GENERALIZED Ψ-GERAGHTY-QUASI CONTRACTIONS IN
b-METRIC SPACES

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Abstract. In this paper we generalize the well-known class of quasi contractions pairs, by using the so-called Geraghty's property and by controlling the contractive inequality with an altering distance function. For this class, in the setting of *b*-metric spaces, we prove the existence and uniqueness a of common fixed point assuming some non commutative notions. Finally, for this class of mappings we prove the *b*-convergence of the Jungck-Mann iterative scheme under suitable conditions on the *b*-metric space and on the altering distance function.

1. INTRODUCTION

The Banach contraction principle (BCP) is one of the most important result in metric fixed point theory. Since its appearance, several interesting generalizations and extensions of this principle have been introduced. Typically, these extensions and generalizations are obtained by either generalizating the domain of the mappings or by extending the contractive conditions of the mappings.

In 1974, L. B. Ćirić ([5]) introduced the notion of a *quasi contraction map*, and prove its fixed point theorem.

Theorem 1 (Ćirić, [5]). *Let (M, d) be a metric S-orbitally complete space and let $S : M \to M$ be a quasi contraction, that is, there exists $q \in (0, 1)$ such that for all $x, y \in M$ the following inequality hold,*

$$d(Sx, Sy) \le q \max\{d(x, y), d(x, Sx), d(y, Sy), (d(x, Sy), d(y, Sx))\}.$$

Then S has a unique fixed point in M.

In 1976, G. Jungck ([11]) generalized the BCP by using the notion of commuting mappings:

Theorem 2 (Jungck, [11]). *Let S, T be two selfmappings of a complete metric space (M, d) such that*

  (i) (S, T) *is a commuting pair.*
 (ii) $SM \subset TM$.
(iii) *T is a continuous mapping.*
 (iv) *There exists* $\alpha \in [0, 1)$ *such that*

$$d(Sx, Sy) \leq \alpha d(Tx, Ty) \quad \text{for all } x, y \in M.$$

Then, S and T have a unique common fixed point in M.

In 1979, K. M. Das and K. V. Naik ([9]) generalized and extended the fixed point theorems of Ćirić and Jungck by proving the following result.

Theorem 3 (Das–Naik, [9]). *Let (M, d) be a complete metric space. Let T be a continuous selfmap on M and S be any selfmap on M that commutes with T. Assume that $SM \subset TM$ and that there exists $q \in (0, 1)$ such that*

$$d(Sx, Sy) \leq q \max\{d(Tx, Ty), d(Sx, Tx), d(Sy, Ty), d(Sx, Ty), d(Sy, Tx)\}$$

for all $x, y \in M$. Then S and T have a unique common fixed point.

In this paper we prove, in the setting of *b*-metric spaces, some common fixed point results that extend and generalize the Das–Naik fixed point theorem and other well-known results stated in the usual metric spaces. The approximate common fixed point can be computed by using the so-called Jungck-Mann iteration process, for which we prove its *b*-convergence.

2. PRELIMINARIES

The b-metric spaces

In this section we recall some concepts, results and properties of the *b*-metric spaces that will be useful in the sequel. The concept of *b*-metric spaces (also known as *quasimetric spaces*) was reintroduced in 1989 by I. A. Bakhtin [4], and used in the metric fixed point theory by S. Czerwik ([6, 7]) in connection with some problems concerning to the convergence of measurable functions with respect to measure. In 2010, M. A. Kamsi [16] and M. A. Kamsi and N. Hussain [15] showed that each cone metric space has a *b*-metric structure, so they reintroduced the *b*-metric space with the name of *metric type space*.

Definition 1. Let M be a non empty set and $s \geq 1$ be given a real number. A function

$$\rho : M \times M \to \mathbb{R}_+ := [0, +\infty)$$

is said to be a *b*-metric if and only if for all $x, y, z \in M$ the following conditions hold:

 (ρ_1) $\rho(x, y) = 0$ if and only if $x = y$.
 (ρ_2) $\rho(x, y) = \rho(y, x)$.
 (ρ_3) $\rho(x, z) \leq s(\rho(x, y) + \rho(y, z))$.

The pair (M, ρ) is called a b-metric space and the real number $s \geq 1$ is called the coefficient of (M, ρ).

The class of b-metric spaces is larger than the class of metric spaces, since these spaces extends the usual metric spaces for the case when $s = 1$. Even more, the classical metric spaces are properly contained in the class of b-metric spaces since there exist b-metric spaces which are not metric spaces (see, Example 1).

On the other hand, as a consequence of the Alexandroff–Urysohn Theorem, the topology induced by a given b-metric on a b-metric space is metrizable (see Definition 2 and see also, [15, 18]), this metrization gives as a consequence that much of the results given in metric spaces are directly valid in b-metric spaces. Nevertheless, the convenience of prove new fixed point results on these spaces takes relevance when we are dealing with b-metric spaces which are not metric spaces.

Example 1. a) Let (M, d) be a metric space, and let the function $\varphi(t) = t^p$ for $t > 0$, and $p > 1$. Notice that this function is continuous and convex. Thus, if we define $\rho(x, y) := \varphi(d(x, y)) = (d(x, y))^p$, then ρ is a b-metric continuous and (M, ρ) is a b-metric space with $s = 2^{p-1}$. However, (M, ρ) is not necessarily a metric space.

In particular, let $M = \mathbb{R}$ be with $d(x, y) = |x - y|$ the usual euclidean metric on $\mathbb{R}$. Then $\rho(x, y) = (x - y)^2$ is a b-metric on $\mathbb{R}$, but it is not a metric since it does not satisfy the triangle inequality of the definition of metric, when $x = 0$, $y = 5$ and $z = 1$.

$$\rho(0, 5) = 25 \geq \rho(0, 1) + \rho(1, 5) = 17.$$

b) Let the sequence space $l_p^{(n)}$ $(0 < p < 1)$ defined by

$$l_p^{(n)} = \left\{ (x_n) \in \mathbb{R}^n \,/\, \sum_{k=1}^{n} |x_k|^p < \infty \right\}$$

together with the function

$$\rho : l_p^{(n)} \times l_p^{(n)} \to \mathbb{R}_+$$

defined by

$$\rho(x, y) = \left(\sum_{k=1}^{n} |x_k - y_k|^p \right)^{1/p}$$

with $x = (x_n)$, $y = (y_n) \in l_p^{(n)}$. This function ρ is a b-metric and $(l_p^{(n)}, \rho)$ is a b-metric space with $s = 2^{p-1}$, but it is not a usual metric space.

c) Let $M = C[0, 1] = \{f \,/\, f : [0, 1] \to \mathbb{R} \quad \text{continuous}\}$ be equipped with the uniform metric

$$d(x, y) = \sup_{t \in [0, 1]} |x(t) - y(t)|, \quad \text{for all } x, y \in M.$$

Now, for $p > 1$ we define

$$\rho(x, y) := (d(x, y))^p = \sup_{t \in [0, 1]} |x(t) - y(t)|^p, \quad \text{for all } x, y \in M.$$

Then, ρ is clearly a b-metric continuous on $C[0,1]$, consequently, (M, ρ) is a b-metric space with $s = 2^{p-1}$.

Definition 2. Let (M, ρ) be a b-metric space with $s \geq 1$. Then

   (1) The open ball of center $x \in M$ and radius $r > 0$ is given by

$$B_\rho(x, r) := \{y \in M \,/\, \rho(x, y) < r\}.$$

   (2) A subset $U \subset M$ is said to be an open set of M if for each $x \in M$ there exists $r > 0$ such that

$$B_\rho(x, r) \subset U.$$

Each b-metric ρ on a nonempty M generates a topology τ_p on M defined by

$$\tau_p = \{U \subset M \,/\, U \text{ is an open set in } M\}.$$

Notice that τ_p is Hausdorff.

Now, we present the notions of a convergent sequence, Cauchy sequence and complete b-metric spaces.

Definition 3. Let (M, ρ) be a b-metric space with $s \geq 1$. Then a sequence (x_n) in M is called:

   (1) $b-$convergent sequence if and only if there exists $x \in M$ such that $\lim_{n\to\infty} \rho(x_n, x) = 0$. In this case we denote it by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

   (2) $b-$Cauchy if $\lim_{n,m\to\infty} \rho(x_n, x_m) = 0$, this is equivalent to: given $\varepsilon > 0$ there exists $n_0(\varepsilon) \in \mathbb{N}$ such that for all $n, m > n_0(\varepsilon)$ implies that $\rho(x_n, x_m) < \varepsilon$.

   (3) A set $A \subset M$ is called closed if and only if for each sequence (x_n) in M which converge to x, we have $x \in A$.

   (4) If every $b-$Cauchy sequence in M is convergent, then (M, ρ) is said to be a complete b-metric space.

Proposition 1. *Let (M, ρ) be a b-metric space with $s \geq 1$. The following assertions hold:*

   (1) *A $b-$convergent sequence has a unique limit.*

   (2) *The subsequences of a $b-$convergent sequence are also $b-$convergent to the limit of the original sequence.*

Definition 4. Let (M, ρ) be a b-metric space with $s \geq 1$. The b-metric is called continuous if $\lim_{n\to\infty} \rho(x_n, x) = 0$ and $\lim_{n\to\infty} \rho(y_n, y) = 0$ then

$$\lim_{n\to\infty} \rho(x_n, y_n) = \rho(x, y).$$

The b-metrics given in Example 2 illustrate that their respective b-metric are continuous functions, but in general a b-metric is not a continuous function in all its variables (see, [24]).

The following results proved by A. Aghajani, M. Abbas and J. R. Rushan in [2], shows some facts about $b-$convergent sequences.

Proposition 2. *Let (M, ρ) be a b-metric space with $s \geq 1$, and suppose that (x_n) and (y_n) are $b-$convergent sequences to x and y respectively. Then we have,*

$$\frac{1}{s^2}\rho(x, y) \leq \liminf_{n \to \infty} \rho(x_n, y_n) \leq \limsup_{n \to \infty} \rho(x_n, y_n) \leq s^2 \rho(x, y).$$

In particular, if $x = y$ then $\lim_{n \to \infty} \rho(x_n, y_n) = 0$. Moreover, for each $z \in M$ we have

$$\frac{1}{s}\rho(x, z) \leq \liminf_{n \to \infty} \rho(x_n, z) \leq \limsup_{n \to \infty} \rho(x_n, z) \leq s\rho(x, z).$$

Proposition 3. *Let (M, ρ) be a b-metric space with $s \geq 1$. Let (x_n) be a sequence in M such that*

$$\lim_{n \to \infty} \rho(x_n, x_{n+1}) = 0.$$

If (x_n) is not a $b-$Cauchy sequence in M, then there exist $\varepsilon > 0$ and sequences of positive integers $(n(k))$ and $(m(k))$ with $n(k) > m(k) > k > 0$ such that

$$\rho(x_{m(k)}, x_{n(k)}) \geq \varepsilon, \quad \rho(x_{m(k)}, x_{n(k)-1}) < \varepsilon$$

and

$$\varepsilon \leq \liminf_{k \to \infty} \rho(x_{m(k)}, x_{n(k)}) \leq \limsup_{k \to \infty} \rho(x_{m(k)}, x_{n(k)}) \leq \varepsilon.$$

$$\frac{\varepsilon}{s^2} \leq \liminf_{k \to \infty} \rho(x_{m(k)}, x_{n(k)-1}) \leq \limsup_{k \to \infty} \rho(x_{m(k)}, x_{n(k)-1}) \leq \varepsilon.$$

$$\frac{\varepsilon}{s^2} \leq \liminf_{k \to \infty} \rho(x_{m(k)-1}, x_{n(k)-1}) \leq \limsup_{k \to \infty} \rho(x_{m(k)-1}, x_{n(k)-1}) \leq \varepsilon s.$$

$$\frac{\varepsilon}{s} \leq \liminf_{k \to \infty} \rho(x_{m(k)-1}, x_{n(k)}) \leq \limsup_{k \to \infty} \rho(x_{m(k)-1}, x_{n(k)}) \leq \varepsilon s^2.$$

Non commutative properties of pair of mappings on b-metric spaces

Now, we present some well-known non commutative definitions and properties of a pair of mappings in the setting of b-metric spaces that will be useful in the proof of our main results.

Definition 5. Let (M, ρ) be a b-metric space with $s \geq 1$. The mappings $S, T : M \to M$ are said to

(1) be compatible, ([14]) if and only if

$$\lim_{n \to \infty} \rho(STx_n, TSx_n) = 0,$$

whenever (x_n) is a sequence in M such that

$$\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \quad \text{for some } t \in M,$$

(2) be non compatible if there exists at least one sequence (x_n) in M such that

$$\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \quad \text{for some } t \in M,$$

but

$$\lim_{n \to \infty} \rho(STx_n, TSx_n) \quad \text{is either non zero or non existent,}$$

(3) be weakly compatible, ([12]) if for all $x \in M$, such that $Sx = Tx$, implies that $STx = TSx$,

(4) Satisfy the b-property (EA), ([1]) if there exists a sequence in M such that

$$\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t, \quad \text{for some } t \in M,$$

(5) Satisfy the $b-$limit range property with respect to T, (in short $b - CLR_T-$ property), ([23]), if there exists a sequence (x_n) in M such that

$$\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = Tt, \quad \text{for some } t \in M.$$

Weakly compatible selfmappings play a key role in this paper since it is a minimal requirement to prove the existence of common fixed point for mappings of contractive type, (for a discussion, see, [19, 20]). Thus, the following result will be useful in the sequel.

Proposition 4 (Jungck–Rhoades, [13]). *Let S and T be weakly compatible selfmaps of a nonempty set M. If S and T have a unique point of coincidence (POC); that is, a unique $u \in M$ such that $z = Su = Tu$, then z is the unique common fixed point of S and T.*

Remark 1. Note the following relations between some of the mentioned notions:

(1) If S and T are compatible, then S and T are weakly compatible.
(2) If S and T are non compatible, then S and T satisfy the b-Property (EA).
(3) The weak compatibility and the b-Property (EA) are independent to each other.

3. GENERALIZED Ψ-GERAGHTY-QUASI-CONTRACTIONS

In this section we introduce the class of generalized Ψ-Geraghty-quasi-contraction mappings in b-metric spaces, by using altering distance functions, Geraghty's property and the quasi-contraction mappings for a pair of maps.

We recall that in 1973, M. Geraghty [10] introduce a condition called *Geraghty's property* which was used to prove a fixed point theorems that generalize the (BCP) in usual metric spaces.

For its use in the framework of b-metric spaces, we consider the class of functions $\mathcal{B}_s$, where we say that $\beta \in \mathcal{B}_s$ if $\beta : \mathbb{R}_+ \to [0, 1/s)$, $s \geq 1$ and it has the Geraghty's

property if

$$\lim_{n\to\infty}\beta(t_n)=\frac{1}{s} \quad \text{implies that} \quad \lim_{n\to\infty}t_n=0. \quad \text{for any } (t_n)\subset\mathbb{R}_+. \qquad (3.1)$$

In 1984, M. S. Khan, M. Swalech and S. Sessa [17] introduce a new type of contraction mapping with the help of a control function which they called *altering distance functions*.

A function $\psi:\mathbb{R}_+\to\mathbb{R}_+$ is called an altering distance function if

   (i) ψ is monotonic increasing function.
   (ii) ψ is a continuous mappings.
   (iii) $\psi(t)=0$ if and only if $t=0$.

By Ψ we denote the set of all altering distance functions.

Definition 6. Let (M,ρ) be a b-metric space with $s\geq 1$. Two mappings $S,T:M\to M$ are called generalized $\Psi-$Geraghty-quasi-contractions, if there exist $\psi\in\Psi$ and $\beta\in\mathcal{B}_s$ such that

$$\psi[s\rho(Sx,Sy)]\leq\beta[\psi(N(x,y))]\psi(N(x,y)), \quad \text{for all } x,y\in M. \qquad (3.2)$$

Where,

$$N(x,y)=\max\{\rho(Tx,Ty),\ \rho(Sx,Tx),\ \rho(Sy,Ty),\ \rho(Sx,Ty),\ \rho(Sy,Tx)\}.$$

Remark 2. Since $\beta\in\mathcal{B}_s$ are strictly smaller than $1/s$ for some $s\geq 1$, we have

$$\beta[\psi(N(x,y))]<1/s, \quad \text{for all } x,y\in M.$$

Proposition 5. *Let* (M,ρ) *be a b-metric space with* $s\geq 1$ *and let* S,T *be two selfmaps of* M. *Assume that* S *and* T *are generalized* $\Psi-$*Geraghty-quasi-contractions. If* S *and* T *have (POC) in* M, *then it is unique.*

Proof. Suppose that z and w are two (POC) of S and T. Therefore, there exists $u,v\in M$ such that $Su=Tu=z$ and $Sv=Tv=w$. Suppose $z\neq w$. From inequality (3.2), we have

$$\psi[\rho(z,w)]\leq\psi[s\rho(z,w)]=\psi[s\rho(Su,Sv)]\leq\beta[\psi(N(u,v))]\psi[N(u,v)]$$

for all $u,v\in M$, where

$$\begin{aligned}N(u,v)&=\max\{\rho(Tu,Tv),\ \rho(Su,Tu),\ \rho(Sv,Tv),\ \rho(Su,Tv),\ \rho(Su,Tu)\}\\&=\max\{\rho(z,w),\ \rho(z,z),\ \rho(w,w),\ \rho(z,w),\ \rho(w,z)\}\\&=\rho(z,w).\end{aligned}$$

Therefore,

$$\begin{aligned}\psi[\rho(z,w)]\leq\psi[s\rho(z,w)]&\leq\beta[\psi(\rho(z,w))]\psi(\rho(z,w))\\&<(1/s)\psi(\rho(z,w))\leq\psi(\rho(z,w)),\end{aligned}$$

which is a contradiction. Hence $\rho(z,w)=0$, consequently, $z=w$. $\qquad\square$

Now, we show a condition that guarantee the existence of a unique (POC).

Theorem 4. *Let* (M, ρ) *be a b-metric space with* $s \geq 1$ *and let* $S, T : M \to M$ *be generalized* Ψ-*Geraghty-quasi-contractions. If* $TM \subset M$ *is a complete subspace of* M *and* S *and* T *satisfy the b-property (EA), then* S *and* T *have a unique (POC).*

Proof. Since S and T satisfy the b-property (EA), there exists a sequence (x_n) in M such that

$$\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = z \quad \text{for some } z \in M.$$

Since $TM \subset M$ is complete, we have

$$\lim_{n \to \infty} Sx_n = z = Tu = \lim_{n \to \infty} Tx_n \quad \text{for some } u \in M.$$

Now we assert that $Su = Tu$. Suppose that $Su \neq Tu$. From inequality (3.2) and Proposition 2, we obtain

$$\psi[\rho(Su, Tu)] \leq \psi[s\rho(Su, Tu)] \leq \psi\left[\frac{s^2\rho(Su, Tu)}{s}\right]$$

$$\leq \limsup_{n \to \infty} \psi[s^2\rho(Sx_n, Su)]$$

$$\leq s \limsup_{n \to \infty} \beta[\psi(N(x_n, u))] \limsup_{n \to \infty} \psi(N(x_n, u))$$

where

$$N(x_n, u) = \max\{\rho(Tu, Tx_n), \ \rho(Sx_n, Tx_n), \ \rho(Su, Tu), \ \rho(Sx_n, Tu), \ \rho(Su, Tx_n)\}.$$

Taking upper limit as $k \to \infty$ and applying Proposition 2, we get

$$\limsup_{n \to \infty} N(x_n, u) \leq \max\{s\rho(z, z), \ s^2\rho(z, z), \ \rho(Su, Tu), \ s\rho(z, z), \ s\rho(Su, Tu)\}$$

$$= \max\{\rho(Su, Tu), \ s\rho(Su, Tu)\} = s\rho(Su, Tu).$$

Then we have,

$$\psi(s\rho(Su, Tu)) \leq s\beta[\psi(s\rho(Su, Tu))]\psi[s\rho(Su, Tu)]$$

$$< \psi(s\rho(Su, Tu)).$$

Since $\psi \in \Psi$, we have a contradiction. Therefore, $\rho(Su, Tu) = 0$, that is, $Su = Tu$. This shows that u is a coincidence point of S and T and $z = Su = Tu$ is a (POC) of S and T. From Proposition 5, z is the unique (POC) of S and T. $\qquad\square$

4. COMMON FIXED POINTS

In this section we are going to prove common fixed point results for generalized Ψ-Geraghty-quasi-contractions. To attain such goal we use the notions of weakly compatible and the b-property (EA).

Theorem 5. *Let* (M, ρ) *be a b-metric space with* $s \geq 1$ *and let* $S, T : M \to M$ *be generalized* Ψ*-Geraghty-quasi-contractions. If* $TM \subset M$ *is a complete subspace of M and S and T satisfy the b-property (EA) and they are weakly compatible, then S and T have a unique common fixed point.*

Proof. Since *S* and *T* satisfy the *b*-property (EA), from Theorem 4, they have a unique (POC). Now, since *S* and *T* are weakly compatible, from Proposition 4 we obtain the conclusion. □

The following example supports our result.

Example 2. Let $M = [0, 1]$ be equipped with the *b*-metric defined by $\rho(x, y) = |x - y|^2 = (x - y)^2$. Consider the mappings defined by

$$Sx = \begin{cases} \dfrac{x}{9} & \text{if} \quad x \in [0, 1/2] \\ 1/9 & \text{if} \quad x \in (1/2, 1] \end{cases} \quad \text{and} \quad Tx = \begin{cases} \dfrac{x}{3} & \text{if} \quad x \in [0, 1/2] \\ 1/3 & \text{if} \quad x \in (1/2, 1]. \end{cases}$$

Let $\psi(t) = \sqrt{t}$ and $\beta(t) = \dfrac{1}{2} \quad \forall t \in \mathbb{R}_+$. Then, by considering the sequence $(x_n) = (1/n)$ in *M* we have that

$$\lim_{n \to \infty} \rho(Sx_n, 0) = \lim_{n \to \infty} \rho(Tx_n, 0) = 0.$$

Therefore, *S* and *T* satisfy the *b*-property (EA). Notice that $C(S, T) = \{x \in M \mid Sx = Tx\} = \{0\}$ and $ST0 = 0 = TS0$. Hence, *S* and *T* are weakly compatible selfmaps. Also notice that $TM \subset M$ is complete. On the other hand,

$$\begin{aligned} N(x, y) &= \max\{\rho(Tx, Ty), \rho(Sx, Tx), \rho(Sy, Ty), \rho(Sx, Ty), \rho(Sy, Tx)\} \\ &= \max\left\{\frac{1}{9}|x - y|^2, \frac{4}{27}x^2, \frac{4}{27}y^2, \frac{1}{81}|x - 3y|^2, \frac{1}{8}|y - 3|^2\right\} \\ &= \frac{1}{9}|x - y|^2. \end{aligned}$$

Thus, $\psi(N(x, y)) = 1/3|x - y|$, from which we obtain

$$\psi[\rho(Sx, Sy)] = \frac{1}{9}|x - y|.$$

Now, from (3.1), we have

$$\begin{aligned} \frac{1}{9}|x - y| = \psi(\rho(Sx, Sy)) &\leq \beta[1/3|x - y|](1/3|x - y|) \\ &< 1/2(1/3|x - y|) = 1/6|x - y|. \end{aligned}$$

Then all conditions of Theorem 5 holds. It is clear that $x = 0$ is the unique common fixed point of *S* and *T*.

Taking $\psi = \text{Id}$, (identity map) in the generalized Ψ-Geraghty-quasi-contraction of Theorem 5, we obtain

Corollary 1. *Let (M, ρ) be a metric space with $s \geq 1$ and let $S, T : M \to M$ be mappings that satisfy the following inequality*

$$\rho(Sx, Sy) \leq \beta(N(x, y))N(x, y) \tag{4.1}$$

for all $x, y \in M$, where

$$N(x, y) = \max\{\rho(Tx, Ty), \ \rho(Sx, Sy), \ \rho(Sy, Ty), \ \rho(Sx, Ty), \ \rho(Sy, Tx)\}.$$

If S and T satisfy the b-property (EA) and if $TM \subset M$ is complete, then

   (1) *S and T have a unique (POC) and*
   (2) *If S and T are weakly compatible, then S and T have a unique common fixed point in M.*

Since two non compatible selfmap of a *b*-metric space (M, ρ) satisfy the *b*-property (EA), we get the following consequence.

Corollary 2. *Let (M, ρ) be a b-metric space with $s \geq 1$ and let $S, T : M \to M$ be maps that satisfy the following conditions:*

   (i) *S and T are non compatible selfmaps.*
  (ii) *$TM \subset M$ is complete.*
 (iii) *S and T satisfy any of the following inequalities (3.2) or (4.1).*

Then,

   (1) *S and T have a unique (POC) and*
   (2) *If S and T are weakly compatible, then S and T have a unique common fixed point in M.*

In the next results we replace the closedness of the range of T with the *b*-CLR$_T$-property.

Theorem 6. *Let (M, ρ) be a b-metric space with $s \geq 1$ and let $S, T : M \to M$ be maps holding the following conditions:*

   (i) *S and T satisfy the b-CLR$_T$-property.*
  (ii) *S and T are generalized $\Psi-$Geraghty-quasi-contraction.*

Then,

   (1) *S and T have a unique (POC).*
   (2) *If S and T are weakly compatible then S and T have a unique common fixed point in M.*

Proof. Since S and T satisfy the *b*-CLR$_T$-property, then there exists a sequence (x_n) in M such that

$$\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = Tu \quad \text{for some } u \in M.$$

Therefore, there exists $z \in M$ such that $z = Tu$. The rest of the proof follows as in the proof of Theorem 4.1. $\qquad\square$

Corollary 3. *Let* (M, ρ) *be a b-metric space with* $s \geq 1$ *and let* $S, T : M \to M$ *be maps holding the following conditions:*

    (i) *S and T satisfy the b-CLR$_T$-property.*
   (ii) *S and T satisfy any of following inequalities* (3.2) *or* (4.1).

Then,

    (1) *S and T have a unique (POC), and*
    (2) *if S and T are weakly compatible, then S and T have a unique common fixed point in M.*

5. ON THE JUNGCK-MANN ITERATIVE SCHEME FOR GENERALIZED Ψ-GERAGHTY-QUASI CONTRACTIONS

Effective algorithms to compute approximate common fixed points for pairs of mappings is a crucial tool in the fixed point theory. In this section, we will prove the *b*-convergence of the so-called *Jungck-Mann iterative scheme* ([22]):

$$Tx_{n+1} = (1 - \alpha_n)Sx_n + \alpha_n Tx_n, \tag{5.1}$$

where (α_n) is a sequence in $[0, 1]$.

In order to define this iteration process in our setting, we are going to consider a *b*-metric space induced by a *quasinormed space* (M, ρ); that is, a vector space M endowed with a function $\rho : M \longrightarrow \mathbb{R}_+$ satisfying

    (ρ_1) $\rho(x) \geq 0$ and $\rho(x) = 0$ iff $x = \mathbf{0}$.
    (ρ_2) $\rho(\lambda x) = |\lambda|\rho(x)$, $\lambda \in \mathbb{R}$.
    (ρ_3) $\rho(x + y) \leq s(\rho(x) + \rho(y))$, $s \geq 1$.

(M, ρ) is metrizable in virtue of the Aoki-Rolewicz theorem ([3, 21]) and it becomes a *b*-metric space with the induced *b*-metric $\rho(x, y) := \rho(x - y)$.

The spaces given in Example 1 are *b*-normed spaces, as well as, some useful spaces in functional analysis as: *Lebesgue spaces with variable exponent* and scales of *weak Lebesgue spaces, Lorentz spaces, Hardy spaces, weak Hardy spaces, Lorentz-based Hardy spaces, Besov spaces, Triebel Lizorkin space* among others.

To prove the convergence of the Jungck-Mann iterative scheme, in the next result we are going to assume that the coefficient *s* of the *b*-normed space is greater or equal than 2 and $\psi \in \Psi$ satisfies the following conditions:

    (i) There exists a function ϕ continuous on $[0, \infty)$ with $\psi(ab) \leq \phi(a)\psi(b)$ for $a > 0$ and $b \geq 0$.
   (ii) $\psi(a + b) \leq \psi(a) + \psi(b)$ for all $a, b \geq 0$.
  (iii) $\psi(s) < \frac{1}{2}$, where *s* is the coefficient of (M, ρ).

A nondecreasing function satisfying (i) it is said to belong to the class H. This class was introduced by F.M. Dannan in [8] in relation to the study of nonlinear generalizations of the Gronwall-Bellman inequality and its applications to the asymptotic behavior of differential equations.

Notice that the function $\psi(t) = \theta\sqrt{t}$, with $\theta < 1/(2f(s))$, for any function f satisfying $f(t) \geq \sqrt{t}$, satisfies conditions (i)–(iii) above.

Theorem 7. *Let (M, ρ) be a b-normed space with $s \geq 2$ and let $S, T : M \to M$ be generalized Ψ-Geraghty-quasi-contractions with $S(M) \subset T(M)$, and let us suppose that the pair (S,T) has a common fixed point p. Then, for any $x_0 \in M$, the iterative process* (5.1) *b-converges to p.*

Proof. Let $p \in M$ such that $p = Sp = Tp$ and $x_o \in M$ arbitrary. Then,

$$\rho(Tx_{n+1} - p) = \rho((1 - \alpha_n)Sx_n + \alpha_n Tx_n + (1 - \alpha_n + \alpha_n)p)$$
$$\leq s(1 - \alpha_n)\rho(Sx_n - p) + s\alpha_n\rho(Tx_n - p).$$

Since $\psi \in \Psi$ is subadditive, we conclude

$$\psi(\rho(Tx_{n+1} - p)) \leq \psi(s(1 - \alpha_n)\rho(Sx_n - p)) + \psi(s\alpha_n\rho(Tx_n - p))$$
$$\leq \psi(s\rho(Sx_n - p)) + \psi(s\rho(Tx_n - p)).$$

On the other hand, since S is a Ψ-Geraghty-quasi contraction, we have

$$\psi(s\rho(Sx_n - p)) = \psi(s\rho(Sx_n - Sp)) \leq \beta(\psi(N(x_n,p)))\psi(N(x_n,p)),$$

where

$$N(x_n, p) = \max\{\rho(Tx_n - p), \rho(Sx_n - Tx_n), \rho(p - p), \rho(Sx_n - p), \rho(p - Tx_n)\}$$
$$\leq \max\{\rho(Tx_n - p), s[\rho(Sx_n - p) + \rho(Tx_n - p)], \rho(Sx_n - p)\}$$
$$= s[\rho(Sx_n - p) + \rho(Tx_n - p)].$$

Thus,

$$\psi(N(x_n,p)) \leq \psi(s\rho(Sx_n - p)) + \psi(s\rho(Tx_n - p)).$$

From the fact that $\beta(t) < s^{-1}$ and $s \geq 2$, we obtain the following estimate:

$$\psi(s\rho(Sx_n - p)) < \frac{1}{s}\psi(s\rho(Sx_n - p)) + \frac{1}{s}\psi(s\rho(Tx_n - p))$$
$$\leq \frac{1}{2}\psi(s\rho(Sx_n - p)) + \frac{1}{2}\psi(s\rho(Tx_n - p)).$$

Then, we have

$$\psi(s\rho(Sx_n - p)) < \psi(s\rho(Tx_n - p)).$$

With these estimates we obtain that

$$\psi(\rho(Tx_{n+1} - p)) < 2\psi(s\rho(Tx_n - p))$$
$$\leq 2\phi(s)(\rho(Tx_n - p))$$
$$\leq (2\phi(s))^n(\rho(Tx_0 - p)).$$

Therefore, we conclude that $\lim_{n \to \infty} \psi(\rho(Tx_{n+1} - p)) = 0$, since $2\phi(s) < 1$. This implies that $\lim_{n \to \infty} \rho(Tx_{n+1} - p) = 0$, as $n \to \infty$. $\qquad\square$

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