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BEST APPROXIMATION AND CHARACTERIZATION OF HILBERT SPACES

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Abstract. It is well known that for any nonempty closed convex subset C of a Hilbert space, any best approximation $y \in C$ of the point x satisfies the inequality $\|x-y\|^2 + \|z-y\|^2 \le \|x-z\|^2$ for all $z \in C$. In this paper, we first introduce and study a new subset of best approximations involving this inequality in general metric spaces. Then, we provide some equivalent conditions which characterize Hilbert spaces.

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1. Introduction and preliminaries

Let (X,d) be a metric space and let C be a nonempty subset of X. For each $x \in X$, the distance between the point x and the set C is denoted by $\operatorname{dist}(x,C)$ and is defined by the following minimum equation

$$\operatorname{dist}(x,C) := \inf_{z \in C} d(x,z).$$

We call $y \in C$ with the property $d(x,y) = \operatorname{dist}(x,C)$ a best approximation of x in C. The set of all best approximation of x in C defines a set valued mapping

$$P_C x := \{y \in C : d(x,y) = \operatorname{dist}(x,C)\}, \tag{1.1}$$

which is called the metric projection operator. It is easily seen that

$$P_C x = \{y \in C : d(x,y) \le d(x,z), \ \ \forall z \in C\}. \tag{1.2}$$

It is clear that $P_C x$ is a closed subset of C if C is closed. If $P_C x \neq \varnothing$ for every $x \in X$, then C is called proximal. If $P_C x$ is a singleton for every $x \in X$, then C is said to be a Chebyshev set. For more properties of metric projection we refer the reader to [4, 10]. Metric projection operators are widely used in different areas of mathematics such as functional and numerical analysis, theory of optimization and approximation and for problems of optimal control (see e.g., [4]).

One of the important characterizations of best approximations is the following theorem which is known as Kolmogorov's criterion.

Theorem 1 ([4, Theorem 3.1]). *Let H be a Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$, and let C be a nonempty closed convex subset of H. Then for $x \in H$ and $y \in C$ the following conditions are equivalent:*

(i) $y = P_C(x)$;
(ii) $\langle x-y, z-y\rangle \leq 0$ *for all* $z \in C$;
(iii) $\|x-y\|^2 + \|z-y\|^2 \leq \|x-z\|^2$ *for all* $z \in C$.

The following example shows that the inequality (iii) in Theorem 1 does not hold in general Banach spaces.

Example 1. Let $X = \mathbb{R}^2 = \{x = (u,v) : u, v \in \mathbb{R}\}$ endowed with the norm

$$\|(u,v)\| = |u-v| + \sqrt{u^2+v^2}$$

and $C = \{(u,0) : |u| \leq 1\}$. Then C is a Chebyshev set. For $x = (1,1)$, we have $y = P_C x = (0,0)$ and the inequality (iii) in Theorem 1 for $z = (1,0) \in C$ does not hold:

$$\|x-y\|^2 + \|z-y\|^2 = 2+4 > 4 = \|x-z\|^2.$$

A natural question arises here. Does the inequality (iii) in Theorem 1 characterize Hilbert spaces? To answer this question we rewrite the equivalence $(i) \Leftrightarrow (ii)$ as equality of two sets. Let C be a nonempty subset of a metric space X. For each $x \in X$, we define the set $Q_C x$ by

$$Q_C x := \left\{y \in C : d^2(x,y) + d^2(z,y) \leq d^2(x,z), \ \ \forall z \in C\right\}. \tag{1.3}$$

It is possible to describe $Q_C x$ by similar way in (1.1). To do this let $y \in C$ and

$$\operatorname{dist}(x,y,C) = \inf\left\{d^2(x,z) - d^2(y,z) : z \in C\right\} \tag{1.4}$$

which may take $-\infty$. Using this notation we have

$$Q_C x = \left\{y \in C : d^2(x,y) = \operatorname{dist}(x,y,C)\right\}. \tag{1.5}$$

Note that if $\operatorname{dist}(x,y,C) < 0$, then $Q_C x = \varnothing$.

Comparison of (1.2) and (1.3) shows that $Q_C x \subseteq P_C x$. By Theorem 1, this turns to equality when C is a nonempty closed convex subset of a Hilbert space.

The main aim of the paper is to study properties of the set $Q_C x$ and to characterize Hilbert spaces by the equality $Q_C x = P_C x$. Some recent characterizations of Hilbert spaces can be found in [2, 8]

2. Main results

We begin with properties of $Q_C x$ in general metric spaces.

Proposition 1. *Let C be a nonempty subset of a metric space (X,d) and $x \in X$. Then the following statements hold:*

(i) $Q_C x \subseteq P_C x$;
(ii) $Q_C x$ *is empty or singleton;*
(iii) *if* $Q_C x \neq \emptyset$, *then* $P_C x$ *is singleton;*
(iv) $Q_C x$ *is empty or* $Q_C x = P_C x$.

Proof. The assertion (i) is clear. To prove (ii) let $Q_C x$ be nonempty and $y_1, y_2 \in Q_C x$. Then we have

$$d^2(x,y_1) + d^2(y_1,y_2) \le d^2(x,y_2),$$

or equivalently

$$d^2(y_1,y_2) \le d^2(x,y_2) - d^2(x,y_1). \tag{2.1}$$

Interchanging the role of y_1 and y_2 gives us

$$d^2(y_1,y_2) \le d^2(x,y_1) - d^2(x,y_2). \tag{2.2}$$

It follows from (2.1) and (2.2) that

$$d^2(y_1,y_2) \le \min\left\{d^2(x,y_2) - d^2(x,y_1), -(d^2(x,y_2) - d^2(x,y_1))\right\} \le 0.$$

Therefore $d(y_1,y_2) = 0$ and so $y_1 = y_2$.
Next, we prove (iii). Let $y \in Q_C x$. It follows from (i) that $y \in P_C x$. Now, let $z \in P_C x$. Since $d(x,y) = \operatorname{dist}(x,C) = d(x,z)$, using the inequality

$$d^2(x,y) + d^2(z,y) \le d^2(x,z),$$

we get $d(y,z) = 0$ and so $y = z$ which implies that $P_C x$ is singleton. Finally, the assertion (iv) follows from (i) and (iii). □

The above proposition clarify the structure of $Q_C x$. Next, we verify that in which metric space (X,d) and for which subset C, the set $Q_C x$ is nonempty. First, we focus on normed linear spaces.

Let $(X, \|\cdot\|)$ be a normed linear space. The subsets $S_X := \{x \in X : \|x\| = 1\}$ and $B_X := \{x \in X : \|x\| \le 1\}$ are called the unit sphere and the unit ball of X, respectively. Also, the closed ball with center at $x \in X$ and radius r is denoted by $\overline{B}(x,r)$. A subset A of X is said to be admissible [9] if it is the intersection of a family of closed balls. Obviously, any admissible set is a bounded, closed and convex set. Let A be an admissible set of X. A complete family of centers of A is a set S such that

$$A = \cap\left\{\overline{B}(x,r(x)) : x \in S\right\}$$

for a mapping $r : S \to [0,\infty)$. A closed subspace D of X is called a diametral space of A if it contains a complete family of centers of A.

Lemma 1. *Let $(X, \|\cdot\|)$ be a normed linear space and A be an admissible subset of X. If D is a diametral space of A such that $P_D x = Q_D x$ for each $x \in A$, then $P_D(A) \subseteq A$.*

Proof. Since D is a diametral space of the admissible set A, there exists complete family of centers S such that $S \subseteq D$ and

$$A = \cap \left\{ \overline{B}(u, r(u)) : u \in S \right\}.$$

Let $x \in A$. For each $u \in S$, we have $x \in \overline{B}(u, r(u))$ and so

$$\|x - u\| \le r(u). \tag{2.3}$$

Since $P_D x = Q_D x$, for each $y \in P_D x$ we have

$$\|x - y\|^2 + \|z - y\|^2 \le \|x - z\|^2, \quad \forall z \in D.$$

This together with the fact that $S \subseteq D$ implies that

$$\|u - y\|^2 \le \|x - y\|^2 + \|u - y\|^2 \le \|x - u\|^2, \quad \forall u \in S. \tag{2.4}$$

It follows from (2.3) and (2.4) that for each $u \in S$,

$$\|y - u\| \le r(u).$$

That is, $y \in A$ and so $P_D(A) \subseteq A$. □

In a normed linear space $(X, \|\cdot\|)$, a vector x is said to be Birkhoff orthogonal to a vector y ($x \perp_B y$) if the inequality $\|x\| \le \|x + \alpha y\|$ holds for any real number α. It is easy to see that $x \perp_B y$ if and only if $0 \in P_{\overline{0y}} x$ where $\overline{0y}$ is the line $\{\alpha y : \alpha \in \mathbb{R}\}$. Birkhoff orthogonality is said to be symmetric if $x \perp_B y$ implies $y \perp_B x$. Also, Birkhoff orthogonality is said to be homogeneous if $x \perp_B y$ implies $\lambda x \perp_B \mu y$ for any real numbers λ and μ. It is well-known that Birkhoff orthogonality is homogeneous in any normed linear space [1, Theorem 4.5]. The following theorem is the origin of many characterizations of Hilbert spaces. For more details on Birkhoff orthogonality see the survey [1].

Theorem 2 ([5])**.** *Let $(X, \|\cdot\|)$ be a normed linear space, whose dimension is at least three. If Birkhoff orthogonality is symmetric and there is at most one orthogonal from a given line to a point not on that line then X is an inner product space.*

Now, we state and prove the characterization results.

Theorem 3. *Let $(X, \|\cdot\|)$ be a Banach space, whose dimension is at least three. Then the followings are equivalent:*

(i) *X is a Hilbert space;*
(ii) *$Q_C x$ is nonempty for each nonempty closed convex subset C of X and $x \in X \backslash C$;*

(iii) $P_C x = Q_C x$ *for every nonempty closed convex subset* C *of* X *and* $x \in X \backslash C$*;*
(iv) $P_L x = Q_L x$ *for every line* L *in* X *and* $x \in X \backslash L$*;*
(v) *Birkhoff orthogonality is symmetric;*
(vi) $P_D(A) \subseteq A$ *for every admissible set* A *and every diametral space* D *of* A*.*

Proof. $(i) \Rightarrow (ii)$ and $(i) \Rightarrow (iii)$ follows from Theorem 1. Also, $(ii) \Rightarrow (iii)$ is an immediate consequence of (iv) in Proposition 1. $(iii) \Rightarrow (iv)$ is trivial.
$(iv) \Rightarrow (v)$ Let $P_L x = Q_L x$ for every line L in X. It follows from Proposition 1 that $P_L x$ have at most one point for each line L. Now, let $x \perp_B y$. Then $0 \in P_{\overline{0y}} x$. Since $P_{\overline{0y}} x = Q_{\overline{0y}} x$, we have

$$\|x\|^2 + \|\alpha y\|^2 \leq \|x - \alpha y\|^2, \qquad \forall \alpha \in \mathbb{R},$$

which implies that

$$\left\|\frac{1}{\alpha} x\right\|^2 + \|y\|^2 \leq \left\|\frac{1}{\alpha} x - y\right\|^2, \qquad \forall \alpha \in \mathbb{R} \backslash \{0\}.$$

Hence for all $\beta \in \mathbb{R}$ we get $\|y\| \leq \|y + \beta x\|$ and so $y \perp_B x$.
$(iii) \Rightarrow (vi)$ and $(v) \Rightarrow (i)$ are Lemma 1 and Theorem 2, respectively. Finally, $(vi) \Rightarrow (i)$ follows from [9, Theorem 1]. □

In two dimensional case we have the following theorem.

Theorem 4. *Let* $(X, \|\cdot\|)$ *be a Banach space, whose dimension is at least two. Then the followings are equivalent:*

(i) X *is a Hilbert space;*
(ii) $Q_C x$ *is nonempty for each nonempty closed convex subset* C *of* X *and* $x \in X \backslash C$*;*
(iii) $P_C x = Q_C x$ *for every nonempty closed convex subset* C *of* X *and* $x \in X \backslash C$*;*
(iv) $P_L x = Q_L x$ *for every line* L *in* X *and* $x \in X \backslash L$*;*
(v) $x, y \in S_X,\ x \perp_B y \quad \Rightarrow \quad \|x + y\| \geq \sqrt{2}$.

Proof. We first prove $(iv) \Rightarrow (v)$. Let $P_L x = Q_L x$ for every line L in X. Now, let $x, y \in S_X$ and $x \perp_B y$. Then, $0 \in P_{\overline{0y}} x$. Since $P_{\overline{0y}} x = Q_{\overline{0y}} x$, we have

$$\|x\|^2 + \|\alpha y\|^2 \leq \|x - \alpha y\|^2, \qquad \forall \alpha \in \mathbb{R},$$

specially for $\alpha = -1$,

$$2 = \|x\|^2 + \|y\|^2 \leq \|x + y\|^2.$$

The equivalence of (i) and (v) follows from $(10.3')$ of [3]. The equivalence of others are the same as Theorem 3. □

As a consequence of Theorems 3 and 4 we would mention that the Hilbert spaces are the only Banach spaces for which $Q_C x$ is nonempty. Of course, there are other metric spaces with this property, namely $CAT(0)$ metric spaces. To see this, we refer the interested readers to consider [6, Proposition 2.4] and [7, Theorem 2.2], and to follow the same argument as in proof of Theorem 1.

Next, we define a new geometric constant for normed linear spaces.

Definition 1. Let $(X, \|\cdot\|)$ be a normed linear space. Define geometric constant $C(X)$ of X as follows:

$$C(X) = \inf\left\{ \frac{\|x+\alpha y\|^2}{\|x\|^2 + \|\alpha y\|^2} : x \perp_B y, \|x\| = \|y\|, x \neq -\alpha y, \alpha \in \mathbb{R} \right\}. \tag{2.5}$$

It is easily seen that for every normed linear space $0 \leq C(X) \leq 1$. If X is an inner product space, then $x \perp_B y$ implies that $\langle x, y \rangle = 0$. Therefore, $\|x+\alpha y\|^2 = \|x\|^2 + \|\alpha y\|^2$ and so $C(X) = 1$. Also, by the homogeneity property of Birkhoff orthogonality we have

$$C(X) = \inf\left\{ \frac{\|x+\alpha y\|^2}{1+\alpha^2} : \; x, y \in S_X, \; x \perp_B y, \; \alpha \in \mathbb{R} \right\}. \tag{2.6}$$

Theorem 5. *Let $(X, \|\cdot\|)$ be a normed linear space and $x, y \in X$. If $C(X) = 1$ and $x \perp_B y$, then*

(i) $0 \in Q_{\overline{0y}} x$;
(ii) $y \perp_B x$;
(iii) *if $x, y \in S_X$, then $\|x+y\| \geq \sqrt{2}$.*

Proof. Let $x, y \in X$ be such that $x \perp_B y$. Without loss of generality we may assume that $y \neq 0$. If x lies on the line $\overline{0y}$, then $x = -\lambda y$ for some $\lambda \in \mathbb{R}$ and it follows from $x \perp_B y$ that $x = 0$ which turns the assertions trivial. Otherwise, for every $\alpha \in \mathbb{R}$ we have $x \neq -\alpha y$. By the homogeneity property of Birkhoff orthogonality we obtain

$$x \perp_B \left(\frac{\|x\|}{\|y\|} \right) y.$$

Since $C(X) = 1$, then for each $\alpha \in \mathbb{R}$ we have

$$1 \leq \frac{\left\| x + \alpha \frac{\|x\|}{\|y\|} y \right\|^2}{\|x\|^2 + \left\| \alpha \frac{\|x\|}{\|y\|} y \right\|^2},$$

or equivalently

$$\|x\|^2 + \left\| \alpha \frac{\|x\|}{\|y\|} y \right\|^2 \leq \left\| x + \alpha \frac{\|x\|}{\|y\|} y \right\|^2. \tag{2.7}$$

Now, for each real umber β, by taking $\alpha = \beta \frac{\|y\|}{\|x\|}$ in (2.7), we have

$$\|x\|^2 + \|\beta y\|^2 \leq \|x+\beta y\|^2, \tag{2.8}$$

which implies that $0 \in Q_{\overline{0y}}x$.
Next, we prove (ii). For each nonzero real umber β, by taking $\alpha = \frac{\|y\|}{\beta\|x\|}$ in (2.7) and then multiplying both sides by $|\beta|$, we have

$$\|y\|^2 + \|\beta x\|^2 \leq \|y + \beta x\|^2,$$

which implies that $y \perp_B x$.
Finally, if $x, y \in S_X$, for $\beta = 1$ in (2.8) we get $\|x + y\| \geq \sqrt{2}$. □

Using Theorems 5, 3 and 4 we have the following corollary.

Corollary 1. *Let $(X, \|\cdot\|)$ be a Banach space. Then X is a Hilbert space if and only if $C(X) = 1$.*

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