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SOME RESULTS ON HYBRID RELATIVES OF THE SHEFFER POLYNOMIALS VIA OPERATIONAL RULES

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Abstract. The intended objective of this paper is to introduce a new class of polynomials, namely the extended Laguerre-Gould-Hopper-Sheffer polynomials. The generating function and operational rule are derived by making use of integral transform. Their quasi-monomial properties and determinant forms are also established. Examples of certain members belonging to the extended Laguerre-Gould-Hopper-Sheffer polynomials are constructed and their corresponding results are established.

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1. INTRODUCTION AND PRELIMINARIES

Theory of special functions performs an essential role in the formalism of mathematical physics. They provide indeed a unique tool for developing simplified yet realistic models of physical problems, thus allowing for analytic solutions and hence a deeper insight into the problem under study. The specific physical problem indeed can suggest investigating new aspects of the well-established theory of special functions as well as introducing new family of special polynomials, which usually exhibit deeper features and thereby appear many times in new roles in numerous branches of mathematics and physics. Consequently, reformulating a physical problem in terms of special functions allows a more elegant mathematical model and then for an easier reading and numerical handling of the relevant equations as well as for the discovery of unsuspected connections with other fields of Physics.

The Sheffer polynomials are one of the most important class of polynomial sequences and have been extensively studied not only due to the fact that they arise in numerous branches of mathematics but also because of their importance in applied sciences, such as physics and engineering. In view of the result [6, p.17], the Sheffer polynomials can be defined as:

Let $f(t)$ be a delta series and let $g(t)$ be an invertible series of the following form:

$$f(t)=\sum_{n=0}^{\infty} f_n \frac{t^n}{n!}, \qquad f_0=0,\ f_1\neq 0 \tag{1.1a}$$

and

$$g(t)=\sum_{n=0}^{\infty} g_n \frac{t^n}{n!}, \qquad g_0\neq 0, \tag{1.1b}$$

then the sequences $\{s_n(x)\}_{n\in\mathbb{N}}$ is Sheffer for the pair $(g(t), f(t))$ if and only if the following orthogonality condition holds:

$$\left\langle g(t)\ f(t)^k \,|\, s_n(x)\right\rangle = n!\,\delta_{n,k}\ , \qquad \forall\ n,k\geq 0, \tag{1.2}$$

where $\delta_{n,k}$ is the Kronecker delta.

According to Roman [6, p.18 (Theorem 2.3.4)], the polynomial sequence $s_n(x)$ is uniquely determined by two (formal) power series given by equations (1.1a) and (1.1b). The exponential generating function of the Sheffer polynomials $s_n(x)$ is then given by

$$\frac{1}{g\left(f^{-1}(t)\right)}\exp\left(xf^{-1}(t)\right)=\sum_{n=0}^{\infty} s_n(x)\frac{t^n}{n!}, \tag{1.3}$$

for all x in $\mathbb{C}$, where $f^{-1}(t)$ is the compositional inverse of $f(t)$.

Let $(s_n(x))_{n\in\mathbb{N}}$ be Sheffer sequence for $(g(t), f(t))$ satisfying the following condition:

$$x^n=\sum_{k=0}^{n} a_{n,k}\, s_k(x), \tag{1.4}$$

then $s_n(x)$ can be expressed by the following determinant form [9, p.232]:

$$s_0(x)=\frac{1}{a_{0,0}},$$

$$s_n(x)=\frac{(-1)^n}{a_{0,0}\, a_{1,1}\, \ldots\, a_{n,n}}\begin{vmatrix} 1 & x & x^2 & \cdots & x^{n-1} & x^n \\ a_{0,0} & a_{1,0} & a_{2,0} & \cdots & a_{n-1,0} & a_{n,0} \\ 0 & a_{1,1} & a_{2,1} & \cdots & a_{n-1,1} & a_{n,1} \\ 0 & 0 & a_{2,2} & \cdots & a_{n-1,2} & a_{n,2} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & a_{n-1,n-1} & a_{n,n-1} \end{vmatrix}, \tag{1.5}$$

where $a_{n,k}$ is the (n,k) entry of the Riordan array $(g(t), f(t))$ [10].

Operational methods can be exploited to simplify the derivation of properties associated with ordinary and generalized special functions and to define new families of hybrid special polynomials. The combined use of integral transforms and operational methods provides a powerful tool to deal with fractional derivatives. Using the Euler's integral [7, p.218]:

$$a^{-\nu} = \frac{1}{\Gamma(\nu)} \int_0^\infty t^{\nu-1} e^{-at} dt, \qquad \min\{Re(\nu), Re(a)\} \geq 0, \tag{1.6}$$

a new family of special polynomials are introduced in [1]:

$${}_LH_{n,\nu}^{(m,r)}(x,y,z;\alpha) = \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-\alpha t} t^{\nu-1} {}_LH_n^{(m,r)}(x,y,zt)dt, \tag{1.7}$$

where ${}_LH_n^{(m,r)}(x,y,z)$ are the Laguerre-Gould-Hopper polynomials [4].

The polynomials ${}_LH_{n,\nu}^{(m,r)}(x,y,z)$ are also defined by the following operational rule:

$$\left(\alpha - z\frac{\partial^r}{\partial y^r}\right)^{-\nu} \exp\left(D_x^{-1}\frac{\partial^m}{\partial y^m}\right)\{y^n\} = {}_LH_{n,\nu}^{(m,r)}(x,y,z;\alpha). \tag{1.8}$$

Most of the properties of hybrid special polynomials recognized as quasi-monomial, can be deduced by using operational rules associated with the relevant multiplicative and derivative operators. For the multi-variable hybrid special polynomials, the use of operational techniques combined with the monomiality principle provides new means of analysis for the solution of a wide class of partial differential equations often encountered in physical problems.

According to monomiality principle [2, 8], a given polynomial set $r_n(x)$ $(n \in \mathbb{N},\ x \in \mathbb{C})$ can be considered as *quasi-monomial*, if two operators $\hat{M}$ and $\hat{P}$, called "multiplicative" and "derivative" operators respectively, can be defined in such a way that

$$\hat{M}\{r_n(x)\} = r_{n+1}(x), \tag{1.9}$$

$$\hat{P}\{r_n(x)\} = n r_{n-1}(x), \tag{1.10}$$

for all $n \in \mathbb{N}$. The operators $\hat{M}$ and $\hat{P}$ also satisfy the commutation relation

$$[\hat{P}, \hat{M}] = \hat{P}\hat{M} - \hat{M}\hat{P} = \hat{1} \tag{1.11}$$

and thus display the Weyl group structure.

If the considered polynomial set $\{r_n(x)\}_{n\in\mathbb{N}}$ is quasi-monomial, its properties can easily be derived from those of the $\hat{M}$ and $\hat{P}$ operators.

If $\hat{M}$ and $\hat{P}$ have a differential realization, then

$$\hat{M}\ \hat{P}\{r_n(x)\} = n\ r_n(x), \tag{1.12}$$

can be interpreted as the differential equation satisfied by $r_n(x)$.

The theory of hybrid special polynomials has been one of the most rapidly growing research topic in mathematical analysis. In 2016, N. Raza *et. al.* [5] introduced the Laguerre-Gould-Hopper-Sheffer polynomials
(LGHSP)$_{_LH^{(m,r)}}s_n(x,y,z)$ which are defined by the following generating function:

$$\frac{1}{g\big(f^{-1}(t)\big)} C_0\left(-x\big(f^{-1}(t)\big)^m\right)\exp\left(yf^{-1}(t)+z\big(f^{-1}(t)\big)^r\right) \tag{1.13}$$
$$= \sum_{n=0}^{\infty} {}_LH^{(m,r)}s_n(x,y,z)\frac{t^n}{n!}.$$

The following operational representations for the LGHSP ${}_LH^{(m,r)}s_n(x,y,z)$ hold [5]:

$${}_LH^{(m,r)}s_n(x,y,z) = \exp\left(D_x^{-1}\frac{\partial^m}{\partial y^m} + z\frac{\partial^r}{\partial y^r}\right)s_n(y), \tag{1.14}$$

$${}_LH^{(m,r)}s_n(x,y,z) = \exp\left(z\frac{\partial^r}{\partial y^r}\right){}_mLs_n(x,y) \tag{1.15}$$

and

$${}_LH^{(m,r)}s_n(x,y,z) = \exp\left(D_x^{-1}\frac{\partial^m}{\partial y^m}\right){}_{H^{(r)}}s_n(y,z), \tag{1.16}$$

where $s_n(y), {}_mLs_n(x,y)$ and ${}_{H^{(r)}}s_n(y,z)$ are the Sheffer, 2-variable generalized Laguerre-Sheffer and Gould-Hopper-Sheffer polynomials, respectively.

For suitable choices of the variables and indices the LGHSP ${}_LH^{(m,r)}s_n(x,y,z)$ reduce to certain hybrid special polynomials. These polynomials along with their notation and name are mentioned in Table 1.

This article is an attempt to further stress the importance of the operational methods, Sheffer polynomials and Laguerre-Gould-Hopper polynomials. In Section 2, the extended Laguerre-Gould-Hopper-Sheffer polynomials are introduced by operational rule and generating function. Operational rule providing the connection between the extended Laguerre-Gould-Hopper-Sheffer polynomials and Sheffer polynomials are established. These special polynomials are framed within the context of monomiality principle formalism and their determinant form is also obtained. In the last section, the corresponding results for the extended Laguerre-Gould-Hopper-Gegenbauer polynomials and extended Laguerre-Gould-Hopper-Jacobi polynomials are derived.

TABLE 1. Special cases of the LGHSP ${}_{L}H^{(m,r)}s_n(x,y,z)$.

S. No.	Values of the Indices and Variables	Relation between the LGHSP ${}_{L}H^{(m,r)}s_n(x,y,z)$ and its Special Case	Name of the hybrid special polynomials
I.	$m=1,\ r=2;\ x\to -x$	${}_{L}H^{(1,2)}s_n(-x,y,z)={}_{LH}s_n(x,y,z)$	3-Variable Laguerre-Hermite-Sheffer polynomials
II.	$m=1,\ r=2;\ y=1,$ $z\to y, x\to -x$	${}_{L}H^{(1,2)}s_n(-x,1,y)={}_{\varphi}s_n(x,y)$	Laguerre-Hermite type-Sheffer polynomials
III.	$x=0$	${}_{L}H^{(m,r)}s_n(0,y,z)={}_{H^{(r)}}s_n(y,z)$	Gould-Hopper-Sheffer polynomials
IV.	$z=0$	${}_{L}H^{(m,r)}s_n(x,y,0)={}_{mL_n}s_n(x,y)$	2-Variable generalized Laguerre-Sheffer polynomials
V.	$r=m;\ x=0,$ $y\to -D_x^{-1}, z\to y$	${}_{L}H^{(m,m)}s_n(0,-D_x^{-1},y)={}_{[m]L}s_n(x,y)$	2-Variable generalized Laguerre type-Sheffer polynomials
VI.	$r=m-1;\ x=0,$ $y\to x, z\to y$	${}_{L}H^{(m,m-1)}s_n(0,x,y)={}_{U^{(m)}}s_n(x,y)$	Generalized Chebyshev-Sheffer polynomials
VII.	$m=1;\ z=0,$ $x\to -x$	${}_{L}H^{(1,r)}s_n(-x,y,0)={}_{L}s_n(x,y)$	2-Variable Laguerre-Sheffer polynomials
VIII.	$x=0,\ y\to x,$ $z\to yD_y y$	${}_{L}H^{(m,r)}s_n(0,x,yD_y y)={}_{e^{(r)}}s_n(x,y)$	2-Variable truncated-Sheffer polynomials of order r
IX.	$r=2;\ x=0$	${}_{L}H^{(m,2)}s_n(0,y,z)={}_{H}s_n(y,z)$	2-Variable Hermite-Kampé de Fériet-Sheffer polynomials
X.	$x\to yD_y y,$ $y\to x$	${}_{L}H^{(m,r)}s_n(yD_y\ y,x,z)$ $={}_{H^{(r,m)}}s_n(x,y,z)$	3-Variable generalized Hermite-Sheffer polynomials
XI.	$m=2,\ r=3;\ y\to x,$ $z\to y, x\to zD_z z,$	${}_{L}H_n^{(2,3)}s_n(zD_z\ z,x,y)$ $={}_{H^{(3,2)}}s_n(x,y,z)$	Bell-type-Sheffer polynomials

2. EXTENDED LAGUERRE-GOULD-HOPPER-SHEFFER POLYNOMIALS

In order to introduce the extended Laguerre-Gould-Hopper-Sheffer polynomials (ELGHSP) denoted by ${}_{L}H^{(m,r)}s_{n,\nu}(x,y,z;\alpha)$, first we prove the following theorem:

Theorem 1. *For the extended Laguerre-Gould-Hopper-Sheffer polynomials* ${}_LH^{(m,r)}s_{n,\nu}(x,y,z;\alpha)$, *the following operational rule holds true:*

$$\left(\alpha - z\frac{\partial^r}{\partial y^r}\right)^{-\nu} {}_mLs_n(x,y) = {}_LH^{(m,r)}s_{n,\nu}(x,y,z;\alpha). \tag{2.1}$$

Proof. Replacing a by $\left(\alpha - z\frac{\partial^r}{\partial y^r}\right)$ in relation (1.6) and then operating the resultant equation on 2-variable generalized Laguerre-Sheffer polynomials ${}_mLs_n(x,y)$, it follows that

$$\left(\alpha - z\frac{\partial^r}{\partial y^r}\right)^{-\nu} {}_mLs_n(x,y) = \frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}t^{\nu-1}\exp\left(zt\frac{\partial^r}{\partial y^r}\right) {}_mLs_n(x,y)dt, \tag{2.2}$$

which in view of equation (1.15) gives

$$\left(\alpha - z\frac{\partial^r}{\partial y^r}\right)^{-\nu} {}_mLs_n(x,y) = \frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}t^{\nu-1} {}_LH^{(m,r)}s_n(x,y,zt)dt. \tag{2.3}$$

Denoting the transform on the r.h.s of equation (2.3) by a new class of extended Laguerre-Gould-Hopper-Sheffer polynomials (ELGHSP) ${}_LH^{(m,r)}s_{n,\nu}(x,y,z;\alpha)$, so that we have

$${}_LH^{(m,r)}s_{n,\nu}(x,y,z;\alpha) = \frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}t^{\nu-1} {}_LH^{(m,r)}s_n(x,y,zt)dt. \tag{2.4}$$

In view of equations (2.3) and (2.4), assertion (2.1) follows.

□

Next, the generating function of the ELGHSP ${}_LH^{(m,r)}s_{n,\nu}(x,y,z;\alpha)$ is obtained by proving the following result:

Theorem 2. *The following generating function for the ELGHSP* ${}_LH^{(m,r)}s_{n,\nu}(x,y,z;\alpha)$ *holds true:*

$$\frac{\exp(yf^{-1}(u))C_0(-x(f^{-1}(u))^m)}{g(f^{-1}(u))(\alpha - z(f^{-1}(u))^r)^\nu} = \sum_{n=0}^\infty {}_LH^{(m,r)}s_{n,\nu}(x,y,z;\alpha)\frac{u^n}{n!}. \tag{2.5}$$

Proof. Multiplying on both sides of equation (2.4) by $\frac{u^n}{n!}$ and summing over n, we find

$$\sum_{n=0}^\infty {}_LH^{(m,r)}s_{n,\nu}(x,y,z;\alpha)\frac{u^n}{n!} = \sum_{n=0}^\infty \frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}t^{\nu-1} {}_LH^{(m,r)}s_n(x,y,zt)\frac{u^n}{n!}dt, \tag{2.6}$$

which on using equation (1.13) in the r.h.s. gives

$$\sum_{n=0}^{\infty} {}_LH^{(m,r)}s_{n,\nu}(x,y,z;\alpha)\frac{u^n}{n!} \tag{2.7}$$
$$= \frac{C_0(-x(f^{-1}(u))^m)\exp(yf^{-1}(u))}{g(f^{-1}(u))\Gamma(\nu)}\int_0^\infty e^{-(\alpha-z(f^{-1}(u))^r)t}t^{\nu-1}dt.$$

Making use of equation (1.6) in the r.h.s. of equation (2.7), yields assertion (2.5). □

Remark 1. For $\alpha = 1$, $\nu = 1$ and $z = D_z^{-1}$, the extended Laguerre-Gould-Hopper-Sheffer polynomials ${}_LH^{(m,r)}s_{n,\nu}(x,y,z;\alpha)$ reduce to the Laguerre-Gould-Hopper-Sheffer polynomials ${}_LH^{(m,r)}s_n(x,y,z)$ [5].

Differentiating generating function (2.5) w.r.t α, the following recurrence relation for the ELGHSP ${}_LH^{(m,r)}s_{n,\nu}(x,y,z;\alpha)$ is obtained:

$$\frac{\partial}{\partial\alpha}{}_LH^{(m,r)}s_{n,\nu}(x,y,z;\alpha) = -\nu\ {}_LH^{(m,r)}s_{n,\nu+1}(x,y,z;\alpha). \tag{2.8}$$

In order to derive quasi-monomial properties and operational representation for the ELGHSP ${}_LH^{(m,r)}s_{n,\nu}(x,y,z;\alpha)$, the following operation will be used:

(Θ): *Replacement of z by zt, multiplication by $\frac{1}{\Gamma(\nu)}e^{-\alpha t}t^{\nu-1}$ and then integration with respect to t from $t = 0$ to $t = \infty$.*

To frame the ELGHSP ${}_LH^{(m,r)}s_{n,\nu}(x,y,z;\alpha)$ within the context of monomiality principle, we prove the following result:

Theorem 3. *The ELGHSP ${}_LH^{(m,r)}s_{n,\nu}(x,y,z;\alpha)$ are quasi-monomial with respect to the following multiplicative and derivative operators:*

$$\hat{M}_{LHs_\nu} = \left(y + mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}} - rz\frac{\partial^r}{\partial\alpha\partial y^{r-1}} - \frac{g'(\partial_y)}{g(\partial_y)}\right)\frac{1}{f'(\partial_y)} \tag{2.9}$$

and

$$\hat{P}_{LHs_\nu} = f(\partial_y), \tag{2.10}$$

respectively, where $\partial_y := \frac{\partial}{\partial y}$.

Proof. We recall the following equations [5]:

$$\left(y + mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}} + rz\frac{\partial^{r-1}}{\partial y^{r-1}} - \frac{g'(\partial_y)}{g(\partial_y)}\right)\frac{1}{f'(\partial_y)}\ {}_LH^{(m,r)}s_n(x,y,z) \tag{2.11}$$
$$= {}_LH^{(m,r)}s_{n+1}(x,y,z);$$

$$f\left(\partial_y\right)\,{}_LH^{(m,r)}s_n(x,y,z) = n\,{}_LH^{(m,r)}s_{n-1}(x,y,z). \tag{2.12}$$

Performing the operation (Θ) on equations (2.11) and (2.12) and using recurrence relation (2.8), we obtain assertions (2.9) and (2.10).

□

The following corollary is an immediate consequence of Theorem 3:

Corollary 1. *The following differential equation for the ELGHSP* ${}_LH^{(m,r)}s_{n,\nu}(x,y,z;\alpha)$ *holds true:*

$$\left(\left(y + mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}} - rz\frac{\partial^r}{\partial\alpha\partial y^{r-1}} - \frac{g'\left(\partial_y\right)}{g\left(\partial_y\right)}\right)\frac{f\left(\partial_y\right)}{f'\left(\partial_y\right)} - n\right){}_LH^{(m,r)}s_{n,\nu}(x,y,z;\alpha) = 0 \tag{2.13}$$

Proof. Using equations (2.9) and (2.10) in equation (1.12), assertion (2.13) follows.

□

The operational representation between the ELGHSP ${}_LH^{(m,r)}s_{n,\nu}(x,y,z;\alpha)$ and Sheffer polynomials $s_n(x)$ is obtained in the form of the following result:

Theorem 4. *The following operational representation between the ELGHSP* ${}_LH^{(m,r)}s_{n,\nu}(x,y,z;\alpha)$ *and Sheffer polynomials* $s_n(x)$ *holds true:*

$$\left(\alpha - z\frac{\partial^r}{\partial y^r}\right)^{-\nu}\exp\left(D_x^{-1}\frac{\partial^m}{\partial y^m}\right)\{s_n(y)\} = {}_LH^{(m,r)}s_{n,\nu}(x,y,z;\alpha). \tag{2.14}$$

Proof. Performing operation (Θ) on equation (1.14), we obtain

$$\begin{aligned}\frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}t^{\nu-1}\exp\left(D_x^{-1}\frac{\partial^m}{\partial y^m} + zt\frac{\partial^r}{\partial y^r}\right)s_n(y)dt \\ = \frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}t^{\nu-1}\,{}_LH^{(m,r)}s_n(x,y,zt)dt.\end{aligned} \tag{2.15}$$

Decoupling the exponential operator in the l.h.s. of above equation by using the Weyl identity [3, p. 7]

$$e^{\hat{A}+\hat{B}} = e^{-k/2}e^{\hat{A}}e^{\hat{B}}, \quad k \in \mathbb{C}, \tag{2.16}$$

we get

$$\exp\left(D_x^{-1}\frac{\partial^m}{\partial y^m}\right)\frac{1}{\Gamma(\nu)}\int_0^\infty e^{-(\alpha - z\frac{\partial}{\partial y}^r)t}t^{\nu-1}dt\,s_n(y) = {}_LH^{(m,r)}s_{n,\nu}(x,y,z;\alpha), \tag{2.17}$$

which in view of relation (1.6), yields assertion (2.14).

□

The determinant definition of the Sheffer sequences proposed by W. Wang [9] in 2014, provides motivation to establish the determinant forms of the new hybrid special polynomials. The determinant approach is equivalent to the corresponding approach based on operational methods. This approach is beneficial in detecting the solution of general linear interpolation problems and also suitable for computations. Inspired by the novel work on determinant approaches, the determinant definition of the ELGHSP ${}_{LH^{(m,r)}}s_{n,\nu}(x,y,z;\alpha)$ is established by proving the following result:

Theorem 5. *The ELGHSP* ${}_{LH^{(m,r)}}s_{n,\nu}(x,y,z;\alpha)$ *of degree* n *are defined by*

$$ {}_{LH^{(m,r)}}s_{0,\nu}(x,y,z;\alpha)=\frac{1}{a_{0,0}}\,{}_{L}H_{0,\nu}^{(m,r)}(x,y,z;\alpha), \tag{2.18}$$

$$ {}_{LH^{(m,r)}}s_{n,\nu}(x,y,z;\alpha)=\frac{(-1)^n}{a_{0,0}\,a_{1,1}\,\ldots\,a_{n,n}} \times\begin{vmatrix} {}_{L}H_{0,\nu}^{(m,r)}(x,y,z;\alpha) & {}_{L}H_{1,\nu}^{(m,r)}(x,y,z;\alpha) & \cdots & {}_{L}H_{n-1,\nu}^{(m,r)}(x,y,z;\alpha) & {}_{L}H_{n,\nu}^{(m,r)}(x,y,z;\alpha)\\ a_{0,0} & a_{1,0} & \cdots & a_{n-1,0} & a_{n,0}\\ 0 & a_{1,1} & \cdots & a_{n-1,1} & a_{n,1}\\ 0 & 0 & \cdots & a_{n-1,2} & a_{n,2}\\ . & . & \cdots & . & .\\ . & . & \cdots & . & .\\ 0 & 0 & \cdots & a_{n-1,n-1} & a_{n,n-1} \end{vmatrix}, \tag{2.19}$$

where $a_{n,k}$ *is the* (n,k) *entry of the Riordan array* $\big(g(t),f(t)\big)$.

Proof. Interchanging x by y and operating $\left(\alpha-z\frac{\partial^r}{\partial y^r}\right)^{-\nu}\exp\left(D_x^{-1}\frac{\partial^m}{\partial y^m}\right)$ on both sides of equation (1.4) and using operational rules (1.8) and (2.14) in r.h.s and l.h.s. respectively, we find

$$ {}_{L}H_{n,\nu}^{(m,r)}(x,y,z;\alpha)=\sum_{k=0}^{n}a_{n,k}\;{}_{LH^{(m,r)}}s_{n,\nu}(x,y,z;\alpha). \tag{2.20}$$

The above equality leads to the following system of infinite equations in the unknowns ${}_{LH^{(m,r)}}s_{n,\nu}(x,y,z;\alpha),\quad n=0,1,\ldots,$

$$\begin{cases} a_{0,0}\ {}_{L}H^{(m,r)}s_{0,\nu}(x,y,z;\alpha) = {}_{L}H^{(m,r)}_{0,\nu}(x,y,z;\alpha), \\ a_{1,0}\ {}_{L}H^{(m,r)}s_{0,\nu}(x,y,z;\alpha) + a_{1,1}\ {}_{L}H^{(m,r)}s_{1,\nu}(x,y,z;\alpha) = {}_{L}H^{(m,r)}_{1,\nu}(x,y,z;\alpha), \\ a_{2,0}\ {}_{L}H^{(m,r)}s_{0,\nu}(x,y,z;\alpha) + a_{2,1}\ {}_{L}H^{(m,r)}s_{1,\nu}(x,y,z;\alpha) \\ \qquad + a_{2,2}\ {}_{L}H^{(m,r)}s_{2,\nu}(x,y,z;\alpha) = {}_{L}H^{(m,r)}_{2,\nu}(x,y,z;\alpha), \\ \vdots \\ a_{n,0}\ {}_{L}H^{(m,r)}s_{0,\nu}(x,y,z;\alpha) + a_{n,1}\ {}_{L}H^{(m,r)}s_{1,\nu}(x,y,z;\alpha) + \\ \qquad \cdots + a_{n,n}\ {}_{L}H^{(m,r)}s_{n,\nu}(x,y,z;\alpha) \\ \vdots \end{cases} \tag{2.21}$$

From first equation of system (2.21), assertion (2.18) follows. Applying the Cramer's rule to the first $n+1$ equations, it follows that

$$ {}_{L}H^{(m,r)}s_{n,\nu}(x,y,z;\alpha)$$

$$= \frac{1}{a_{0,0}\, a_{1,1}\, \dots\, a_{n,n}} \begin{vmatrix} a_{0,0} & 0 & \cdots & 0 & {}_{L}H^{(m,r)}_{0,\nu}(x,y,z;\alpha) \\ a_{1,0} & a_{1,1} & \cdots & 0 & {}_{L}H^{(m,r)}_{1,\nu}(x,y,z;\alpha) \\ a_{2,0} & a_{2,1} & \cdots & 0 & {}_{L}H^{(m,r)}_{2,\nu}(x,y,z;\alpha) \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ a_{n-1,0} & a_{n-1,1} & \cdots & a_{n-1,n-1} & {}_{L}H^{(m,r)}_{n-1,\nu}(x,y,z;\alpha) \\ a_{n,0} & a_{n,1} & \cdots & a_{n,n-1} & {}_{L}H^{(m,r)}_{n,\nu}(x,y,z;\alpha) \end{vmatrix}. \tag{2.22}$$

Now, bringing the $(n+1)$-th column to the first place by n transpositions of adjacent columns and noting that the determinant of a square matrix is the same as that of its transpose, assertion (2.19) follows.

□

In the next section, certain members belonging to the ELGHSP ${}_{L}H^{(m,r)}s_{n,\nu}(x,y,z;\alpha)$ are constructed and their analogues results are established.

3. EXAMPLES

The Sheffer family contains several polynomials as their members for different choices of $g(t)$ and $f(t)$. Thus, by taking $g(t)$ and $f(t)$ of the special polynomials belonging to Sheffer family, we get the corresponding members belonging to the ELGHSP family ${}_{L}H^{(m,r)}s_{n,\nu}(x,y,z;\alpha)$.

Example 3.1 Taking $g(u)=\left(\frac{2}{1+\sqrt{1-u^2}}\right)^{\lambda}$, $f(u)=\frac{-u}{1+\sqrt{1-u^2}}$, $f^{-1}(u)=\frac{-2u}{1+u^2}$ and $g(f^{-1}(u))=(1+u^2)^{\lambda}$ (for which the Sheffer polynomials becomes the Gegenbauer polynomials $C_n^{(\lambda)}(x)$ [9]) in Theorem 2- Theorem 4, we find that:

The following generating function for the extended Laguerre-Gould-Hopper Gegenbauer polynomials (ELGHGnP) ${}_{L}H^{(m,r)}C_{n,\nu}^{(\lambda)}(x,y,z;\alpha)$ holds true:

$$\frac{\exp\left(y\left(\frac{-2u}{1+u^2}\right)\right)C_0\left(-x\left(\frac{-2u}{1+u^2}\right)^m\right)}{(1+u^2)^{\lambda}\left(\alpha-z\left(\frac{-2u}{1+u^2}\right)^r\right)^{\nu}}=\sum_{n=0}^{\infty}{}_{L}H^{(m,r)}C_{n,\nu}^{(\lambda)}(x,y,z;\alpha)\frac{u^n}{n!}. \tag{3.1}$$

The ELGHGnP ${}_{L}H^{(m,r)}C_{n,\nu}^{(\lambda)}(x,y,z;\alpha)$ are quasi-monomial with respect to the following multiplicative and derivative operators:

$$\hat{M}_{LHC_{\nu}}=\lambda\partial y+\left(y+mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}}-rz\frac{\partial^r}{\partial\alpha\partial y^{r-1}}\right)\left((\partial y)^2-1-\sqrt{1-(\partial y)^2}\right), \tag{3.2}$$

and

$$\hat{P}_{LHC_{\nu}}=\frac{-\partial y}{1+\sqrt{1-(\partial y)^2}}. \tag{3.3}$$

The ELGHGnP ${}_{L}H^{(m,r)}C_{n,\nu}^{(\lambda)}(x,y,z;\alpha)$ satisfy the following differential equation:

$$\left(\frac{-\lambda(\partial y)^2}{1+\sqrt{1-(\partial y)^2}}+\left(y\frac{\partial}{\partial y}+mD_x^{-1}\frac{\partial^m}{\partial y^m}-rz\frac{\partial^{r+1}}{\partial\alpha\partial y^r}\right)\frac{\left(1-(\partial y)^2+\sqrt{1-(\partial y)^2}\right)}{1+\sqrt{1-(\partial y)^2}}-n\right)$$
$$\times{}_{L}H^{(m,r)}C_{n,\nu}^{(\lambda)}(x,y,z;\alpha)=0. \tag{3.4}$$

The following relation between the Gegenbauer polynomials $C_n^{(\lambda)}(x)$ and ELGHGnP ${}_{L}H^{(m,r)}C_{n,\nu}^{(\lambda)}(x,y,z;\alpha)$ holds true:

$$\left(\alpha-z\frac{\partial^r}{\partial y^r}\right)^{-\nu}\exp\left(D_x^{-1}\frac{\partial^m}{\partial y^m}\right)\{C_n^{(\lambda)}(y)\}={}_{L}H^{(m,r)}C_{n,\nu}^{(\lambda)}(x,y,z;\alpha). \tag{3.5}$$

Taking

$$a_{n,k} = \begin{cases} 0, & \text{if } n-k \text{ is odd,} \\ \frac{c_n(-1)^k(\lambda+k)}{c_k 2^n(\lambda+n)}\binom{\lambda+n}{n-k/2}, & \text{if } n-k \text{ is even,} \end{cases} \tag{3.6}$$

where $c_n = 1/\binom{-\lambda}{n}$ and $a_{n,n} = \left(-\frac{1}{2}\right)^n$ in equations (2.18) and (2.19), we find that the ELGHGnP ${}_{LH^{(m,r)}}C_{n,\nu}^{(\lambda)}(x,y,z;\alpha)$ for $n=4$ are defined by

$${}_{LH^{(m,r)}}C_{0,\nu}^{(\lambda)}(x,y,z;\alpha) = {}_LH_{0,\nu}^{(m,r)}(x,y,z;\alpha), \tag{3.7}$$

$${}_{LH^{(m,r)}}C_{n,\nu}^{(\lambda)}(x,y,z;\alpha) = \frac{128\lambda(\lambda+1)(\lambda+2)(\lambda+3)}{3}$$

$$\times \begin{vmatrix} {}_LH_{0,\nu}^{(m,r)}(x,y,z;\alpha) & {}_LH_{1,\nu}^{(m,r)}(x,y,z;\alpha) & {}_LH_{2,\nu}^{(m,r)}(x,y,z;\alpha) & {}_LH_{3,\nu}^{(m,r)}(x,y,z;\alpha) & {}_LH_{4,\nu}^{(m,r)}(x,y,z;\alpha) \\ 1 & 0 & \frac{1}{2(\lambda+1)} & 0 & \frac{3}{4(\lambda+1)(\lambda+2)} \\ 0 & -\frac{1}{2} & 0 & \frac{-3}{4(\lambda+2)} & 0 \\ 0 & 0 & \frac{1}{4} & 0 & \frac{3}{4(\lambda+3)} \\ 0 & 0 & 0 & -\frac{1}{8} & 0 \end{vmatrix}. \tag{3.8}$$

Example 3.2 Taking $g(u) = \left(\frac{2}{1+\sqrt{1+2u}}\right)^{1+\alpha+\beta}$, $f(u) = \frac{u}{1+u+\sqrt{1+2u}}$, $f^{-1}(u) = \frac{2u}{(1-u)^2}$ and $g(f^{-1}(u)) = (1-u)^{1+\alpha+\beta}$ (for which the Sheffer polynomials becomes the Jacobi polynomials $J_n(x)$ [9]) in Theorem 2 - Theorem 4, we find that:

The following generating function for the extended Laguerre-Gould-Hopper-Jacobi polynomials (ELGHJP) ${}_{LH^{(m,r)}}J_{n,\nu}(x,y,z;\alpha)$ holds true:

$$\frac{\exp\left(y\left(\frac{2u}{(1-u)^2}\right)\right)C_0\left(-x\left(\frac{2u}{(1-u)^2}\right)^m\right)}{(1-u)^{1+\alpha+\beta}\left(\alpha - z\left(\frac{2u}{(1-u)^2}\right)^r\right)^\nu} = \sum_{n=0}^{\infty} {}_{LH^{(m,r)}}J_{n,\nu}(x,y,z;\alpha)\frac{u^n}{n!}. \tag{3.9}$$

The ELGHJP ${}_{LH^{(m,r)}}J_{n,\nu}(x,y,z;\alpha)$ are quasi-monomial with respect to the following multiplicative and derivative operators:

$$\begin{aligned} \hat{M}_{LHJ_\nu} = &\left(y + mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}} - rz\frac{\partial^r}{\partial\alpha\partial y^{r-1}} + \frac{(1+\alpha+\beta)}{1+2\partial y+\sqrt{1+2\partial y}}\right) \\ &\times\left(1+\partial y+\sqrt{1+2\partial y}\right)\sqrt{1+2\partial y} \end{aligned} \tag{3.10}$$

and

$$\hat{P}_{LHJ_\nu} = \frac{\partial y}{1+\partial y+\sqrt{1+2\partial y}}. \tag{3.11}$$

The ELGHJP ${}_{LH^{(m,r)}}J_{n,\nu}(x,y,z;\alpha)$ satisfy the following differential equation:

$$\left(\left(y\frac{\partial}{\partial y}+mD_x^{-1}\frac{\partial^m}{\partial y^m}-rz\frac{\partial^{r+1}}{\partial y^r\partial\alpha}-\frac{(1+\alpha+\beta)\partial y}{1+2\partial y+\sqrt{1+2\partial y}}\right)\sqrt{1+2\partial y}-n\right)$$
$$\times{}_{LH^{(m,r)}}J_{n,\nu}(x,y,z;\alpha)=0. \tag{3.12}$$

The following relation between the Jacobi polynomials $J_n(x)$ and ELGHJP ${}_{LH^{(m,r)}}J_{n,\nu}(x,y,z;\alpha)$ holds true:

$$\left(\alpha-z\frac{\partial^r}{\partial y^r}\right)^{-\nu}\exp\left(D_x^{-1}\frac{\partial^m}{\partial y^m}\right)\{J_n(y)\}={}_{LH^{(m,r)}}J_{n,\nu}(x,y,z;\alpha). \tag{3.13}$$

Taking

$$a_{n,k}=\frac{(-1)^{n-k}}{2^n}\frac{c_n}{c_k}\frac{1+\alpha+\beta+2k}{1+\alpha+\beta+2n}\binom{1+\alpha+\beta+2n}{n-k}, \tag{3.14}$$

where $c_n=\frac{4^n(\alpha+n)_n n!}{(\alpha+\beta+2n)_{2n}}$ and $a_{n,n}=\left(\frac{1}{2}\right)^n$ in equations (2.18) and (2.19), we find that the ELGHJP ${}_{LH^{(m,r)}}J_{n,\nu}(x,y,z;\alpha)$ for $n=4$ are defined by

$${}_{LH^{(m,r)}}J_{0,\nu}(x,y,z;\alpha)={}_LH_{0,\nu}^{(m,r)}(x,y,z;\alpha), \tag{3.15}$$

$${}_{LH^{(m,r)}}J_{n,\nu}(x,y,z;\alpha)=1024$$
$$\times\begin{vmatrix} {}_LH_{0,\nu}^{(m,r)}(x,y,z;\alpha) & {}_LH_{1,\nu}^{(m,r)}(x,y,z;\alpha) & {}_LH_{2,\nu}^{(m,r)}(x,y,z;\alpha) & {}_LH_{3,\nu}^{(m,r)}(x,y,z;\alpha) & {}_LH_{4,\nu}^{(m,r)}(x,y,z;\alpha)\\ 1 & -\frac{2(\alpha+1)}{\alpha+\beta+2} & \frac{4(\alpha+2)_2}{(\alpha+\beta+3)_2} & -\frac{8(\alpha+3)_3}{(\alpha+\beta+4)_3} & \frac{16(\alpha+4)_4}{(\alpha+\beta+5)_4}\\ 0 & \frac{1}{2} & -\frac{2(\alpha+2)}{\alpha+\beta+4} & \frac{6(\alpha+3)_2}{(\alpha+\beta+5)_2} & -\frac{16(\alpha+4)_3}{(\alpha+\beta+6)_3}\\ 0 & 0 & \frac{1}{4} & -\frac{3(\alpha+3)}{2(\alpha+\beta+6)} & \frac{6(\alpha+4)_2}{(\alpha+\beta+7)_2}\\ 0 & 0 & 0 & \frac{1}{8} & -\frac{\alpha+4}{\alpha+\beta+8}\end{vmatrix}. \tag{3.16}$$

The above examples show that the operational rules provide a mechanism to obtain the results for the members belonging to the ELGHSP ${}_{LH^{(m,r)}}s_{n,\nu}(x,y,z;\alpha)$ and prove the usefulness of the method adopted in this paper. The operational techniques can be used for a more general insight into the theory of hybrid special polynomials and for their extensions. The appropriate combination of methods relevant to generalized operational calculus and to special functions can be very useful tool to treat a large body of problems both in physics and mathematics. Thus, we conclude that the method based on the operational rules may provide powerful tools to deal with the possibilities offered by extended forms of the hybrid special polynomials.

APPENDIX

We have mentioned several special cases of the LGHSP ${}_LH^{(m,r)}s_n(x,y,z)$ in Table 1. Now, for the same choice of the variables and indices, the ELGHSP ${}_LH^{(m,r)}s_{n,\nu}(x,y,z;\alpha)$ reduce to the corresponding special case. These new hybrid special polynomials related to the Sheffer polynomials are given in Table 2.

TABLE 2. Special cases of the ELGHSP ${}_LH^{(m,r)}s_{n,\nu}(x,y,z;\alpha)$.

S. No.	Values of the Indices and Variables	Name of the hybrid special polynomials	Generating function
I.	$m=1,\ r=2;\ x\to -x$	Extended 3-variable Laguerre-Hermite-Sheffer polynomials (E3VLHSP)	$\frac{\exp(yf^{-1}(u))C_0(xf^{-1}(u))}{g(f^{-1}(u))(\alpha-z(f^{-1}(u))^2)^\nu} = \sum_{n=0}^{\infty} {}_LHs_{n,\nu}(x,y,z;\alpha)\frac{u^n}{n!}$
II.	$m=1,\ r=2;\ y=1,$ $z\to y, x\to -x$	Extended Laguerre-Hermite type-Sheffer polynomials (ELHtSP)	$\frac{\exp(f^{-1}(u))C_0(xf^{-1}(u))}{g(f^{-1}(u))(\alpha-y(f^{-1}(u))^2)^\nu} = \sum_{n=0}^{\infty} {}_\varphi s_{n,\nu}(x,y;\alpha)\frac{u^n}{n!}$
III.	$x=0$	Extended Gould-Hopper-Sheffer polynomials (EGHSP)	$\frac{\exp(yf^{-1}(u))}{g(f^{-1}(u))(\alpha-z(f^{-1}(u))^r)^\nu} = \sum_{n=0}^{\infty} {}_{H^{(r)}}s_{n,\nu}(y,z;\alpha)\frac{u^n}{n!}$
IV.	$r=1;\ y=0,$ $z\to y$	Extended 2-Variable generalized Laguerre-Sheffer polynomials (E2VGLSP)	$\frac{C_0(-x(f^{-1}(u))^m)}{g(f^{-1}(u))(\alpha-y(f^{-1}(u)))^\nu} = \sum_{n=0}^{\infty} {}_mL_{n,\nu}s_n(x,y;\alpha)\frac{u^n}{n!}$
V.	$r=m;\ x=0,$ $y\to -D_x^{-1}, z\to y$	Extended 2-Variable generalized Laguerre type-Sheffer polynomials (E2VGLtSP)	$\frac{C_0(xf^{-1}(u))}{g(f^{-1}(u))(\alpha-y(f^{-1}(u))^m)^\nu} = \sum_{n=0}^{\infty} {}_{[m]L}s_{n,\nu}(x,y;\alpha)\frac{u^n}{n!}$
VI.	$r=m-1;\ x=0,$ $y\to x, z\to y$	Extended generalized Chebyshev-Sheffer polynomials (EGCSP)	$\frac{\exp(xf^{-1}(u))}{g(f^{-1}(u))(\alpha-y(f^{-1}(u))^{m-1})^\nu} = \sum_{n=0}^{\infty} {}_{U^{(m)}}s_{n,\nu}(x,y;\alpha)\frac{u^n}{n!}$
VII.	$m=1;\ z=0,$ $x\to -x$	Extended 2-Variable Laguerre-Sheffer polynomials (E2VLSP)	$\frac{C_0(xf^{-1}(u))}{g(f^{-1}(u))(\alpha-y(f^{-1}(u)))^\nu} = \sum_{n=0}^{\infty} {}_Ls_{n,\nu}(x,y;\alpha)\frac{u^n}{n!}$
VIII.	$x=0,\ y\to x,$ $z\to yD_yy$	Extended 2-Variable truncated-Sheffer polynomials of order r (E2VTSP)	$\frac{\exp(xf^{-1}(u))}{g(f^{-1}(u))(\alpha-yD_yy(f^{-1}(u))^r)^\nu} = \sum_{n=0}^{\infty} {}_{e^{(r)}}s_{n,\nu}(x,y;\alpha)\frac{u^n}{n!}$
IX.	$r=2;\ x=0$	Extended 2-Variable Hermite-Kampé de Fériet-Sheffer polynomials (E2VHKdFSP)	$\frac{\exp(yf^{-1}(u))}{g(f^{-1}(u))(\alpha-z(f^{-1}(u))^2)^\nu} = \sum_{n=0}^{\infty} {}_Hs_{n,\nu}(y,z;\alpha)\frac{u^n}{n!}$
X.	$x\to yD_yy,$ $y\to x$	Extended 3-Variable generalized Hermite-Sheffer polynomials (E3VGHSP)	$\frac{\exp\left(xf^{-1}(u)+y(f^{-1}(u))^m\right)}{g(f^{-1}(u))(\alpha-z(f^{-1}(u))^r)^\nu} = \sum_{n=0}^{\infty} {}_{H^{(r,m)}}s_{n,\nu}(x,y,z;\alpha)\frac{u^n}{n!}$
XI.	$m=2,\ r=3;\ y\to x,$ $z\to y, x\to zD_zz,$	Extended Bell-type-Sheffer polynomials (EBtSP)	$\frac{\exp\left(xf^{-1}(u)+z(f^{-1}(u))^2\right)}{g(f^{-1}(u))(\alpha-y(f^{-1}(u))^3)^\nu} = \sum_{n=0}^{\infty} {}_{H^{(3,2)}}s_{n,\nu}(x,y,z;\alpha)\frac{u^n}{n!}$

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