2x\cdot P_{n+1}(x)+P_n(x))\\
&\quad+\varepsilon(2x\cdot P_{n+2}(x)+P_{n+1}(x))+\mathbf{h}(2x\cdot P_{n+3}(x)+P_{n+2}(x))\\
&=Q_n(x)+\mathbf{i}Q_{n+1}(x)+\varepsilon Q_{n+2}(x)+\mathbf{h}Q_{n+3}(x)\\
&=QH_n(x),
\end{aligned}$$

so the result follows. □

Theorem 12 ([3])**.** *Let $n \geq 1$ be an integer. Then*

$$Q_{n+1}(x)+Q_{n-1}(x)=4(x^2+1)P_n(x). \tag{2.6}$$

Theorem 13. *Let $n \geq 1$ be an integer. Then*

$$QH_{n+1}(x)+QH_{n-1}(x)=4(x^2+1)PH_n(x).$$

Proof. Using (2.6) and proceeding in the same way as in the Theorem 11 the result follows. □

Theorem 14 ([3])**.** *Let $n \geq 2$ be an integer. Then*

$$\sum_{l=1}^{n-1}P_l(x)=\frac{P_n(x)+P_{n-1}(x)-1}{2x}. \tag{2.7}$$

Theorem 15. *Let $n \geq 2$ be an integer. Then*

$$\sum_{l=1}^{n-1}PH_l(x)=\frac{PH_n(x)+PH_{n-1}(x)-PH_0(x)-PH_1(x)}{2x}.$$

Proof. For an integer $n \geq 2$ we have

$$\begin{aligned}
\sum_{l=1}^{n-1} PH_l(x) &= PH_1(x) + PH_2(x) + \ldots + PH_{n-1}(x) \\
&= P_1(x) + \mathbf{i}P_2(x) + \varepsilon P_3(x) + \mathbf{h}P_4(x) \\
&\quad + P_2(x) + \mathbf{i}P_3(x) + \varepsilon P_4(x) + \mathbf{h}P_5(x) \\
&\quad + \cdots \\
&\quad + P_{n-1}(x) + \mathbf{i}P_n(x) + \varepsilon P_{n+1}(x) + \mathbf{h}P_{n+2}(x) \\
&= P_1(x) + P_2(x) + \cdots + P_{n-1}(x) \\
&\quad + \mathbf{i}(P_2(x) + P_3(x) + \cdots + P_n(x) + P_1(x) - P_1(x)) \\
&\quad + \varepsilon(P_3(x) + P_4(x) + \cdots + P_{n+1}(x) + P_1(x) + P_2(x) \\
&\quad - P_1(x) - P_2(x)) \\
&\quad + \mathbf{h}(P_4(x) + P_5(x) + \cdots + P_{n+2}(x) + P_1(x) + P_2(x) + P_3(x) \\
&\quad - P_1(x) - P_2(x) - P_3(x)).
\end{aligned}$$

Using (2.7) we obtain

$$\begin{aligned}
\sum_{l=1}^{n-1} PH_l(x) &= \frac{P_n(x) + P_{n-1}(x) - 1}{2x} \\
&\quad + \mathbf{i}\left(\frac{P_{n+1}(x) + P_n(x) - 1}{2x} - P_1(x)\right) \\
&\quad + \varepsilon\left(\frac{P_{n+2}(x) + P_{n+1}(x) - 1}{2x} - P_1(x) - P_2(x)\right) \\
&\quad + \mathbf{h}\left(\frac{P_{n+3}(x) + P_{n+2}(x) - 1}{2x} - P_1(x) - P_2(x) - P_3(x)\right).
\end{aligned}$$

Bringing to the common denominator we have

$$\begin{aligned}
\sum_{l=1}^{n-1} PH_l(x) &= \frac{P_n(x) + P_{n-1}(x) - 1}{2x} \\
&\quad + \mathbf{i}\left(\frac{P_{n+1}(x) + P_n(x) - 1 - 2x}{2x}\right) \\
&\quad + \varepsilon\left(\frac{P_{n+2}(x) + P_{n+1}(x) - 1 - 2x - 4x^2}{2x}\right) \\
&\quad + \mathbf{h}\left(\frac{P_{n+3}(x) + P_{n+2}(x) - 1 - 2x - 4x^2 - 2x(4x^2+1)}{2x}\right)
\end{aligned}$$

and finally

$$\sum_{l=1}^{n-1} PH_l(x) = \frac{P_n(x) + \mathbf{i}P_{n+1}(x) + \varepsilon P_{n+2}(x) + \mathbf{h}P_{n+3}(x)}{2x}$$

$$+ \frac{P_{n-1}(x) + \mathbf{i}P_n(x) + \varepsilon P_{n+1}(x) + \mathbf{h}P_{n+2}(x)}{2x}$$

$$+ \frac{-(0+1) - \mathbf{i}(1+2x) - \varepsilon(2x + (4x^2+1)) - \mathbf{h}((4x^2+1) + (8x^3+4x))}{2x}$$

$$= \frac{PH_n(x) + PH_{n-1}(x) - PH_0(x) - PH_1(x)}{2x}.$$

Thus the Theorem is proved. □

Theorem 16 ([3])**.** *Let $n \geq 2$ be an integer. Then*

$$\sum_{l=1}^{n-1} Q_l(x) = \frac{Q_n(x) + Q_{n-1}(x) - 2 - 2x}{2x}. \tag{2.8}$$

Theorem 17. *Let $n \geq 2$ be an integer. Then*

$$\sum_{l=1}^{n-1} QH_l(x) = \frac{QH_n(x) + QH_{n-1}(x) - QH_0(x) - QH_1(x)}{2x}.$$

Proof. Using (2.8) and proceeding in the same way as in the Theorem 15 the result follows. □

Next we shall give the generating function for Pell hybrinomials.

Theorem 18. *The generating function for Pell hybrinomial sequence $\{PH_n(x)\}$ is*

$$G(t) = \frac{\mathbf{i} + \varepsilon \cdot (2x) + \mathbf{h} \cdot (4x^2+1) + (1 + \varepsilon + \mathbf{h} \cdot (2x))t}{1 - 2xt - t^2}.$$

Proof. Assume that the generating function of the Pell hybrinomial sequence $\{PH_n(x)\}$ has the form $G(t) = \sum_{n=0}^{\infty} PH_n(x)t^n$. Then

$$G(t) = PH_0(x) + PH_1(x)t + PH_2(x)t^2 + \ldots$$

Multiply the above equality on both sides by $-2xt$ and then by $-t^2$ we obtain

$$-G(t) \cdot (2x)t = -PH_0(x) \cdot (2x)t - PH_1(x) \cdot (2x)t^2 - PH_2(x) \cdot (2x)t^3 - \ldots$$

$$-G(t)t^2 = -PH_0(x)t^2 - PH_1(x)t^3 - PH_2(x)t^4 - \dots$$

By adding these three equalities above, we will get

$$G(t)(1-2xt-t^2) = PH_0(x) + (PH_1(x) - PH_0(x)\cdot(2x))t$$

since $PH_n(x) = 2x \cdot PH_{n-1}(x) + PH_{n-2}(x)$ (see (2.1)) and the coefficient of t^n, for $n \geq 2$, are equal to zero. Moreover, $PH_0(x) = \mathbf{i} + \varepsilon \cdot (2x) + \mathbf{h} \cdot (4x^2+1)$, $PH_1(x) - PH_0(x) \cdot (2x) = 1 + \varepsilon + \mathbf{h} \cdot (2x)$. □

In the same way we obtain the generating function $g(t)$ for Pell-Lucas hybrinomials.

Theorem 19. *The generating function for the Pell-Lucas hybrinomial sequence* $\{QH_n(x)\}$ *is*

$$g(t) = \frac{QH_0(x) + (QH_1(x) - QH_0(x)\cdot(2x))t}{1-2xt-t^2},$$

where $QH_0(x) = 2 + \mathbf{i} \cdot (2x) + \varepsilon \cdot (4x^2+2) + \mathbf{h} \cdot (8x^3+6x)$,
and $QH_1(x) - QH_0(x) \cdot (2x) = -2x + 2\mathbf{i} + \varepsilon \cdot (2x) + \mathbf{h} \cdot (4x^2+2)$.

We will give the matrix representation of Pell hybrinomials.

Theorem 20. *Let* $n \geq 0$ *be an integer. Then*

$$\begin{bmatrix} PH_{n+2}(x) & PH_{n+1}(x) \\ PH_{n+1}(x) & PH_n(x) \end{bmatrix} = \begin{bmatrix} PH_2(x) & PH_1(x) \\ PH_1(x) & PH_0(x) \end{bmatrix} \cdot \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix}^n.$$

Proof. (by induction on n)
If $n = 0$ then assuming that the matrix to the power 0 is the identity matrix the result is obvious. Now suppose that for any $n \geq 0$ holds

$$\begin{bmatrix} PH_{n+2}(x) & PH_{n+1}(x) \\ PH_{n+1}(x) & PH_n(x) \end{bmatrix} = \begin{bmatrix} PH_2(x) & PH_1(x) \\ PH_1(x) & PH_0(x) \end{bmatrix} \cdot \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix}^n.$$

We shall show that

$$\begin{bmatrix} PH_{n+3}(x) & PH_{n+2}(x) \\ PH_{n+2}(x) & PH_{n+1}(x) \end{bmatrix} = \begin{bmatrix} PH_2(x) & PH_1(x) \\ PH_1(x) & PH_0(x) \end{bmatrix} \cdot \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix}^{n+1}.$$

By simple calculation using induction's hypothesis we have

$$\begin{aligned} &\begin{bmatrix} PH_2(x) & PH_1(x) \\ PH_1(x) & PH_0(x) \end{bmatrix} \cdot \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix}^n \cdot \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} PH_{n+2}(x) & PH_{n+1}(x) \\ PH_{n+1}(x) & PH_n(x) \end{bmatrix} \cdot \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2x \cdot PH_{n+2}(x) + PH_{n+1}(x) & PH_{n+2}(x) \\ 2x \cdot PH_{n+1}(x) + PH_n(x) & PH_{n+1}(x) \end{bmatrix} \\ &= \begin{bmatrix} PH_{n+3}(x) & PH_{n+2}(x) \\ PH_{n+2}(x) & PH_{n+1}(x) \end{bmatrix}, \end{aligned}$$

which ends the proof. □

In the same way we obtain the matrix representation for Pell-Lucas hybrinomials.

Theorem 21. *Let $n \geq 0$ be an integer. Then*

$$\left[\begin{array}{cc} QH_{n+2}(x) & QH_{n+1}(x) \\ QH_{n+1}(x) & QH_n(x) \end{array}\right] = \left[\begin{array}{cc} QH_2(x) & QH_1(x) \\ QH_1(x) & QH_0(x) \end{array}\right] \cdot \left[\begin{array}{cc} 2x & 1 \\ 1 & 0 \end{array}\right]^n .$$

Compliance with ethical standards

Conflict of Interest: The authors declare that they have no conflict of interest.

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