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STABILITY AND HYPERSTABILITY OF MULTI-ADDITIVE-CUBIC MAPPINGS

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Abstract. In this article, we introduce the multi-additive-cubic mappings and then unify the system of functional equations defining a multi-additive-cubic mapping to a single equation. Using a fixed point theorem, we study the generalized Hyers-Ulam stability of such equation. As a result, we show that the multi-additive-cubic functional equation can be hyperstable.

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1. INTRODUCTION

A classical question in the theory of functional equation is the following: "when is it true that a function which approximately satisfies a functional equation $\mathcal{F}$ must be close to an exact solution of $\mathcal{F}$?" If the problem accepts a solution, we say that the equation $\mathcal{F}$ is stable.

A stimulating and famous talk presented by Ulam [22] in 1940, motivated the study of stability problems for various functional equations. He gave a wide range talk before a Mathematical Colloquium at the University of Wisconsin in which he presented a list of unsolved problems. In 1941, Hyers [14] was the first Mathematician to present the result concerning the stability of functional equations. He brilliantly answered the question of Ulam, the problem for the case of approximately additive mappings on Banach spaces. In course of time, the theorem formulated by Hyers was generalized by Aoki [1], Th. M. Rassias [20] and J. M. Rassias [19] for additive mappings.

Let V be a commutative group, W be a linear space, and $n \geq 2$ be an integer. Recall from [11] that a mapping $f : V^n \longrightarrow W$ is said to be *multi-additive* if it is additive (satisfies Cauchy's functional equation $A(x+y) = A(x) + A(y)$) in each variable. Some facts on such mappings can be found in [17] and many other sources. Ciepliński in [11] showed that f is multi-additive if and only if it satisfying the equation

$$f(x_1+x_2)=\sum_{j_1,j_2,\dots,j_n\in\{1,2\}} f(x_{1j_1},x_{2j_2},\dots,x_{nj_n}), \tag{1.1}$$

where $x_j=(x_{1j},x_{2j},\dots,x_{nj})\in V^n$ with $j\in\{1,2\}$. Moreover, f is called *multi-quadratic* if it is quadratic (satisfies the quadratic functional equation $Q(x+y)+Q(x-y)=2Q(x)+2Q(y)$) in each variable [12]. In [23], Zhao et al. proved that the mapping $f:V^n\longrightarrow W$ is multi-quadratic if and only if the following relation holds

$$\sum_{t\in\{-1,1\}^n} f(x_1+tx_2)=2^n\sum_{j_1,j_2,\dots,j_n\in\{1,2\}} f(x_{1j_1},x_{2j_2},\dots,x_{nj_n}). \tag{1.2}$$

For the generalized form and Jensen type of multi-quadratic mappings refer to [6] and [21], respectively. In [11] and [12], Ciepliński studied the generalized Hyers-Ulam stability of multi-additive and multi-quadratic mappings in Banach spaces, respectively (see also [23]). Furthermore, the mentioned mapping f is also called a *multi-cubic* if it is cubic in each variable, i.e., satisfies the equation

$$C(2x+y)+C(2x-y)=2C(x+y)+2C(x-y)+12C(x) \tag{1.3}$$

in each variable [15]. In [7], the second author and Shojaee, introduced the multi-cubic mappings and proved the multi-cubic functional equations can be hyperstable, that is, every approximately multi-cubic mapping under some conditions is multi-cubic; for other forms of cubic functional equations and their stabilities refer to [4, 5, 16, 19]. Various versions of multi-cubic mappings and functional equations which are recently studied can be found in [13] and [18].

In this paper, we define the multi-additive-cubic mappings and present a characterization of such mappings. In other words, we reduce the system of n equations defining the multi-additive-cubic mappings to obtain a single functional equation. We also prove the generalized Hyers-Ulam stability for multi-additive-cubic functional equations by using the fixed point method which was introduced and used for the first time by Brzdęk [8]; for more applications of this approach and alternative version for the stability of multi-Cauchy-Jensen mappings in Banach spaces and 2-Banach spaces see [2, 3] and [10], respectively.

2. Characterization of multi-additive-cubic mappings

Throughout this paper, $\mathbb{N}$ stands for the set of all positive integers, $\mathbb{N}_0:=\mathbb{N}\cup\{0\}$, $\mathbb{R}_+:=[0,\infty)$. For any $l\in\mathbb{N}_0$, $m\in\mathbb{N}$, $t=(t_1,\dots,t_m)\in\{-1,1\}^m$ and $x=(x_1,\dots,x_m)\in V^m$ we write $lx:=(lx_1,\dots,lx_m)$ and $tx:=(t_1x_1,\dots,t_mx_m)$, where ra stands, as usual, for the rth power of an element a of the linear space V.

Let V and W be linear spaces, $n\in\mathbb{N}$ and $k\in\{0,\dots,n\}$. A mapping $f:V^n\longrightarrow W$ is called k-additive and $n-k$-cubic (briefly, multi-additive-cubic) if f is additive in each of some k variables and is cubic in each of the other variables (see equation (1.3)). In this note, we suppose for simplicity that f is additive in each of the first k variables, but one can obtain analogous results without this assumption. Let us note that for $k = n$ ($k = 0$), the above definition leads to the so-called multi-additive (multi-cubic) mappings.

In what follows, we assume that V and W are vector spaces over the rationals. Moreover, we identify $x = (x_1,\dots,x_n) \in V^n$ with $(x^k, x^{n-k}) \in V^k \times V^{n-k}$, where $x^k := (x_1,\dots,x_k)$ and $x^{n-k} := (x_{k+1},\dots,x_n)$, and we adopt the convention that $(x^n, x^0) := x^n := (x^0, x^n)$. Put $x_i^k = (x_{i1},\dots,x_{ik}) \in V^k$ and $x_i^{n-k} = (x_{i,k+1}\dots,x_{in}) \in V^{n-k}$ where $i \in \{1,2\}$. In addition, we put

$$\mathcal{M} = \left\{(N_{k+1},\dots,N_n) |\, N_j \in \{x_{1j} \pm x_{2j}, x_{1j}\},\ j \in \{k+1,\dots,n\}\right\}.$$

Consider

$$\mathcal{M}_m^{n-k} := \left\{\mathfrak{N}_{n-k} = (N_{k+1},\dots,N_n) \in \mathcal{M} |\, \mathrm{Card}\{N_j : N_j = x_{1j}\} = m\right\},$$

for any $m \in \{0,\dots,n-k\}$. From now on, we use the following notation:

$$f\left(x_i^k, \mathcal{M}_m^{n-k}\right) := \sum_{\mathfrak{N}_{n-k} \in \mathcal{M}_m^{n-k}} f\left(x_i^k, \mathfrak{N}_{n-k}\right) \qquad (i \in \{1,2\}).$$

Note that in the above notations, if $k = 0$ then we obtain the same notation for multi-cubic mappings which are used in [7]. Here, we reduce the system of n equations defining the k-additive and $n-k$-cubic mapping to obtain a single functional equation.

Proposition 1. *Let $n \in \mathbb{N}$ and $k \in \{0,\dots,n\}$. If the mapping $f : V^n \longrightarrow W$ is k-additive and $n-k$-cubic mapping, then f satisfies the equation*

$$\sum_{t \in \{-1,1\}^{n-k}} f\left(x_1^k + x_2^k, 2x_1^{n-k} + t x_2^{n-k}\right) = \sum_{m=0}^{n-k} 2^{n-k-m} 12^m \sum_{i \in \{1,2\}} f\left(x_i^k, \mathcal{M}_m^{n-k}\right) \tag{2.1}$$

for all $x_i^k = (x_{i1},\dots,x_{ik}) \in V^k$ and $x_i^{n-k} = (x_{i,k+1}\dots,x_{in}) \in V^{n-k}$ where $i \in \{1,2\}$.

Proof. Without loss of generality, we assume that $k \in \{0,\dots,n-1\}$. For any $x^{n-k} \in V^{n-k}$, define the mapping $g_{x^{n-k}} : V^k \longrightarrow W$ by $g_{x^{n-k}}\left(x^k\right) := f\left(x^k, x^{n-k}\right)$ for $x^k \in V^k$. By assumption, $g_{x^{n-k}}$ is k-additive, and hence Theorem 2 from [11] implies that

$$g_{x^{n-k}}\left(x_1^k + x_2^k\right) = \sum_{j_1,j_2,\dots,j_k \in \{1,2\}} g_{x^{n-k}}\left(x_{j_1 1}, x_{j_2 2},\dots,x_{j_k k}\right), \qquad \left(x_1^k, x_2^k \in V^k\right).$$

It now follows from the above equality that

$$f\left(x_1^k + x_2^k, x^{n-k}\right) = \sum_{j_1,j_2,\dots,j_k \in \{1,2\}} f\left(x_{j_1 1}, x_{j_2 2},\cdots,x_{j_k k}, x^{n-k}\right) \tag{2.2}$$

for all x_1^k, $x_2^k \in V^k$ and $x^{n-k} \in V^{n-k}$. Similarly to the above, for any $x^k \in V^k$ consider the mapping $h_{x^k} : V^{n-k} \longrightarrow W$ defined through $h_{x^k}\left(x^{n-k}\right) := f(x^k, x^{n-k})$, $x^{n-k} \in V^{n-k}$

which is $n-k$-cubic and so we conclude from Proposition 2.2 of [7] that

$$\sum_{t\in\{-1,1\}^{n-k}} h_{x^k}\left(2x_1^{n-k}+tx_2^{n-k}\right) = \sum_{m=0}^{n-k} 2^{n-k-m}12^m h_{x^k}\left(\mathcal{M}_k^{n-k}\right). \tag{2.3}$$

for all x_1^{n-k}, $x_2^{n-k} \in V^{n-k}$, where

$$h_{x^k}\left(\mathcal{M}_k^{n-k}\right) := \sum_{\mathfrak{N}_{n-k}\in\mathcal{M}_k^{n-k}} h_{x^k}\left(\mathfrak{N}_{n-k}\right).$$

By the definition of h_{x^k}, relation (2.3) is equivalent to

$$\sum_{t\in\{-1,1\}^{n-k}} f\left(x^k, 2x_1^{n-k}+tx_2^{n-k}\right) = \sum_{m=0}^{n-k} 2^{n-k-m}12^m f\left(x^k, \mathcal{M}_k^{n-k}\right) \tag{2.4}$$

for all $x_1^{n-k}, x_2^{n-k} \in V^{n-k}$ and $x^k \in V^k$. Plugging equality (2.2) into (2.4), we get

$$\begin{aligned}\sum_{t\in\{-1,1\}^{n-k}} f\Big(x_1^k+x_2^k, 2x_1^{n-k}+tx_2^{n-k}\Big) &= \sum_{m=0}^{n-k} 2^{n-k-m}12^m f\left(x_1^k+x_2^k, \mathcal{M}_m^{n-k}\right)\\ &= \sum_{m=0}^{n-k} 2^{n-k-m}12^m \sum_{j_1,j_2,\ldots,j_n\in\{1,2\}} f\left(x_{j_1 1}, x_{j_2 2},\ldots,x_{j_k k}, \mathcal{M}_m^{n-k}\right)\\ &= \sum_{m=0}^{n-k} 2^{n-k-m}12^m \sum_{i\in\{1,2\}} f\left(x_i^k, \mathcal{M}_m^{n-k}\right)\end{aligned}$$

for all $x_i^k = (x_{i1},\ldots,x_{ik}) \in V^k$ and $x_i^{n-k} = (x_{i,k+1}\ldots,x_{in}) \in V^{n-k}$, which proves that f satisfies equation (2.1). $\square$

We remember that in Proposition 1, if $k=0$, we arrive to the upcoming equation. In other words, it is proved in [7, Proposition 2.2] that every multi-cubic mapping $f: V^n \longrightarrow W$ satisfying

$$\sum_{t\in\{-1,1\}^n} f(2x_1+tx_2) = \sum_{m=0}^{n} 2^{n-m}12^m f(\mathcal{M}_m^n). \tag{2.5}$$

In the sequel, $\binom{n}{k}$ is the binomial coefficient defined for all $n,k\in\mathbb{N}$ with $n\geq k$ by $\frac{n!}{k!(n-k)!}$. We say the mapping $f: V^n \longrightarrow W$ satisfies *the 3-power condition* in the jth variable if

$$f(z_1,\ldots,z_{j-1},2z_j,z_{j+1},\ldots,z_n) = 8f(z_1,\ldots,z_{j-1},z_j,z_{j+1},\ldots,z_n),\ ((z_1,\ldots,z_n)\in V^n).$$

Remark 1. It is easily verified that if the mapping C satisfying equation (1.3), then

$$C(2x) = 8C(x). \tag{2.6}$$

But the converse is not true. Let $(\mathcal{A}, \|\cdot\|)$ be a Banach algebra. Fix the vector a_0 in $\mathcal{A}$ (not necessarily unit). Define the mapping $h: \mathcal{A} \longrightarrow \mathcal{A}$ by $h(a) = \|a\|^3 a_0$ for any $a \in \mathcal{A}$. Obviously, for each $x \in \mathcal{A}$, (2.6) is true while (1.3) does not hold for h. Therefore, condition (2.6) does not imply that f is a cubic mapping.

Lemma 1. *Suppose that the mapping $f : V^n \longrightarrow W$ satisfies equation (2.1). Then, $f(2x) = 2^{3n-2k} f(x)$. In particular,*

(i) *$f(0) = 0$;*

(ii) *if f satisfies equation (1.1) or equivalently is multi-additive (the case $k = n$), then $f(2x) = 2^n f(x)$;*

(iii) *if f satisfies equation (2.5) (the case $k = 0$), then $f(2x) = 2^{3n} f(x)$.*

Proof. We firstly rewrite (2.1) as follows:

$$\sum_{t\in\{-1,1\}^{n-k}} f\left(x_1^k + x_2^k, 2x_1^{n-k} + tx_2^{n-k}\right) = \sum_{m=0}^{n-k} 2^{n-k-m} 12^m$$

$$\sum_{j_1,j_2,\dots,j_n\in\{1,2\}} f(x_{j_1 1},\dots,x_{j_k k},\overbrace{x_{1,k+1} \pm x_{2,k+1},\dots,x_{1n} \pm x_{2n}}^{n-k-m-\text{times}},\overbrace{x_{1,k+1},\dots,x_{1n}}^{m-\text{times}}). \tag{2.7}$$

Note that by the definition of $\mathcal{M}_m^{n-k}$, the elements of set $\{x_{1,k+1},\dots,x_{1n}\}$ have m choice in the last $n-k$ components. Putting $x_1^k = x_2^k = x^k$ and $x_1^{n-k} = x^{n-k}$, $x_2^{n-k} = 0$ in (2.7), we have

$$\begin{aligned}
2^{n-k} f(2x) &= \sum_{m=0}^{n-k} \binom{n-k}{m} 2^{n-k-m} 12^m 2^k 2^{n-k-m} f(x) \\
&= 2^{2n-k} \sum_{m=0}^{n-k} \binom{n-k}{m} 3^m 1^{n-k-m} f(x) \\
&= 2^{2n-k} (3+1)^{n-k} f(x) \\
&= 2^{4n-3k} f(x).
\end{aligned} \tag{2.8}$$

Therefore, $f(2x) = 2^{3n-2k} f(x)$. □

Let $0 \le p \le k$ and $0 \le q \le n-k$. Put

$$\mathcal{K}_{(p,q)} = \left\{ {}_{(p,q)}x := (\overbrace{0,\dots,0,x_{i_1},0,\dots,0,x_{i_p},0,\dots,0}^{k-\text{times}},\overbrace{0,\dots,0,x_{j_1},0,\dots,0,x_{j_q},0,\dots,0}^{n-k-\text{times}}) \in V^n \right\}$$

where $1 \le i_1 < \cdots < i_p \le k$ and $1 \le j_1 < \cdots < j_q \le n-k$. In other words, $\mathcal{K}_{(p,q)}$ is the set of all vectors in V^n that exactly their $p+q$-components are non-zero such that p components of them are coordinates of x^k and q components of them are just coordinates of x^{n-k}.

We wish to show that if the mapping $f : V^n \longrightarrow W$ satisfies equation (2.1), then it is multi-additive-cubic. In order to do this, we bring the next lemma.

Lemma 2. *If the mapping $f : V^n \longrightarrow W$ fulfilling equation (2.1) and the 3-power condition in the last $n-k$ variables, then $f(x) = 0$ for any $x \in V^n$ with at least one component which is equal to zero.*

Proof. We argue by induction on $p+q$ that for each ${}_{(p,q)}x \in \mathcal{K}_{(p,q)}$, $f\left({}_{(p,q)}x\right) = 0$ for $0 \le p \le k$ and $0 \le q \le n-k$. For $p+q = 0$, it follows from Lemma 1 that $f(0,\ldots,0) = 0$. Assume that for each ${}_{(p,q)}x \in \mathcal{K}_{(p,q)}$, $f\left({}_{(p,q)}x\right) = 0$ with $p+q = s-1$. We show that if ${}_{(p,q)}x \in \mathcal{K}_{(p,q)}$, then $f\left({}_{(p,q)}x\right) = 0$ for $p+q = s$. By a suitable replacement in (2.1), that is p coordinates of x^k and q coordinates of x^{n-k} are non-zero and using the assumption, we get

$$\begin{aligned}
2^{n-k}2^{3q} f\left({}_{(p,q)}x\right) &= \sum_{m=0}^{n-k-q} \binom{n-k-q}{m} 2^{n-k-m} 12^m 2^{k-p} 2^{n-k-m} f\left({}_{(p,q)}x\right) \\
&= 2^{2n-k-p} \sum_{m=0}^{n-k-q} \binom{n-k-q}{m} 3^m 1^{n-k-q-m} f\left({}_{(p,q)}x\right) \\
&= 2^{2n-k-p} (3+1)^{n-k-q} f\left({}_{(p,q)}x\right) \\
&= 2^{4n-3k-p-2q} f\left({}_{(p,q)}x\right).
\end{aligned}$$

Hence, $f\left({}_{(p,q)}x\right) = 0$. Note that we have used the same computations of (2.8) of Lemma 1 in the above relations. This shows that $f(x) = 0$ for any $x \in V^n$ with at least one component which is equal to zero. □

It follows from Remark 1 that the 3-power condition does not imply f is cubic in the jth variable. Adding this condition for f, we show that if f satisfies equation (2.1), then it is k-additive and $n-k$-cubic (multi-additive-cubic) mapping as follows.

Proposition 2. *If the mapping $f : V^n \longrightarrow W$ satisfies equation (2.1) and the 3-condition in the last $n-k$ variables, then it is multi-additive-cubic mapping.*

Proof. Putting $x_2^{n-k} = (0,\cdots,0)$ in the left side of (2.1) and applying the hypothesis, we obtain

$$2^{n-k} f\left(x_1^k + x_2^k, 2x_1^{n-k}\right) = 2^{n-k} \times 2^{3(n-k)} f\left(x_1^k + x_2^k, x_1^{n-k}\right). \tag{2.9}$$

On the other hand, by using Lemma 2, the right side of (2.1) will be

$$\begin{aligned}
&\sum_{m=0}^{n-k} \binom{n-k}{m} 2^{n-k-m} 12^m 2^{n-k-m} \sum_{j_1,j_2,\cdots,j_n \in \{1,2\}} f\left(x_{j_1 1}, x_{j_2 2}, \ldots, x_{j_k k}, x_1^{n-k}\right) \\
&= \sum_{m=0}^{n-k} \binom{n-k}{m} 4^{n-k-m} 12^m \sum_{j_1,j_2,\cdots,j_k \in \{1,2\}} f\left(x_{j_1 1}, x_{j_2 2}, \ldots, x_{j_k k}, x_1^{n-k}\right) \\
&= 2^{4(n-k)} \sum_{j_1,j_2,\ldots,j_k \in \{1,2\}} f\left(x_{j_1 1}, x_{j_2 2}, \ldots, x_{j_k k}, x_1^{n-k}\right).
\end{aligned} \tag{2.10}$$

Comparing relations (2.9) and (2.10), we find

$$f\left(x_1^k+x_2^k,x_1^{n-k}\right)=\sum_{j_1,j_2,\cdots,j_n\in\{1,2\}} f\left(x_{j_11},x_{j_22},\ldots,x_{j_nn},x_1^{n-k}\right)$$

for all $x_1^k,x_2^k\in V^n$ and $x_1^{n-k}\in V^{n-k}$. In light of [11, Theorem 2], we see that f is additive in each of the k first variables. Furthermore, by putting $x_2^k=(0,\ldots,0)$ in (2.1) and using Lemma 2, we have

$$\sum_{t\in\{-1,1\}^{n-k}} f\left(x_1^k,2x_1^{n-k}+tx_2^{n-k}\right)=\sum_{m=0}^{n-k}2^{n-k-m}12^m f\left(x_1^k,\mathcal{M}_m^{n-k}\right)$$

for all $x_1^k\in V^k$ and $x_1^{n-k},x_2^{n-k}\in V^{n-k}$, and thus [7, Proposition 2.3] now completes the proof. □

3. STABILITY OF (2.1)

In this section, we prove the generalized Hyers-Ulam stability of equation (2.1) by a fixed point result (Theorem 1) in Banach spaces. Throughout, for two sets X and Y, the set of all mappings from X to Y is denoted by Y^X. Here, we introduce the oncoming three hypotheses:

(A1) Y is a Banach space, $\mathcal{S}$ is a nonempty set, $j\in\mathbb{N}$, $g_1,\ldots,g_j:\mathcal{S}\longrightarrow\mathcal{S}$ and $L_1,\ldots,L_j:\mathcal{S}\longrightarrow\mathbb{R}_+$,

(A2) $\mathcal{T}:Y^{\mathcal{S}}\longrightarrow Y^{\mathcal{S}}$ is an operator satisfying the inequality

$$\|\mathcal{T}\lambda(x)-\mathcal{T}\mu(x)\|\leq\sum_{i=1}^{j}L_i(x)\left\|\lambda(g_i(x))-\mu(g_i(x))\right\|,\quad \lambda,\mu\in Y^{\mathcal{S}},x\in\mathcal{S},$$

(A3) $\Lambda:\mathbb{R}_+^{\mathcal{S}}\longrightarrow\mathbb{R}_+^{\mathcal{S}}$ is an operator defined through

$$\Lambda\delta(x):=\sum_{i=1}^{j}L_i(x)\delta(g_i(x))\qquad \delta\in\mathbb{R}_+^{\mathcal{S}},x\in\mathcal{S}.$$

In the next theorem, we present a fundamental result in fixed point theory [9, Theorem 1]. This result plays a key tool to obtain our aim in this paper.

Theorem 1. *Let hypotheses* (A1)-(A3) *hold and the function* $\theta:\mathcal{S}\longrightarrow\mathbb{R}_+$ *and the mapping* $\phi:\mathcal{S}\longrightarrow Y$ *fulfils the following two conditions:*

$$\|\mathcal{T}\phi(x)-\phi(x)\|\leq\theta(x),\quad \theta^*(x):=\sum_{l=0}^{\infty}\Lambda^l\theta(x)<\infty\qquad(x\in\mathcal{S}).$$

Then, there exists a unique fixed point ψ *of* $\mathcal{T}$ *such that*

$$\|\phi(x)-\psi(x)\|\leq\theta^*(x)\qquad(x\in\mathcal{S}).$$

Moreover, $\psi(x)=\lim_{l\to\infty}\mathcal{T}^l\phi(x)$ *for all* $x\in\mathcal{S}$.

Here and subsequently, for the mapping $f : V^n \longrightarrow W$, we consider the difference operator $\mathcal{D}f : V^n \times V^n \longrightarrow W$ by

$$\mathcal{D}f(x_1,x_2) := \sum_{q\in\{-1,1\}^{n-k}} f\left(x_1^k + x_2^k, 2x_1^{n-k} + qx_2^{n-k}\right)$$
$$- \sum_{m=0}^{n-k} 2^{n-k-m} 12^m \sum_{i\in\{1,2\}} f\left(x_i^k, \mathcal{M}_m^{n-k}\right)$$

for all $x_i^k = (x_{i1},\dots,x_{ik}) \in V^k$ and $x_i^{n-k} = (x_{i,k+1}\dots,x_{in}) \in V^{n-k}$. We have the following stability result for the functional equation (2.1).

Theorem 2. *Let $\beta \in \{-1,1\}$, V be a linear space and W be a Banach space. Suppose that $\phi : V^n \times V^n \longrightarrow \mathbb{R}_+$ is a mapping satisfying the inequality*

$$\sum_{l=0}^{\infty} \left(\frac{1}{2^{(3n-2k)\beta}}\right)^l \phi\left(2^{\beta l-\frac{|\beta-1|}{2}} x_1, 2^{\beta l-\frac{|\beta-1|}{2}} x_2\right) < \infty \tag{3.1}$$

for all $x_1,x_2 \in V^n$ and

$$\Phi(x) := \frac{1}{2^{(3n-2k)\frac{|\beta+1|}{2}+n-k}} \sum_{l=0}^{\infty} \left(\frac{1}{2^{(3n-2k)\beta}}\right)^l \phi\left(2^{\beta l-\frac{|\beta-1|}{2}} x, \left(2^{\beta l-\frac{|\beta-1|}{2}} x^k, 0\right)\right) < \infty$$

for all $x \in V^n$. Assume also $f : V^n \longrightarrow W$ is a mapping fulfilling the inequality

$$\|\mathcal{D}f(x_1,x_2)\| \leqslant \phi(x_1,x_2) \tag{3.2}$$

for all $x_1,x_2 \in V^n$. Then, there exists a unique solution $\mathcal{F} : V^n \longrightarrow W$ of (2.1) such that

$$\|f(x) - \mathcal{F}(x)\| \leq \Phi(x) \tag{3.3}$$

for all $x = (x^k, x^{n-k}) \in V^n$.

Proof. Putting $x_1^k = x_2^k = x^k$, $x_1^{n-k} = x^{n-k}$, $x_2^{n-k} = 0$ in (3.2) and by relation (2.8) of Lemma 1, we have

$$\left\|2^{n-k} f(2x) - 2^{4n-3k} f(x)\right\| \leq \phi\left(x, \left(x^k, 0\right)\right) \tag{3.4}$$

and so

$$\left\|f(2x) - 2^{3n-2k} f(x)\right\| \leq \frac{1}{2^{n-k}} \phi\left(x, \left(x^k, 0\right)\right). \tag{3.5}$$

for all $x = x_1 \in V^n$. Set

$$\xi(x) := \frac{1}{2^{(3n-2k)\frac{|\beta+1|}{2}+n-k}} \phi\left(\frac{x}{2^{\frac{|\beta-1|}{2}}}, \left(\frac{x^k}{2^{\frac{|\beta-1|}{2}}}, 0\right)\right)$$

and $\mathcal{T}\xi(x) := \frac{1}{2^{(3n-2k)\beta}} \xi(2^\beta x)$ where $\xi \in W^{V^n}$. Then, relation (3.5) can be modified as

$$\|f(x) - \mathcal{T}f(x)\| \leq \xi(x) \qquad (x \in V^n).$$

Define $\Lambda\eta(x) := \frac{1}{2^{(3n-2k)\beta}}\eta(2^{\beta}x)$ for all $\eta \in \mathbb{R}_{+}^{V^n}, x = (x^k, x^{n-k}) \in V^n$. We now see that Λ has the form described in (A3) with $\mathcal{S} = V^n$, $g_1(x) = 2^{\beta}x$ and $L_1(x) = \frac{1}{2^{(3n-2k)\beta}}$ for all $x \in V^n$. On the other hand, for each $\lambda, \mu \in W^{V^n}$ and $x \in V^n$, we get

$$\|\mathcal{T}\lambda(x) - \mathcal{T}\mu(x)\| = \left\| \frac{1}{2^{(3n-2k)\beta}} \left[\lambda\left(2^{\beta}x\right) - \mu\left(2^{\beta}x\right)\right] \right\| \leq L_1(x)\,\|\lambda(g_1(x)) - \mu(g_1(x))\|.$$

The above relation shows that the hypothesis (A2) holds. By induction on l, one can check for any $l \in \mathbb{N}_0$ and $x \in V^n$ that

$$\begin{aligned}
\Lambda^l\xi(x) :&= \left(\frac{1}{2^{(3n-2k)\beta}}\right)^l \xi\left(2^{\beta l}x\right) \\
&= \frac{1}{2^{(3n-2k)\frac{|\beta+1|}{2}+n-k}} \left(\frac{1}{2^{(3n-2k)\beta}}\right)^l \phi\left(2^{\beta l-\frac{|\beta-1|}{2}}x, \left(2^{\beta l-\frac{|\beta-1|}{2}}x^k, 0\right)\right)
\end{aligned} \tag{3.6}$$

for all $x \in V^n$. The relations (3.1) and (3.6) necessitate that all assumptions of Theorem 1 are satisfied. Hence, there exists a unique mapping $\mathcal{F} : V^n \longrightarrow W$ such that

$$\mathcal{F}(x) = \lim_{l\to\infty}\left(\mathcal{T}^l f\right)(x) = \frac{1}{2^{(3n-2k)\beta}}\mathcal{F}\left(2^{\beta}x\right) \qquad (x \in V^n),$$

and (3.3) holds as well. We shall to prove that

$$\|\mathcal{D}(\mathcal{T}^l f)(x_1, x_2)\| \leq \left(\frac{1}{2^{(3n-2k)\beta}}\right)^l \phi\left(2^{\beta l}x_1, 2^{\beta l}x_2\right) \tag{3.7}$$

for all $x_1, x_2 \in V^n$ and $l \in \mathbb{N}_0$. We argue by induction on l. The validity of (3.7) for $l = 0$ obtains by (3.2). Assume that (3.7) is true for an $l \in \mathbb{N}_0$. Then

$$\begin{aligned}
\left\|\mathcal{D}\left(\mathcal{T}^{l+1}f\right)(x_1,x_2)\right\| &= \Bigg\| \sum_{q\in\{-1,1\}^{n-k}} \left(\mathcal{T}^{l+1}f\right)\left(x_1^k + x_2^k, 2x_1^{n-k} + qx_2^{n-k}\right) \\
&\qquad - \sum_{m=0}^{n-k} 2^{n-k-m}12^m \sum_{i\in\{1,2\}} \left(\mathcal{T}^{l+1}f\right)\left(x_i^k, \mathcal{M}_m^{n-k}\right) \Bigg\| \\
&= \frac{1}{2^{(3n-2k)\beta}} \Bigg\| \sum_{q\in\{-1,1\}^{n-k}} \left(\mathcal{T}^l f\right)\left(2^{\beta}\left(x_1^k + x_2^k\right), 2^{\beta}\left(2x_1^{n-k} + qx_2^{n-k}\right)\right) \\
&\qquad - \sum_{m=0}^{n-k} 2^{n-k-m}12^m \sum_{i\in\{1,2\}} \left(\mathcal{T}^l f\right)\left(2^{\beta}x_i^k, 2^{\beta}\mathcal{M}_m^{n-k}\right) \Bigg\| \\
&= \frac{1}{2^{(3n-2k)\beta}} \left\|\mathcal{D}\left(\mathcal{T}^l f\right)\left(2^{\beta}x_1, 2^{\beta}x_2\right)\right\| \\
&\leq \left(\frac{1}{2^{(3n-2k)\beta}}\right)^{l+1} \phi\left(2^{\beta(l+1)}x_1, 2^{\beta(l+1)}x_2\right)
\end{aligned}$$

for all $x_1, x_2 \in V^n$. Letting $l \to \infty$ in (3.7) and applying (3.1) we arrive at $\mathcal{D}\mathcal{F}(x_1, x_2) = 0$ for all $x_1, x_2 \in V^n$. This means that the mapping $\mathcal{F}$ satisfies (2.1) which finishes the proof. $\square$

In the next corollary, we show that the functional equation (2.1) is stable when the norm $\mathcal{D}f(x_1,x_2)$ is controlled by a small positive real number.

Corollary 1. *Given $\delta > 0$. Let also V be a normed space and W be a Banach space. If $f : V^n \longrightarrow W$ is a mapping satisfying the inequality*

$$\|\mathcal{D}f(x_1,x_2)\| \leq \delta$$

for all $x_1,x_2 \in V^n$, then there exists a unique solution $\mathcal{F} : V^n \longrightarrow W$ of (2.1) such that

$$\|f(x) - \mathcal{F}(x)\| \leq \frac{\delta}{2^{n-k}(2^{3n-2k}-1)}$$

for all $x \in V^n$.

Proof. Setting the constant function $\phi(x_1,x_2) = \delta$ for all $x_1,x_2 \in V^n$, and applying Theorem 2 in the case $\beta = 1$, one can obtain the desired result. $\square$

Corollary 2. *Let $\alpha \in \mathbb{R}$ with $\alpha \neq 3n-2k$. Let also V be a normed space and W be a Banach space. If $f : V^n \longrightarrow W$ is a mapping satisfying the inequality*

$$\|\mathcal{D}f(x_1,x_2)\| \leq \sum_{i=1}^{2}\sum_{j=1}^{n}\|x_{ij}\|^{\alpha}$$

for all $x_1,x_2 \in V^n$, then there exists a unique solution $\mathcal{F} : V^n \longrightarrow W$ of (2.1) such that

$$\|f(x) - \mathcal{F}(x)\| \leq \frac{1}{2^{n-k}(|2^{3n-2k}-2^{\alpha}|)}\left(2\sum_{j=1}^{k}\|x_{1j}\|^{\alpha} + \sum_{j=k+1}^{n}\|x_{1j}\|^{\alpha}\right)$$

for all $x = x_1 \in V^n$.

Proof. The result can be obtained by choosing the function $\phi(x_1,x_2) = \sum_{i=1}^{2}\sum_{j=1}^{n}\|x_{ij}\|^{\alpha}$ for all $x_1,x_2 \in V^n$ and using Theorem 2. $\square$

Recall that a functional equation Γ is *hyperstable* if any mapping f satisfying the equation Γ approximately is a true solution of Γ. Under some conditions the functional equation (2.1) can be hyperstable as follows.

Corollary 3. *Let $\delta > 0$. Suppose that $\alpha_{ij} > 0$ for $i \in \{1,2\}$ and $j \in \{1,\cdots,n\}$ fulfill $\sum_{i=1}^{2}\sum_{j=1}^{n}\alpha_{ij} \neq 3n-2k$. Let V be a normed space and W be a Banach space. If $f : V^n \longrightarrow W$ is a mapping satisfying the inequality*

$$\|\mathcal{D}f(x_1,x_2)\| \leq \delta\prod_{i=1}^{2}\prod_{j=1}^{n}\|x_{ij}\|^{\alpha_{ij}}$$

for all $x_1,x_2 \in V^n$, then f satisfies equation (2.1). In particular, if f satisfies the 3-power condition in the last $n-k$ variables, then it is multi-additive-cubic.

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