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ON THE GENERALIZED BI-PERIODIC FIBONACCI AND LUCAS QUATERNIONS

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Abstract. In this paper we introduce the generalized bi-periodic Fibonacci and Lucas quaternions which are the further generalizations of the bi-periodic Fibonacci and Lucas quaternions considered in the literature. For those quaternions, we derive the generating functions, Binet's formulas and Catalan's identities.

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1. INTRODUCTION

As is well known, the Fibonacci sequence $\{F_n\}$ is generated from the recurrence relation $F_n = F_{n-1} + F_{n-2}$ $(n \geq 2)$ with $F_0 = 0,\ F_1 = 1$, and the Lucas sequence $\{L_n\}$ is generated from the recurrence relation $L_n = L_{n-1} + L_{n-2}$ $(n \geq 2)$ with $L_0 = 2,\ L_1 = 1$. The Binet's formulas for $\{F_n\}$ and $\{L_n\}$ are respectively given by

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$
$$L_n = \alpha^n + \beta^n,$$

where $\alpha(>0)$ and $\beta(<0)$ are roots of the equation $x^2 - x - 1 = 0$.

Many authors generalized the Fibonacci and Lucas sequences by changing initial conditions and/or recurrence relations. In particular, Edson and Yayenie [5] introduced the bi-periodic Fibonacci sequence $\{p_n\}$ defined by

$$p_0 = 0,\ p_1 = 1,\ \ p_n = \begin{cases} ap_{n-1} + p_{n-2}, & \text{if } n \text{ is even} \\ bp_{n-1} + p_{n-2}, & \text{if } n \text{ is odd} \end{cases} \ (n \geq 2). \tag{1.1}$$

The Binet's formula for $\{p_n\}$ is given by [5]

$$p_n = \frac{a^{\xi(n+1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}}\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right), \tag{1.2}$$

where $\alpha(>0)$ and $\beta(<0)$ are roots of the equation $x^2-abx-ab=0$, and $\zeta(\cdot)$ is the parity function such that $\zeta(n)=0$ if n is even and $\zeta(n)=1$ if n is odd.

The bi-periodic Fibonacci sequence $\{p_n\}$ given in (1.1) includes many sequences as special cases. For $a=b=1$, $\{p_n\}$ becomes the Fibonacci sequence. For $a=b=2$, $\{p_n\}$ becomes the Pell sequence. If $a=b=k$, then $\{p_n\}$ denotes the k-Fibonacci sequence defined in [8], etc.

On the other hand, Bilgici [2] generalized the Lucas sequence by introducing the bi-periodic Lucas sequence $\{u_n\}$ defined by

$$u_0=2,\ u_1=a,\ u_n=\begin{cases} bu_{n-1}+u_{n-2}, & \text{if } n \text{ is even} \\ au_{n-1}+u_{n-2}, & \text{if } n \text{ is odd} \end{cases} \quad (n\geq 2). \tag{1.3}$$

If $a=b=1$, then $\{u_n\}$ becomes the Lucas sequence $\{L_n\}$. If $a=b=k$, then $\{u_n\}$ becomes the k-Lucas sequence in [7].

The Binet's formula for $\{u_n\}$ is given by [2]

$$u_n=\frac{a^{\zeta(n)}}{(ab)^{\lfloor\frac{n+1}{2}\rfloor}}(\alpha^n+\beta^n), \tag{1.4}$$

where α and β are as defined in (1.2).

A quaternion q is defined by

$$q=q_0e_0+q_1e_1+q_2e_2+q_3e_3,$$

where $q_0,q_1,q_2,q_3\in\mathbb{R}$, $e_0=1$, and e_1,e_2 and e_3 are the standard basis in $\mathbb{R}^3$ such that $e_i^2=-1$, $i=1,2,3$, and

$$e_1e_2=-e_2e_1=e_3,\ e_2e_3=-e_3e_2=e_1,\ e_3e_1=-e_1e_3=e_2.$$

As noted in the literature [1, 6, 9, 20], quaternions are widely used in the fields of engineering and physics as well as mathematics, and attracted sustained attention from many researchers. In particular, a variety of results are available in the literature on the properties of quaternions related to the sequences described earlier. Horadam [13] defined the Fibonacci quaternion sequence $\{G_n\}$ and Lucas quaternion sequence $\{H_n\}$ as

$$\begin{aligned} G_n &= F_ne_0+F_{n+1}e_1+F_{n+2}e_2+F_{n+3}e_3, \\ H_n &= L_ne_0+L_{n+1}e_1+L_{n+2}e_2+L_{n+3}e_3, \end{aligned}$$

where F_n and L_n are respectively the nth Fibonacci and Lucas numbers.

Following the work of Horadam [13], diverse results have appeared in the literature. Halici [10] obtained the generating functions, Binet's formulas and some combinatorial properties of the Fibonacci and Lucas quaternions. Halici [11] also introduced the complex Fibonacci quaternions. Ramirez [15] studied the properties of the k-Fibonacci and k-Lucas quaternions. Çimen and İpek [4], Szynal-Liana and Włoch [17] investigated the Pell and Pell-Lucas quaternions. Szynal-Liana and Włoch [17] introduced the Jacobsthal and Jacobsthal-Lucas quaternions also. Catarino [3] considered the modified Pell and modified k-Pell quaternions. Halici and Karataş [12] defined a general quaternion which includes several quaternions mentioned above as special cases.

Recently Tan et al. [19] introduced the bi-periodic Fibonacci quaternion sequence $\{P_n\}$ defined by

$$P_n = p_n e_0 + p_{n+1} e_1 + p_{n+2} e_2 + p_{n+3} e_3, \tag{1.5}$$

where p_n is the nth bi-periodic Fibonacci number.

The Binet's formula for $\{P_n\}$ is given by [19]

$$P_n = \begin{cases} \frac{1}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^* \alpha^n - \beta^* \beta^n}{\alpha - \beta} \right), & \text{if } n \text{ is even} \\ \frac{1}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^{**} \alpha^n - \beta^{**} \beta^n}{\alpha - \beta} \right), & \text{if } n \text{ is odd} \end{cases} \tag{1.6}$$

where α and β are as defined in (1.2), and

$$\alpha^* = \sum_{l=0}^{3} \frac{a^{\xi(l+1)}}{(ab)^{\lfloor \frac{l}{2} \rfloor}} \alpha^l e_l,$$

$$\beta^* = \sum_{l=0}^{3} \frac{a^{\xi(l+1)}}{(ab)^{\lfloor \frac{l}{2} \rfloor}} \beta^l e_l,$$

$$\alpha^{**} = \sum_{l=0}^{3} \frac{a^{\xi(l)}}{(ab)^{\lfloor \frac{l+1}{2} \rfloor}} \alpha^l e_l,$$

$$\beta^{**} = \sum_{l=0}^{3} \frac{a^{\xi(l)}}{(ab)^{\lfloor \frac{l+1}{2} \rfloor}} \beta^l e_l.$$

Tan et al. [18] also introduced the bi-periodic Lucas quaternion sequence $\{U_n\}$ as follows:

$$U_n = u_n e_0 + u_{n+1} e_1 + u_{n+2} e_2 + u_{n+3} e_3, \tag{1.7}$$

where u_n is the nth bi-periodic Lucas number.

The Binet's formula for $\{U_n\}$ is given by [18]

$$U_n = \begin{cases} \frac{1}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}} (\alpha^{**} \alpha^n + \beta^{**} \beta^n), & \text{if } n \text{ is even} \\ \frac{1}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}} (\alpha^* \alpha^n + \beta^* \beta^n), & \text{if } n \text{ is odd} \end{cases} \tag{1.8}$$

where α, β are as defined in (1.2), and α^*, β^*, α^{**} and β^{**} are as defined in (1.6).

If we use the initial condition $P_0 = e_1 + e_2 + 2e_3$ and $P_1 = e_0 + e_1 + 2e_2 + 3e_3$ in (1.5), then $\{P_n\}$ is the same as the generalized Fibonacci quaternion sequence considered in [14]. Also if we set $P_0 = 2e_0 + e_1 + 3e_2 + 4e_3$ and $P_1 = e_0 + 3e_1 + 4e_2 +$

$7e_3$ in (1.5), then $\{P_n\}$ is the same as the generalized Lucas quaternion sequence considered in [14].

In this paper we introduce the generalized bi-periodic Fibonacci and Lucas quaternion sequences which include $\{P_n\}$ and $\{U_n\}$ as special cases. For those quaternions, we derive the generating functions, Binet's formulas and Catalan's identities.

2. Main results

2.1. *Generalized bi-periodic Fibonacci quaternion*

Consider the generalized bi-perioodic Fibonacci sequence $\{q_n\}$ defined by Sahin [16] and Yayenie [21] as

$$q_0 = 0,\ q_1 = 1,\ \ q_n = \begin{cases} aq_{n-1} + cq_{n-2}, & \text{if } n \text{ is even} \\ bq_{n-1} + dq_{n-2}, & \text{if } n \text{ is odd} \end{cases} \quad (n \geq 2). \tag{2.1}$$

The Binet's formula for $\{q_n\}$ is given by [21]

$$q_n = \frac{a^{\xi(n+1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^{\lfloor \frac{n}{2} \rfloor}(\alpha + d - c)^{n - \lfloor \frac{n}{2} \rfloor} - \beta^{\lfloor \frac{n}{2} \rfloor}(\beta + d - c)^{n - \lfloor \frac{n}{2} \rfloor}}{\alpha - \beta} \right), \tag{2.2}$$

where $\alpha(>0)$ and $\beta(<0)$ are roots of the equation $x^2 - (ab + c - d)x - abd = 0$.

Definition 1. We define the generalized bi-periodic Fibonacci quaternion sequence $\{Q_n\}$ by

$$Q_n = q_n e_0 + q_{n+1} e_1 + q_{n+2} e_2 + q_{n+3} e_3, \tag{2.3}$$

where q_n is the nth generalized bi-periodic Fibonacci number.

If $c = d = 1$, then $\{Q_n\}$ becomes the bi-periodic Fibonacci quaternion sequence given in (1.5).

If $a = b = 1$ and $c = d = 2$, then $\{Q_n\}$ becomes the Jacobsthal quaternion sequence defined in [17].

In the rest of the paper, we will use the following identities [21] whenever necessary: (i) $\alpha + \beta = ab + c - d$, (ii) $\alpha\beta = -abd$, (iii) $\alpha(\alpha + d - c) = ab(\alpha + d)$, (iv) $\beta(\beta + d - c) = ab(\beta + d)$, (v) $(\alpha + d)(\beta + d) = cd$.

Theorem 1 (Generating function)**.** *The generating function for the generalized bi-periodic Fibonacci quaternion sequence is*

$$G(x) = \frac{\big(1 - (ab + d)x^2 + bcx^3\big)Q_0 + x(1 + ax - cx^2)Q_1}{1 - (ab + c + d)x^2 + cdx^4}. \tag{2.4}$$

Proof. We can show that $\{Q_n\}$ satisfies the same recurrence relation as $\{q_n\}$ with the initial condition

$$\begin{aligned} Q_0 &= q_0 e_0 + q_1 e_1 + q_2 e_2 + q_3 e_3 \\ &= e_1 + ae_2 + (ab + d)e_3, \end{aligned}$$

$$\begin{aligned} Q_1 &= q_1 e_0 + q_2 e_1 + q_3 e_2 + q_4 e_3 \\ &= e_0 + a e_1 + (ab+d) e_2 + a(ab+c+d) e_3, \end{aligned}$$

and

$$\begin{aligned} Q_{2n} &= (ab+c+d) Q_{2n-2} - cd Q_{2n-4}, \\ Q_{2n+1} &= (ab+c+d) Q_{2n-1} - cd Q_{2n-3}. \end{aligned}$$

Then, proceeding as in the proof of [21, Theorem 7], we can obtain (2.4). □

If $a = b$ and $c = d$, then

$$\begin{aligned} G(x) &= \frac{(1-ax)Q_0 + xQ_1}{1-ax-cx^2} \\ &= \frac{xe_0 + e_1 + (a+x)e_2 + (a^2+c+acx)e_3}{1-ax-cx^2}. \end{aligned}$$

Hence, for $a = b = c = d = 1$, we get the generating function for the Fibonacci quaternion

$$G(x) = \frac{xe_0 + e_1 + (1+x)e_2 + (2+x)e_3}{1-x-x^2},$$

as in [10], and, for $a = b = k$ and $c = d = 1$, we obtain the generating function for the k-Fibonacci quaternion

$$G(x) = \frac{xe_0 + e_1 + (k+x)e_2 + (k^2+1+kx)e_3}{1-kx-x^2},$$

which is given in [15].

Theorem 2 (Binet's formula)**.** *The Binet's formula for the generalized bi-periodic Fibonacci quaternion sequence is*

$$Q_n = \begin{cases} \frac{1}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha_e \alpha^{\lfloor \frac{n}{2} \rfloor} (\alpha+d-c)^{n-\lfloor \frac{n}{2} \rfloor} - \beta_e \beta^{\lfloor \frac{n}{2} \rfloor} (\beta+d-c)^{n-\lfloor \frac{n}{2} \rfloor}}{\alpha-\beta} \right), & \text{if } n \text{ is even} \\ \frac{1}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha_o \alpha^{\lfloor \frac{n}{2} \rfloor} (\alpha+d-c)^{n-\lfloor \frac{n}{2} \rfloor} - \beta_o \beta^{\lfloor \frac{n}{2} \rfloor} (\beta+d-c)^{n-\lfloor \frac{n}{2} \rfloor}}{\alpha-\beta} \right), & \text{if } n \text{ is odd} \end{cases} \tag{2.5}$$

where

$$\begin{aligned} \alpha_e &= \sum_{l=0}^{3} \frac{a^{\xi(l+1)}}{(ab)^{\lfloor \frac{l}{2} \rfloor}} \alpha^{\lfloor \frac{l}{2} \rfloor} (\alpha+d-c)^{\lfloor \frac{l+1}{2} \rfloor} e_l, \\ \beta_e &= \sum_{l=0}^{3} \frac{a^{\xi(l+1)}}{(ab)^{\lfloor \frac{l}{2} \rfloor}} \beta^{\lfloor \frac{l}{2} \rfloor} (\beta+d-c)^{\lfloor \frac{l+1}{2} \rfloor} e_l, \\ \alpha_o &= \sum_{l=0}^{3} \frac{a^{\xi(l)}}{(ab)^{\lfloor \frac{l+1}{2} \rfloor}} \alpha^{\lfloor \frac{l+1}{2} \rfloor} (\alpha+d-c)^{\lfloor \frac{l}{2} \rfloor} e_l, \end{aligned}$$

$$\beta_o = \sum_{l=0}^{3} \frac{a^{\zeta(l)}}{(ab)^{\lfloor \frac{l+1}{2} \rfloor}} \beta^{\lfloor \frac{l+1}{2} \rfloor} (\beta + d - c)^{\lfloor \frac{l}{2} \rfloor} e_l.$$

Proof. Firstly we note that $\lfloor \frac{n}{2} \rfloor = \frac{n-\zeta(n)}{2}$.
¿From (2.2) and (2.3), we have

$$\begin{aligned}
Q_n &= \frac{a^{\zeta(n+1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^{\lfloor \frac{n}{2} \rfloor}(\alpha+d-c)^{n-\lfloor \frac{n}{2} \rfloor} - \beta^{\lfloor \frac{n}{2} \rfloor}(\beta+d-c)^{n-\lfloor \frac{n}{2} \rfloor}}{\alpha-\beta} \right) e_0 \\
&+ \frac{a^{\zeta(n)}}{(ab)^{\zeta(n)}(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^{\zeta(n)}(\alpha+d-c)^{\zeta(n+1)}\alpha^{\lfloor \frac{n}{2} \rfloor}(\alpha+d-c)^{n-\lfloor \frac{n}{2} \rfloor}}{\alpha-\beta} \right. \\
&\qquad \left. - \frac{\beta^{\zeta(n)}(\beta+d-c)^{\zeta(n+1)}\beta^{\lfloor \frac{n}{2} \rfloor}(\beta+d-c)^{n-\lfloor \frac{n}{2} \rfloor}}{\alpha-\beta} \right) e_1 \\
&+ \frac{a^{\zeta(n+1)}}{ab(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha(\alpha+d-c)\alpha^{\lfloor \frac{n}{2} \rfloor}(\alpha+d-c)^{n-\lfloor \frac{n}{2} \rfloor}}{\alpha-\beta} \right. \\
&\qquad \left. - \frac{\beta(\beta+d-c)\beta^{\lfloor \frac{n}{2} \rfloor}(\beta+d-c)^{n-\lfloor \frac{n}{2} \rfloor}}{\alpha-\beta} \right) e_2 \\
&+ \frac{a^{\zeta(n)}}{(ab)^{1+\zeta(n)}(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^{1+\zeta(n)}(\alpha+d-c)^{1+\zeta(n+1)}\alpha^{\lfloor \frac{n}{2} \rfloor}(\alpha+d-c)^{n-\lfloor \frac{n}{2} \rfloor}}{\alpha-\beta} \right. \\
&\qquad \left. - \frac{\beta^{1+\zeta(n)}(\beta+d-c)^{1+\zeta(n+1)}\beta^{\lfloor \frac{n}{2} \rfloor}(\beta+d-c)^{n-\lfloor \frac{n}{2} \rfloor}}{\alpha-\beta} \right) e_3,
\end{aligned}$$

or

$$Q_n = \frac{1}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha_n \alpha^{\lfloor \frac{n}{2} \rfloor}(\alpha+d-c)^{n-\lfloor \frac{n}{2} \rfloor} - \beta_n \beta^{\lfloor \frac{n}{2} \rfloor}(\beta+d-c)^{n-\lfloor \frac{n}{2} \rfloor}}{\alpha-\beta} \right),$$

where

$$\begin{aligned}
\alpha_n &= a^{\zeta(n+1)} e_0 + \frac{a^{\zeta(n)}\alpha^{\zeta(n)}(\alpha+d-c)^{\zeta(n+1)}}{(ab)^{\zeta(n)}} e_1 \\
&\quad + \frac{a^{\zeta(n+1)}\alpha(\alpha+d-c)}{ab} e_2 + \frac{a^{\zeta(n)}\alpha^{1+\zeta(n)}(\alpha+d-c)^{1+\zeta(n+1)}}{(ab)^{1+\zeta(n)}} e_3, \\
\beta_n &= a^{\zeta(n+1)} e_0 + \frac{a^{\zeta(n)}\beta^{\zeta(n)}(\beta+d-c)^{\zeta(n+1)}}{(ab)^{\zeta(n)}} e_1 \\
&\quad + \frac{a^{\zeta(n+1)}\beta(\beta+d-c)}{ab} e_2 + \frac{a^{\zeta(n)}\beta^{1+\zeta(n)}(\beta+d-c)^{1+\zeta(n+1)}}{(ab)^{1+\zeta(n)}} e_3.
\end{aligned}$$

If n is even, then

$$\begin{aligned}\alpha_n &= ae_0 + (\alpha+d-c)e_1 + \frac{a\alpha(\alpha+d-c)}{ab}e_2 + \frac{\alpha(\alpha+d-c)^2}{ab}e_3\\ &= \sum_{l=0}^{3}\frac{a^{\xi(l+1)}}{(ab)^{\lfloor\frac{l}{2}\rfloor}}\alpha^{\lfloor\frac{l}{2}\rfloor}(\alpha+d-c)^{\lfloor\frac{l+1}{2}\rfloor}e_l,\\ \beta_n &= ae_0 + (\beta+d-c)e_1 + \frac{a\beta(\beta+d-c)}{ab}e_2 + \frac{\beta(\beta+d-c)^2}{ab}e_3\\ &= \sum_{l=0}^{3}\frac{a^{\xi(l+1)}}{(ab)^{\lfloor\frac{l}{2}\rfloor}}\beta^{\lfloor\frac{l}{2}\rfloor}(\beta+d-c)^{\lfloor\frac{l+1}{2}\rfloor}e_l.\end{aligned}$$

Similarly, if n is odd, then

$$\alpha_n = \sum_{l=0}^{3}\frac{a^{\xi(l)}}{(ab)^{\lfloor\frac{l+1}{2}\rfloor}}\alpha^{\lfloor\frac{l+1}{2}\rfloor}(\alpha+d-c)^{\lfloor\frac{l}{2}\rfloor}e_l,$$

$$\beta_n = \sum_{l=0}^{3}\frac{a^{\xi(l)}}{(ab)^{\lfloor\frac{l+1}{2}\rfloor}}\beta^{\lfloor\frac{l+1}{2}\rfloor}(\beta+d-c)^{\lfloor\frac{l}{2}\rfloor}e_l,$$

and the proof is completed. □

If $c = d = 1$, then (2.5) becomes the Binet's formula for the bi-periodic Fibonacci quaternion given in (1.6).

Theorem 3 (Catalan's identity)**.** *The Catalan's identity for the generalized bi-periodic Fibonacci quaternion sequence is*

$$Q_n^2 - Q_{n+2r}Q_{n-2r} = \begin{cases} (cd)^{\frac{n-2r}{2}}\left(\frac{\alpha_e\beta_e\left((\alpha+d)^{2r}-(cd)^r\right)+\beta_e\alpha_e\left((\beta+d)^{2r}-(cd)^r\right)}{(\alpha-\beta)^2}\right), & if\ n\ is\ even,\\ abc(cd)^{\frac{n-2r-1}{2}}\left(\frac{\alpha_o\beta_o\left((cd)^r-(\alpha+d)^{2r}\right)+\beta_o\alpha_o\left((cd)^r-(\beta+d)^{2r}\right)}{(\alpha-\beta)^2}\right), & if\ n\ is\ odd.\end{cases} \quad (2.6)$$

Proof. Firstly, assume that n is even, and let

$$X_1 = (\alpha-\beta)^2(ab)^nQ_n^2,$$
$$X_2 = (\alpha-\beta)^2(ab)^nQ_{n+2r}Q_{n-2r}.$$

Then

$$\begin{aligned}X_1 &= \left(\alpha_e\alpha^{\frac{n}{2}}(\alpha+d-c)^{\frac{n}{2}} - \beta_e\beta^{\frac{n}{2}}(\beta+d-c)^{\frac{n}{2}}\right)^2\\ &= \alpha_e^2\alpha^n(\alpha+d-c)^n + \beta_e^2\beta^n(\beta+d-c)^n\end{aligned}$$

$$
\begin{aligned}
&-(\alpha_e\beta_e+\beta_e\alpha_e)\alpha^{\frac{n}{2}}(\alpha+d-c)^{\frac{n}{2}}\beta^{\frac{n}{2}}(\beta+d-c)^{\frac{n}{2}},\\
&=\alpha_e^2\alpha^n(\alpha+d-c)^n+\beta_e^2\beta^n(\beta+d-c)^n\\
&-(\alpha_e\beta_e+\beta_e\alpha_e)(ab)^n(\alpha+d)^{\frac{n}{2}}(\beta+d)^{\frac{n}{2}},\\
&=\alpha_e^2\alpha^n(\alpha+d-c)^n+\beta_e^2\beta^n(\beta+d-c)^n\\
&-(\alpha_e\beta_e+\beta_e\alpha_e)(ab)^n(cd)^{\frac{n}{2}},
\end{aligned}
$$

and

$$
\begin{aligned}
X_2&=\left(\alpha_e\alpha^{\frac{n+2r}{2}}(\alpha+d-c)^{\frac{n+2r}{2}}-\beta_e\beta^{\frac{n+2r}{2}}(\beta+d-c)^{\frac{n+2r}{2}}\right)\\
&\times\left(\alpha_e\alpha^{\frac{n-2r}{2}}(\alpha+d-c)^{\frac{n-2r}{2}}-\beta_e\beta^{\frac{n-2r}{2}}(\beta+d-c)^{\frac{n-2r}{2}}\right)\\
&=\alpha_e^2\alpha^n(\alpha+d-c)^n+\beta_e^2\beta^n(\beta+d-c)^n\\
&-\alpha_e\beta_e\alpha^{\frac{n+2r}{2}}(\alpha+d-c)^{\frac{n+2r}{2}}\beta^{\frac{n-2r}{2}}(\beta+d-c)^{\frac{n-2r}{2}}\\
&-\beta_e\alpha_e\alpha^{\frac{n-2r}{2}}(\alpha+d-c)^{\frac{n-2r}{2}}\beta^{\frac{n+2r}{2}}(\beta+d-c)^{\frac{n+2r}{2}},\\
&=\alpha_e^2\alpha^n(\alpha+d-c)^n+\beta_e^2\beta^n(\beta+d-c)^n\\
&-\alpha_e\beta_e(ab)^n(\alpha+d)^{\frac{n+2r}{2}}(\beta+d)^{\frac{n-2r}{2}}\\
&-\beta_e\alpha_e(ab)^n(\alpha+d)^{\frac{n-2r}{2}}(\beta+d)^{\frac{n+2r}{2}},\\
&=\alpha_e^2\alpha^n(\alpha+d-c)^n+\beta_e^2\beta^n(\beta+d-c)^n\\
&-\alpha_e\beta_e(ab)^n(cd)^{\frac{n-2r}{2}}(\alpha+d)^{2r}\\
&-\beta_e\alpha_e(ab)^n(cd)^{\frac{n-2r}{2}}(\beta+d)^{2r}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
X_1-X_2&=\alpha_e\beta_e(ab)^n(cd)^{\frac{n-2r}{2}}\left((\alpha+d)^{2r}-(cd)^r\right)\\
&+\beta_e\alpha_e(ab)^n(cd)^{\frac{n-2r}{2}}\left((\beta+d)^{2r}-(cd)^r\right),
\end{aligned}
$$

and the proof is completed for the case where n is even.

When n is odd, we can proceed similarly, and details are omitted. □

If $c=d=1$, then $\alpha^2=ab(\alpha+1)$ and

$$
\begin{aligned}
(\alpha+1)^{2r}-1&=\frac{(ab)^{2r}(\alpha+1)^{2r}-(ab)^{2r}}{(ab)^{2r}}\\
&=\frac{\alpha^{4r}-(ab)^{2r}}{(ab)^{2r}}.
\end{aligned}
$$

Similarly

$$(\beta+1)^{2r}-1=\frac{\beta^{4r}-(ab)^{2r}}{(ab)^{2r}},$$

and Theorem 3 reduces to [19, Theorem 5].

2.2. *Generalized bi-periodic Lucas quaternion*

Consider the generalized bi-periodic Lucas sequence $\{v_n\}$ defined by Bilgici [2] as

$$v_0=\frac{d+1}{d},\ v_1=a,\ v_n=\begin{cases} bv_{n-1}+dv_{n-2}, & \text{if } n \text{ is even} \\ av_{n-1}+cv_{n-2}, & \text{if } n \text{ is odd} \end{cases} \quad (n\geq 2). \tag{2.7}$$

The Binet's formula for $\{v_n\}$ is given by [2]

$$v_n=\frac{a^{\xi(n)}}{(ab)^{\lfloor\frac{n-1}{2}\rfloor}}\left(\frac{(\alpha+d+1)\alpha^{\lfloor\frac{n-1}{2}\rfloor}(\alpha+d-c)^{\lfloor\frac{n}{2}\rfloor}-(\beta+d+1)\beta^{\lfloor\frac{n-1}{2}\rfloor}(\beta+d-c)^{\lfloor\frac{n}{2}\rfloor}}{\alpha-\beta}\right), \tag{2.8}$$

where α and β are as defined in (2.2).

Definition 2. The generalized bi-periodic Lucas quaternion sequence $\{V_n\}$ is defined by

$$V_n=v_ne_0+v_{n+1}e_1+v_{n+2}e_2+v_{n+3}e_3, \tag{2.9}$$

where v_n is the nth generalized bi-periodic Lucas number.

If $c=d=1$, $\{V_n\}$ becomes the bi-periodic Lucas quaternion sequence given in (1.7).

Theorem 4 (Generating function)**.** *The generating function for the generalized bi-periodic Lucas quaternion sequence is*

$$H(x)=\frac{\left(1-(ab+c)x^2+adx^3\right)V_0+x(1+bx-dx^2)V_1}{1-(ab+c+d)x^2+cdx^4}. \tag{2.10}$$

Proof. Replacing Q_0, Q_1, a, b, c and d by V_0, V_1, b, a, d and c in (2.4), we obtain (2.10). □

If $a=b$ and $c=d$, then

$$\begin{aligned} H(x) &= \frac{(1-ax)V_0+xV_1}{1-ax-cx^2} \\ &= \frac{\frac{c+1-ax}{c}e_0+\left(a+(c+1)x\right)e_1+(a^2+c+1+acx)e_2}{1-ax-cx^2} \\ &\quad +\frac{\left(a^3+2ac+a+(a^2+c^2+1)x\right)e_3}{1-ax-cx^2}. \end{aligned}$$

Hence, for $a = b = k$ and $c = d = 1$, we obtain the generating function for the k-Lucas quaternion

$$H(x) = \frac{(2-kx)e_0+(k+2x)e_1+(k^2+2+kx)e_2+\big(k^3+3k+(k^2+2)x\big)e_3}{1-kx-x^2},$$

as in [15].

Theorem 5 (Binet's formula)**.** *The Binet's formula for the generalized bi-periodic Lucas quaternion sequence is*

$$V_n = \begin{cases} \frac{1}{(ab)^{\lfloor\frac{n-1}{2}\rfloor}} V_{ne}, & \text{if } n \text{ is even} \\ \frac{1}{(ab)^{\lfloor\frac{n-1}{2}\rfloor}} V_{no}, & \text{if } n \text{ is odd} \end{cases} \tag{2.11}$$

where

$$V_{ne} = \frac{\alpha_o(\alpha+d+1)\alpha^{\lfloor\frac{n-1}{2}\rfloor}(\alpha+d-c)^{\lfloor\frac{n}{2}\rfloor} - \beta_o(\beta+d+1)\beta^{\lfloor\frac{n-1}{2}\rfloor}(\beta+d-c)^{\lfloor\frac{n}{2}\rfloor}}{\alpha-\beta},$$

$$V_{no} = \frac{\alpha_e(\alpha+d+1)\alpha^{\lfloor\frac{n-1}{2}\rfloor}(\alpha+d-c)^{\lfloor\frac{n}{2}\rfloor} - \beta_e(\beta+d+1)\beta^{\lfloor\frac{n-1}{2}\rfloor}(\beta+d-c)^{\lfloor\frac{n}{2}\rfloor}}{\alpha-\beta},$$

with α_e, β_e, α_o and β_o as defined in (2.5).

Proof. Using the Binet's formula for $\{v_n\}$ and proceeding as in the proof of Theorem 2, we can easily obtain (2.11). □

If $c = d = 1$, then

$$\begin{aligned}(\alpha+2)\alpha^{\lfloor\frac{n-1}{2}\rfloor+\lfloor\frac{n}{2}\rfloor} &= (\alpha+2)\alpha^{n-1} \\ &= \left(1+\frac{2}{\alpha}\right)\alpha^n \\ &= \left(1-\frac{2\beta}{ab}\right)\alpha^n \\ &= \frac{(\alpha-\beta)\alpha^n}{ab},\end{aligned}$$

and

$$\begin{aligned}(\beta+2)\beta^{\lfloor\frac{n-1}{2}\rfloor+\lfloor\frac{n}{2}\rfloor} &= (\beta+2)\beta^{n-1} \\ &= \left(1+\frac{2}{\beta}\right)\beta^n \\ &= \left(1-\frac{2\alpha}{ab}\right)\beta^n \\ &= \frac{(\beta-\alpha)\beta^n}{ab}.\end{aligned}$$

Hence (2.11) reduces to the Binet's formula for the bi-periodic Lucas quaternion given in (1.8).

We verify (2.11) for $n = 1$. From (2.7) and the definition of $\{V_n\}$, we have

$$\begin{aligned} V_1 &= v_1 e_0 + v_2 e_1 + v_3 e_2 + v_4 e_3 \\ &= a e_0 + (ab + d + 1)e_1 + a(ab + c + d + 1)e_2 \\ &\quad + (a^2b^2 + abc + 2abd + ab + d^2 + d)e_3. \end{aligned}$$

On the other hand, if $n = 1$, then (2.11) becomes

$$V_1 = \frac{\alpha_e(\alpha + d + 1) - \beta_e(\beta + d + 1)}{\alpha - \beta}.$$

In this case, α_e and β_e respectively can be written as

$$\begin{aligned} \alpha_e &= a e_o + (\alpha + d - c)e_1 + a(\alpha + d)e_2 + (\alpha + d)(\alpha + d - c)e_3, \\ \beta_e &= a e_o + (\beta + d - c)e_1 + a(\beta + d)e_2 + (\beta + d)(\beta + d - c)e_3. \end{aligned}$$

Let

$$\alpha_e(\alpha + d + 1) - \beta_e(\beta + d + 1) = E_0 e_0 + E_1 e_1 + E_2 e_2 + E_3 e_3.$$

Then

$$\begin{aligned} E_0 &= a(\alpha - \beta), \\ E_1 &= (\alpha + d + 1)(\alpha + d - c) - (\beta + d + 1)(\beta + d - c) \\ &= (\alpha + d)^2 - (\beta + d)^2 - (c - 1)(\alpha - \beta) \\ &= (\alpha + \beta + 2d)(\alpha - \beta) - (c - 1)(\alpha - \beta) \\ &= (ab + d + 1)(\alpha - \beta), \\ E_2 &= a\big((\alpha + d)(\alpha + d - c) - (\beta + d)(\beta + d - c)\big) \\ &= a\big((\alpha + d)^2 - (\beta + d)^2 + (\alpha - \beta)\big) \\ &= a\big((\alpha + \beta + 2d)(\alpha - \beta) + (\alpha - \beta)\big) \\ &= a(ab + c + d + 1)(\alpha - \beta), \\ E_3 &= (\alpha + d)(\alpha + d + 1)(\alpha + d - c) - (\beta + d)(\beta + d + 1)(\beta + d - c) \\ &= (\alpha + d)^3 - (\beta + d)^3 - (c - 1)\big((\alpha + d)^2 - (\beta + d)^2\big) - c(\alpha - \beta) \\ &= \big((\alpha + \beta + 2d)^2 - (\alpha + d)(\beta + d)\big)(\alpha - \beta) \\ &\quad - (c - 1)(\alpha + \beta + 2d)(\alpha - \beta) - c(\alpha - \beta) \\ &= \big((ab + c + d)^2 - cd\big)(\alpha - \beta) - (c - 1)(ab + c + d)(\alpha - \beta) - c(\alpha - \beta) \\ &= (a^2b^2 + abc + 2abd + ab + d^2 + d)(\alpha - \beta). \end{aligned}$$

Hence (2.11) is true for $n = 1$.

Theorem 6 (Catalan's identity)**.** *The Catalan's identity for the generalized bi-periodic Lucas quaternion sequence is*

$$V_n^2 - V_{n+2r}V_{n-2r}$$
$$= \begin{cases} \dfrac{(cd)^{\frac{n-2r}{2}}(d\alpha-\beta-cd+d^2)(d\beta-\alpha-cd+d^2)}{d^2} \\ \times\left(\dfrac{\alpha_o\beta_o\left((\alpha+d)^{2r}-(cd)^r\right)+\beta_o\alpha_o\left((\beta+d)^{2r}-(cd)^r\right)}{(\alpha-\beta)^2}\right), & if\ n\ is\ even, \\ \dfrac{(cd)^{\frac{n-2r-1}{2}}(d\alpha-\beta-cd+d^2)(d\beta-\alpha-cd+d^2)}{abd} \\ \times\left(\dfrac{\alpha_e\beta_e\left((cd)^r-(\alpha+d)^{2r}\right)+\beta_e\alpha_e\left((cd)^r-(\beta+d)^{2r}\right)}{(\alpha-\beta)^2}\right), & if\ n\ is\ odd. \end{cases} \tag{2.12}$$

Proof. Assume that n is even, and let

$$Y_1 = (\alpha-\beta)^2(ab)^{n-2}V_n^2,$$
$$Y_2 = (\alpha-\beta)^2(ab)^{n-2}V_{n+2r}V_{n-2r}.$$

Then

$$Y_1 = \left(\alpha_o(\alpha+d+1)\alpha^{\frac{n-2}{2}}(\alpha+d-c)^{\frac{n}{2}} - \beta_o(\beta+d+1)\beta^{\frac{n-2}{2}}(\beta+d-c)^{\frac{n}{2}}\right)^2$$
$$= \alpha_o^2(\alpha+d+1)^2\alpha^{n-2}(\alpha+d-c)^n + \beta_o^2(\beta+d+1)^2\beta^{n-2}(\beta+d-c)^n$$
$$-(\alpha_o\beta_o+\beta_o\alpha_o)(\alpha+d+1)(\beta+d+1)\alpha^{\frac{n-2}{2}}(\alpha+d-c)^{\frac{n}{2}}\beta^{\frac{n-2}{2}}(\beta+d-c)^{\frac{n}{2}},$$

and

$$Y_2 = \left(\alpha_o(\alpha+d+1)\alpha^{\frac{n+2r-2}{2}}(\alpha+d-c)^{\frac{n+2r}{2}} - \beta_o(\beta+d+1)\beta^{\frac{n+2r-2}{2}}(\beta+d-c)^{\frac{n+2r}{2}}\right)$$
$$\times\left(\alpha_o(\alpha+d+1)\alpha^{\frac{n-2r-2}{2}}(\alpha+d-c)^{\frac{n-2r}{2}} - \beta_o(\beta+d+1)\beta^{\frac{n-2r-2}{2}}(\beta+d-c)^{\frac{n-2r}{2}}\right),$$
$$= \alpha_o^2(\alpha+d+1)^2\alpha^{n-2}(\alpha+d-c)^n + \beta_o^2(\beta+d+1)^2\beta^{n-2}(\beta+d-c)^n$$
$$-\alpha_o\beta_o(\alpha+d+1)(\beta+d+1)\alpha^{\frac{n+2r-2}{2}}(\alpha+d-c)^{\frac{n+2r}{2}}\beta^{\frac{n-2r-2}{2}}(\beta+d-c)^{\frac{n-2r}{2}},$$
$$-\beta_o\alpha_o(\alpha+d+1)(\beta+d+1)\alpha^{\frac{n-2r-2}{2}}(\alpha+d-c)^{\frac{n-2r}{2}}\beta^{\frac{n+2r-2}{2}}(\beta+d-c)^{\frac{n+2r}{2}}.$$

Hence

$$Y_1 - Y_2 = \alpha_o\beta_o A_1 + \beta_o\alpha_o A_2,$$

where

$$A_1 = (\alpha+d+1)(\beta+d+1)\left(\alpha^{\frac{n+2r-2}{2}}(\alpha+d-c)^{\frac{n+2r}{2}}\beta^{\frac{n-2r-2}{2}}(\beta+d-c)^{\frac{n-2r}{2}}\right.$$
$$\left.-\alpha^{\frac{n-2}{2}}(\alpha+d-c)^{\frac{n}{2}}\beta^{\frac{n-2}{2}}(\beta+d-c)^{\frac{n}{2}}\right)$$
$$A_2 = (\alpha+d+1)(\beta+d+1)\left(\alpha^{\frac{n-2r-2}{2}}(\alpha+d-c)^{\frac{n-2r}{2}}\beta^{\frac{n+2r-2}{2}}(\beta+d-c)^{\frac{n+2r}{2}}\right.$$
$$\left.-\alpha^{\frac{n-2}{2}}(\alpha+d-c)^{\frac{n}{2}}\beta^{\frac{n-2}{2}}(\beta+d-c)^{\frac{n}{2}}\right).$$

Since $\alpha\beta = -abd$ and $\alpha+\beta = ab+c-d$, we have

$$\begin{aligned}(\alpha+d+1) &= \left(1+\frac{d+1}{\alpha}\right)\alpha\\ &= \left(1-\frac{(d+1)\beta}{abd}\right)\alpha\\ &= \left(\frac{(\alpha+\beta)d-d(c-d)-(d+1)\beta}{abd}\right)\alpha\\ &= \frac{(d\alpha-\beta-cd+d^2)\alpha}{abd}.\end{aligned}$$

Similarly

$$(\beta+d+1) = \frac{(d\beta-\alpha-cd+d^2)\beta}{abd}.$$

Then

$$\begin{aligned}A_1 &= \frac{(d\alpha-\beta-cd+d^2)(d\beta-\alpha-cd+d^2)\alpha^{\frac{n}{2}}(\alpha+d-c)^{\frac{n}{2}}\beta^{\frac{n}{2}}(\beta+d-c)^{\frac{n}{2}}}{(abd)^2}\\ &\times\left(\frac{\alpha^r(\alpha+d-c)^r}{\beta^r(\beta+d-c)^r}-1\right)\\ &= \frac{(ab)^{n-2}(cd)^{\frac{n}{2}}(d\alpha-\beta-cd+d^2)(d\beta-\alpha-cd+d^2)}{d^2}\\ &\times\frac{\alpha^{2r}(\alpha+d-c)^{2r}-\alpha^r(\alpha+d-c)^r\beta^r(\beta+d-c)^r}{\alpha^r(\alpha+d-c)^r\beta^r(\beta+d-c)^r}\\ &= \frac{(ab)^{n-2}(cd)^{\frac{n-2r}{2}}(d\alpha-\beta-cd+d^2)(d\beta-\alpha-cd+d^2)\big((\alpha+d)^{2r}-(cd)^r\big)}{d^2}.\end{aligned}$$

Similarly we have

$$A_2 = \frac{(ab)^{n-2}(cd)^{\frac{n-2r}{2}}(d\alpha-\beta-cd+d^2)(d\beta-\alpha-cd+d^2)\big((\beta+d)^{2r}-(cd)^r\big)}{d^2},$$

and the proof is completed for the case where n is even.

Using the same procedure, we can also prove (2.12) for the case where n is odd. □

If $c = d = 1$, then Theorem 6 reduces to [18, Theorem 5].

3. Conclusions

In this paper we introduced the generalized bi-periodic Fibonacci and Lucas quaternions which are the further generalizations of the bi-periodic Fibonacci and Lucas quaternions considered in the literature. For those quaternions, we obtained the generating functions, Binet's formulas and Catalan's identities.

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