



LINEAR EQUATIONS WITH ONE CONSTRAINT AND THEIR CONNECTION TO NONLINEAR EQUATIONS OF THE FOURTH ORDER

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Abstract. The purpose of this paper is to present several new results concerning relations between linear differential equations of the fourth order with one constraint and nonlinear differential equations of the fourth order. We consider linear differential equations of the second, the third and the fourth order and nonlinear fourth order differential equations related via the Schwarzian derivative. The method is based on the use of the Schwarzian derivative, which is defined as the ratio of two linearly independent solutions of the linear differential equations of the second or third and fourth order. As a result we obtain new relations between the solutions of these linear and nonlinear equations. To illustrate theorems and our constructive approach we give two examples. The given method may be generalized to differential equations of higher orders.

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1. INTRODUCTION

The Schwarzian derivative is a differential operator that is invariant under all linear fractional transformations, see [1, 8, 10]. It plays a significant role in the theory of modular forms, hypergeometric functions, univalent functions and conformal mappings [1, 8]. It is defined by

$$(Sf)(z) = \left(\frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left(\frac{f''(z)}{f'(z)} \right)^2 = \frac{f'''(z)}{f'(z)} - \frac{3}{2} \left(\frac{f''(z)}{f'(z)} \right)^2.$$

The well-known relation between a second-order linear differential equation of the form

$$y''(z) + Q(z)y(z) = 0$$

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and the Schwarzian derivative of the ratio of two linearly independent solutions y_1, y_2 of the linear equation above is as follows:

$$(S\xi)(z) = 2Q(z),$$

where $\xi = y_1/y_2$ and z is, in general, a complex variable. See [1] and [8] for more details.

If we have a general second order differential equation

$$y''(z) + p(z)y'(z) + q(z)y(z) = 0, \quad (1.1)$$

then substituting $y(z) = \xi(z)y_1(z)$ with the condition that y_1 is also a solution of (1.1), we get an expression for ξ, y_1 and their derivatives (up to order 2 and 1 respectively). Differentiating again and eliminating y_1, y_1' , we get that the function

$$w(z) = (S\xi)(z)$$

satisfies

$$w(z) = \frac{1}{2}(4q(z) - p(z)^2 - 2p'(z)). \quad (1.2)$$

We call expression (1.2) the invariant for the second order linear differential equation (1.1).

On extension of this approach for linear differential equation of the third order see in [10–12]. The generalization of the method for the linear differential equations of the fourth order is given in [6]. In papers [2, 4, 5] special classes of the fourth order linear differential equations and the nonlinear fourth order differential equations related via the Schwarzian derivative are considered and general solutions of both differential equations are found. In paper [3] the generalization of the method for a special type of linear differential equations of the fifth order is given along with a computer realization of this method in Mathematica (www.wolfram.com).

Several questions arise. What happens if the second, the third and the fourth order linear differential equations are related? What happens if we modify the function to be the ratio of solutions of two different equations? These questions were answered during the studies of linear differential equation of the third order in [9].

Similar questions were resolved for the fourth order ordinary differential equations with coefficients satisfying a system of two first order differential equations [7].

The main objective of this paper is to answer these questions for linear differential equation of the fourth order with coefficients that satisfy the differential equation of the first order. The proofs of statements are computational, that is the results can be verified by using any computer algebra system.

2. MAIN RESULTS

In this section we shall present 5 main results concerning the relations between linear

$$y''''(z) + p(z)y'''(z) + q(z)y''(z) + r(z)y'(z) + s(z)y(z) = 0, \quad (2.1)$$

where

$$p' = \frac{1}{12} (8q - 3p^2) \tag{2.2}$$

and nonlinear differential equations.

Theorem 1. *Let y be a solution of the fourth order linear differential equation (2.1) and y_1 be a solution of another fourth order linear differential equation*

$$y''''(z) + p_1(z)y'''(z) + q_1(z)y''(z) + r_1(z)y'(z) + s_1(z)y(z) = 0, \tag{2.3}$$

where

$$p'_1 = \frac{1}{12} (8q_1 - 3p_1^2).$$

If the function $w(z) = (S\xi)(z)$ with $\xi = y/y_1$ solves a nonlinear differential equation

$$\sum_{k=0}^1 \sum_{j=0}^2 \sum_{i=0}^3 \alpha_{ijk}(z)w^i(w')^j(w'')^k w'''' + \sum_{l=0}^2 \sum_{k=0}^3 \sum_{j=0}^4 \sum_{i=0}^6 \beta_{ijkl}(z)w^i(w')^j(w'')^k(w''')^l = 0, \tag{2.4}$$

which is explicitly given by

$$\begin{aligned} & (240ww'' - 300w'^2 - 40\psi w' + 160w^3 + (80q'' + \frac{40}{3}q^2 - 10p^2q + 60pr - 480s - 20\phi)w \\ & - \psi^2)w'''' - 280ww'''' + ((56\psi + 840w')w'' - (1120w^2 - 110\phi + 55p\psi - 20\psi')w' \\ & - \frac{448}{3}\psi w^2 + (10pq^2 - \frac{5}{2}p^3q - \frac{45}{4}p^2\psi - 90\phi' + 45p\psi' - \frac{20}{3}q(3r - 4\psi - 3p'') - 20\psi'')w \\ & + 4\phi\psi - 2p\psi^2 + 2\psi\psi')w'''' - 504w''^3 + (192w^2 - 153\phi + \frac{153}{2}p\psi - 66\psi')w''^2 \\ & + (2040ww'^2 + 448w^4 - 7\phi^2 - \frac{5}{16}p^2\psi^2 - \frac{23}{6}q\psi^2 + \frac{23}{2}\psi\phi' + \phi(7p\psi - 18\psi')) \\ & + \frac{13}{4}p\psi\psi' - 2\psi^2 + (42p\psi - 84\phi + 152\psi')w^2 \\ & + (\frac{55}{3}\psi^2 - \frac{15}{4}p^3\psi - 30r\psi + 15p^2\psi' - 40q\psi' + 60\phi'' + 15p(q\psi - 2\psi''))w \\ & + \psi\psi'' + (\frac{135}{8}p^2\psi - 45q\psi + 528\psi w + 135\phi' - \frac{135}{2}p\psi' + 30\psi'')w' \\ & - 1275w'^4 - 520\psi w'^3 + (-560w^3 + \frac{15}{2}(38\phi - 19p\psi - 4\psi')w + \frac{75}{16}p^3\psi \\ & + \frac{75}{2}r\psi - \frac{611}{12}\psi^2 - \frac{75}{4}p^2\psi' + 50q\psi' - 75\phi'' - \frac{75}{4}p(q\psi - 2\psi''))w'^2 \\ & + (-128\psi w^3 + \frac{35}{32}p^2\phi\psi - \frac{35}{12}q\phi\psi + \frac{5}{64}p^3\psi^2 - \frac{25}{24}pq\psi^2 + 5r\psi^2 - \frac{7}{6}\psi^3 \\ & + \frac{35}{4}\phi\phi' + \frac{5}{16}p^2\psi\psi' + 5q\psi\psi' + 5\phi'\psi' + (32\phi\psi - 16p\psi^2 + 16\psi\psi')w \\ & - 10\psi\phi'' - \frac{5}{8}p(\psi'(7\phi + 4\psi') + \psi(7\phi' - 2\psi'')) \end{aligned}$$

$$\begin{aligned}
& - \left(\frac{45}{2} p^2 \psi - 60 q \psi + 180 \phi' - 90 p \psi' + 40 \psi'' \right) w^2 + \frac{15}{2} \phi \psi'' w' + 64 w^6 \\
& - (8 \phi - 4 p \psi - \frac{272}{3} \psi') w^4 - \left(\frac{5}{2} p^3 \psi + 20 r \psi + \frac{82}{9} \psi^2 - 10 p^2 \psi' + \frac{80}{3} q \psi' - 40 \phi'' \right. \\
& \left. - 10 p (q \psi - 2 \psi'') \right) w^3 + (344 p \psi \psi' - 84 \phi^2 - 105 p^2 \psi^2 + 4 \phi (21 p \psi - 4 \psi')) \\
& + 16 (14 q \psi^2 - 42 \psi \phi' + 31 \psi'^2 - 28 \psi \psi'') \frac{w^2}{48} + \left(\frac{13}{24} \phi \psi^2 - \frac{5}{32} p^3 \phi \psi - \frac{5}{9} q^2 \psi^2 \right. \\
& \left. + \frac{10}{3} q \psi \phi' - 5 \phi'^2 - \frac{5}{3} q \phi \psi' + \frac{5}{12} \psi^2 \psi' - \frac{10}{3} q \psi'^2 - \frac{5}{4} r \psi (\phi + 2 \psi') \right. \\
& \left. + \frac{5}{48} p^2 (q \psi^2 - 12 \psi \phi' + 6 \phi \psi') + \frac{5}{2} \phi \phi'' + 5 \psi' \phi'' + \frac{5}{3} q \psi \psi'' - 5 \phi' \psi'' \right. \\
& \left. + \frac{1}{48} p (30 r \psi^2 - 13 \psi^3 + 240 \phi' \psi' + 10 q \psi (3 \phi + 2 \psi') - 60 \psi \phi'' - 60 \phi \psi'') \right) w \\
& + \frac{1}{32} (3 \phi^2 (p \psi - 4 \psi') - 2 \phi^3 + 2 \psi (2 r \psi^2 - 3 p \psi \phi' + 8 \phi' \psi' - 4 \psi \phi'')) \\
& + 2 \phi (\psi (6 \phi' + 3 p \psi' + 4 \psi'') - 2 q \psi^2 - 8 \psi'^2) = 0,
\end{aligned}$$

then conditions

$$p_1(z) = p(z), \quad q_1(z) = q(z), \quad r_1(z) = r(z), \quad s_1(z) = s(z)$$

and

$$\phi = pr - 16s + 4r', \quad \psi = pq - 6r + 4q' \quad (2.5)$$

hold.

Proof. We substitute $w(z) = (S\xi)(z)$ into equation (2.4) with unknown coefficients and then replace ξ by the ratio of y and y_1 . Replacing the fourth and higher order derivatives of y and y_1 by using the linear equations, we collect the coefficients of y , y_1 and their derivatives up to order 3. In the result we obtain a system of equations on the coefficients of linear and nonlinear equations, from which we get the desired result. \square

Theorem 2. Let y be a solution of equation (2.1) and y_1 be a solution of the third order linear differential equation of the form

$$y'''(z) + q_1(z)y''(z) + r_1(z)y'(z) + s_1(z)y(z) = 0. \quad (2.6)$$

If the function $w(z) = (S\xi)(z)$ with $\xi = y/y_1$ solves the nonlinear differential equation (2.4), then we have conditions (2.5) for (2.4) and three additional conditions on the coefficients of the linear equation (2.6)

$$q'_1 = q_1^2 + q - p q_1 - r_1, \quad r'_1 = r - p r_1 + q_1 r_1 - s_1, \quad s'_1 = s_1 (q_1 - p) + s. \quad (2.7)$$

Proof. We substitute $w(z) = (S\xi)(z)$ into equation (2.4), (2.5) with unknown coefficients and then replace ξ by the ratio of y and y_1 . Replacing the fourth and higher order derivatives of y and the third and higher order derivatives of y_1 by using the linear equations, we collect the coefficients of y , y_1 and their derivatives up to order

3, order 2 and order 1 respectively. In the result we obtain a system of equations on the coefficients of linear and nonlinear equations, from which we get the desired result. \square

Example 1. Assume that we know a particular solution y and coefficient p of equation (2.1)

$$y(z) = z, \quad p(z) = b, \quad (2.8)$$

where b is a constant. We substitute functions (2.8) into equations (2.1), (2.2). Solving the obtained equations we find

$$q(z) = \frac{3b^2}{8}, \quad s(z) = -\frac{r(z)}{z}. \quad (2.9)$$

Assume that we know a particular solution y_1 and coefficient q_1 of the equation (2.6)

$$y_1(z) = z, \quad q_1(z) = b. \quad (2.10)$$

Then from equation (2.6) we find

$$s_1(z) = -\frac{r_1(z)}{z}. \quad (2.11)$$

We substitute functions (2.8) -(2.11) into equations (2.7). Solving the resulting equations we find

$$r_1(z) = \frac{3b^2}{8}, \quad r(z) = -\frac{3b^2}{8z}. \quad (2.12)$$

We substitute functions (2.10), (2.11) into equation (2.6) and obtain

$$8zy''' + 8bzy'' + 3b^2zy' - 3b^2y = 0. \quad (2.13)$$

The substitution

$$y = z \int v(z) dz \quad (2.14)$$

reduces equation (2.13) to the second order differential equation

$$8zv'' + 8(bz + 3)v' + b(3bz + 16)v = 0. \quad (2.15)$$

The general solution of equation (2.15) is of the form

$$v = e^{-\frac{1}{4}i(\sqrt{2}-2i)bz} \left(C_1 U_1 + C_2 L^2_{-\frac{3}{2}-\frac{i}{\sqrt{2}}} \left(\frac{biz}{\sqrt{2}} \right) \right), \quad (2.16)$$

where $U_1 = U \left(\frac{3}{2} + \frac{i}{\sqrt{2}}, 3, \frac{biz}{\sqrt{2}} \right)$ is the confluent hypergeometric function and has the integral representation

$$U(a, b, z) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-xt} t^{a-1} (1+t)^{b-a-1} dt; \quad L^2_{-\frac{3}{2}-\frac{i}{\sqrt{2}}} \left(\frac{biz}{\sqrt{2}} \right)$$

is the Laguerre polynomial, that satisfy the differential equation

$$zv'' + (3-z)v' - \left(\frac{3}{2} + \frac{i}{\sqrt{2}} \right) v = 0,$$

and C_1, C_2 are arbitrary constants.

The general solution of equation (2.13) is of the form (2.14), (2.16). We choose, for example, the values of the arbitrary constants equal to $C_1 = 1, C_2 = 0$. Then the particular solution is

$$y_1 = z \int e^{-\frac{1}{4}i(\sqrt{2}-2i)bz} U\left(\frac{3}{2} + \frac{i}{\sqrt{2}}, 3, \frac{biz}{\sqrt{2}}\right) dz$$

and

$$\xi = \left(\int e^{-\frac{1}{4}i(\sqrt{2}-2i)bz} U\left(\frac{3}{2} + \frac{i}{\sqrt{2}}, 3, \frac{biz}{\sqrt{2}}\right) dz \right)^{-1}.$$

Then

$$\begin{aligned} w &= \frac{\xi'''}{\xi'} - \frac{3}{2} \left(\frac{\xi''}{\xi'} \right)^2 \\ &= \frac{b^2}{16} \left(U^{-2}(3(7+6\sqrt{2}i)U_2^2 + 2U_1(5-2i\sqrt{2})U_2 - (13+8i\sqrt{2})U_3) - 2i\sqrt{2} - 1 \right), \end{aligned} \quad (2.17)$$

where

$$U_2 = U\left(\frac{5}{2} + \frac{i}{\sqrt{2}}, 4, \frac{biz}{\sqrt{2}}\right), \quad U_3 = U\left(\frac{7}{2} + \frac{i}{\sqrt{2}}, 5, \frac{biz}{\sqrt{2}}\right).$$

The differential equation (2.4), (2.5) has coefficients (2.8), (2.9), (2.12) and

$$\phi(z) = -\frac{3b^2(12+bz)}{8z^2}, \quad \psi(z) = \frac{3b^2(6+bz)}{8z}.$$

Then according to Theorem 2 this nonlinear differential equation has a solution (2.17) which can be easily verified by substitution.

Theorem 3. *Let y be a general solution of the third order linear differential equation (2.6), (2.7). Then this solution is a three parameter family of solutions of the fourth order linear differential equation (2.1), (2.2).*

Proof. The proof is computational. □

Theorem 4. *Let y be a solution of equation (2.1) and y_1 be a solution of the second order linear differential equation of the form*

$$y''(z) + r_1(z)y'(z) + s_1(z)y(z) = 0. \quad (2.18)$$

If the function $w(z) = (S\xi)(z)$ with $\xi = y/y_1$ solves a nonlinear differential equation (2.4), then we have condition (2.5) for (2.4) and two additional conditions on the coefficients of the linear equation (2.18)

$$\begin{aligned} r_1'' &= -pr_1' + pr_1^2 - ps_1 - qr_1 + r + 3r_1r_1' + 2r_1s_1 - r_1^3 - 2s_1', \\ s_1'' &= pr_1s_1 - ps_1' - qs_1 + 2s_1r_1' + r_1s_1' - r_1^2s_1 + s + s_1^2. \end{aligned} \quad (2.19)$$

Proof. We substitute $w(z) = (S\xi)(z)$ into equation (2.4), (2.5) with unknown coefficients and then replace ξ by the ratio of y and y_1 . Replacing the fourth and higher order derivatives of y and the second and higher order derivatives of y_1 by using the linear equations, we collect the coefficients of y , y_1 and their derivatives up to order 3 and order 1 respectively. In the result we obtain a system of equations on the coefficients of linear and nonlinear equations, from which we get the desired result. \square

Example 2. Let

$$y(z) = \sqrt{z}, \quad p(z) = \frac{2}{z}, \quad r(z) = \frac{3}{2z^3}. \quad (2.20)$$

We substitute functions (2.20) into equations (2.1), (2.2). Solving the resulting equations we find

$$q(z) = -\frac{3}{2z^2}, \quad s(z) = -\frac{15}{16z^4}. \quad (2.21)$$

Let

$$r_1(z) = -\frac{3}{z}. \quad (2.22)$$

We substitute the relations (2.20)-(2.22) into (2.19). After simplifications we obtain the system

$$s_1''(z) = s_1^2(z) - \frac{5}{z}s_1'(z) - \frac{15}{2z^2}s_1(z) - \frac{5}{16z^4}, \quad 4z^3s_1'(z) + 16z^2s_1(z) = 30. \quad (2.23)$$

We find the following solution of system (2.23):

$$s_1(z) = \frac{15}{4z^2}. \quad (2.24)$$

We substitute the relations (2.22), (2.24) into equation (2.18) and integrate it. We write the general solution in the form

$$y_1 = C_1z^{3/2} + C_2z^{5/2}, \quad (2.25)$$

where C_1, C_2 are arbitrary constants. We choose, for example, the values of arbitrary constants equal to one. Then we obtain $y_1 = z^{3/2} + z^{5/2}$ and $\xi = \frac{1}{z^2+z}$. Then we find the solution

$$w = \frac{\xi'''}{\xi'} - \frac{3}{2} \left(\frac{\xi''}{\xi'} \right)^2 = -\frac{6}{(2z+1)^2}. \quad (2.26)$$

Differential equation (2.4), (2.5) for coefficients (2.20), (2.21) has the form

$$\begin{aligned} &20(8w^3 + 12ww'' - 15w'^2)w'''' - 280ww''''^2 - 280w'(4w^2 - 3w'')w''' - 504w''^3 \\ &+ 192w^2w''^2 + 8(56w^4 + 255ww'^2)w'' - 1275w'^4 - 560w^3w'^2 + 64w^6 = 0. \end{aligned} \quad (2.27)$$

According to Theorem 4 equation (2.27) has the solution (2.26) that can be easily verified by the direct substitution.

If we choose the function (2.25), then the corresponding solution of the equation (2.27) takes the form

$$w = -\frac{6C_2^2}{(2C_2z + C_1)^2}.$$

It is a one-parameter family of solutions. This is easily seen by introducing the substitution $c = C_2/C_1$.

Theorem 5. *Let y be a general solution of the second order linear differential equation (2.18), (2.19). Then this solution is a two parameter family of solutions of the fourth order linear differential equation (2.1), (2.2).*

Proof. The proof is computational. □

3. CONCLUSIONS

One research direction is to replace linear differential equations with nonlinear equations of second and higher order and to consider the Schwarzian derivative of the ratio of 2 solutions. This might give a new insight into the theory of some nonlinear special functions.

Taking into account the obtained results for the known solutions of the fourth and the second order linear equations, we can formulate the corresponding theorems for the known solutions of the fourth order linear equation and the Riccati equation, to which the second order linear equation reduces. Here it seems appropriate to use the results of [9] and the method of V. Orlov [13, 14] for the study of the Riccati equation and nonlinear differential equations of the second order.

From the point of view of programming algorithms for solving the considered problems, the opportunities of Wolfram Research technologies described in [15] are essential. They significantly complement the set of tools for creating, maintaining and distributing dynamic content when constructing and studying solutions of differential equations.

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