Miskolc Mathematical Notes
Vol. 21 (2020), No. 1, pp. 81–89

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2020.2904

COFINITELY $\oplus$-SUPPLEMENTED LATTICES

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Received 20 March, 2019

Abstract. In this work, cofinitely $\oplus-$supplemented and strongly cofinitely $\oplus-$supplemented lattices are defined and investigated some properties of these lattices. Let L be a lattice and $1 = \bigoplus_{i\in I} a_i$ with $a_i \in L$. If $a_i/0$ is cofinitely $\oplus-$supplemented for every $i \in I$, then L is also cofinitely $\oplus-$supplemented. Let L be a distributive lattice and $1 = a_1 \oplus a_2$ with $a_1, a_2 \in L$. If $a_1/0$ and $a_2/0$ are strongly cofinitely $\oplus-$supplemented, then L is also strongly cofinitely $\oplus-$supplemented. Let L be a lattice. If every cofinite element of L lies above a direct summand in L, then L is cofinitely $\oplus-$supplemented.

2010 *Mathematics Subject Classification:* 06C05; 06C15

Keywords: lattices, compact elements, small elements, supplemented lattices

1. Introduction

Throughout this paper, all lattices are complete modular lattices with the smallest element 0 and the greatest element 1. Let L be a lattice, $a, b \in L$ and $a \leq b$. A sublattice $\{x \in L | a \leq x \leq b\}$ is called a *quotient sublattice*, denoted by b/a. An element a' of a lattice L is called a *complement* of a if $a \wedge a' = 0$ and $a \vee a' = 1$, this case we denote $1 = a \oplus a'$ (a and a' also is called *direct summands* of L). L is called a *complemented lattice* if each element has at least one complement in L. An element c of L is said to be *compact* if for every subset X of L such that $c \leq \vee X$, there exists a finite $F \subseteq X$ such that $c \leq \vee F$. A lattice L is said to be *compactly generated* if each of its elements is a join of compact elements. A lattice L is said to be *compact* if 1 is a compact element of L. An element a of a lattice L is said to be *cofinite* if $1/a$ is compact. An element a of L is said to be *small* or *superfluous* and denoted by $a \ll L$ if $b = 1$ for every element b of L such that $a \vee b = 1$. The meet of all the maximal elements ($\neq 1$) of a lattice L is called the *radical* of L and denoted by $r(L)$. An element c of L is called a *supplement* of b in L if it is minimal for $b \vee c = 1$. a is a supplement of b in a lattice L if and only if $a \vee b = 1$ and $a \wedge b \ll a/0$. A lattice L is said to be *supplemented* if every element of L has a supplement in L. L is said to be *cofinitely supplemented* if every cofinite element of L has a supplement in L. L is said to be $\oplus-$*supplemented* if every element of L has a supplement that is a direct summand in L.We say that an element b of L *lies above* an element a of L if $a \leq b$ and $b \ll 1/a$. L is said to be *hollow* if every element $(\neq 1)$ is superfluous in L, and L is said to be *local* if L has the greatest element $(\neq 1)$. An element a of L is called a *weak supplement* of b in L if $a \vee b = 1$ and $a \wedge b \ll L$. A lattice L is said to be *weakly supplemented*, if every element of L has a weak supplement in L. L is said to be *cofinitely weak supplemented*, if every cofinite element of L has a weak supplement in L. An element $a \in L$ has *ample supplements* in L if for every $b \in L$ with $a \vee b = 1$, a has a supplement b' in L with $b' \leq b$. L is called an *amply supplemented lattice*, if every element of L has ample supplements in L. It is clear that every supplemented lattice is weakly supplemented and every amply supplemented lattice is supplemented. A lattice L is said to be *distributive* if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for every $a, b, c \in L$. Let L be a lattice. It is defined β_* relation on the elements of L by $a\beta_* b$ with $a, b \in L$ if and only if for each $t \in L$ such that $a \vee t = 1$ then $b \vee t = 1$ and for each $k \in L$ such that $b \vee k = 1$ then $a \vee k = 1$.

More details about (amply) supplemented lattices are in [1, 2, 7]. The definitions of cofinitely (weak) supplemented lattices and some properties of these lattices are in [1, 2]. The definition of $\oplus-$supplemented lattices and some properties of these lattices are in [5]. More results about (amply) supplemented modules are in [6, 11]. Some important properties of $\oplus-$supplemented modules are in [8, 9]. The definition of $\oplus-$cofinitely supplemented modules and some properties of these modules are in [4]. The definition of β_* relation on lattices and some properties of this relation are in [10]. The definition of β^* relation on modules and some properties of this relation are in [3].

Lemma 1. *Let L be a lattice and $a, b, c \in L$ with $a \leq b$. If c is a supplement of b in L, then $a \vee c$ is a supplement of b in $1/a$.*

Proof. Similar to proof of [7, Proposition12.2(7)]. □

Lemma 2 ([7, Lemma 7.4])**.** *Let L be a lattice, $a, b \in L$ and $a \leq b$. If $a \ll b/0$ then $a \ll L$.*

Lemma 3 ([7, Lemma 7.5])**.** *In a lattice L let $c' \ll c/0$ and $d' \ll d/0$. Then $c' \vee d' \ll (c \vee d)/0$.*

Lemma 4 ([7, Exercise 7.3])**.** *If L is a lattice and $a \in L$, then $r(a/0) \leq r(L)$.*

Lemma 5 ([7, Lemma 12.3])**.** *In any modular lattice $[(c \vee d) \wedge b] \leq [c \wedge (b \vee d)] \vee [d \wedge (b \vee c)]$ holds for every $b, c, d \in L$.*

Lemma 6 (See also [5])**.** *Let L be a lattice, $a, b \in L$ and $a \leq b$. Then b lies above a if and only if $a\beta_* b$.*

Proof. $(\Longrightarrow)$ See [10, Theorem 3].

$(\Longleftarrow)$ Let $b \vee t = 1$ with $t \in 1/a$. Since $a\beta_* b$, $a \vee t = 1$ and since $a \leq t$, $t = 1$. Hence $b \ll 1/a$ and b lies above a. □

2. Cofinitely $\oplus-$ supplemented lattices

Definition 1. Let L be a lattice. L is called a cofinitely $\oplus-$supplemented lattice, if every cofinite element of L has a supplement that is a direct summand of L.

Clearly we can see that every $\oplus-$supplemented lattice is cofinitely $\oplus-$supplemented and every cofinitely $\oplus-$supplemented lattice is cofinitely supplemented.

Proposition 1. *Let L be a lattice. Then L is cofinitely $\oplus-$supplemented if and only if for every cofinite $b \in L$, there exists a direct summand c of L such that $b \vee c = 1$ and $b \wedge c \ll c/0$.*

Proof. Clear from definition. □

Proposition 2. *Let L be a lattice. If every cofinite element of L has a weak supplement that is a direct summand of L, then L is cofinitely $\oplus-$supplemented.*

Proof. Let a be a cofinite element of L and b be a weak supplement of a in L that is a direct summand of L. Since b is a weak supplement of a in L, $a \wedge b \ll L$ and since b is a direct summand of L, $a \wedge b \ll b/0$. Hence b is a supplement of a in L and L is cofinitely $\oplus-$supplemented. □

Lemma 7 (See also [5])**.** *Let L be a lattice, and $a, b \in L$. If x is a supplement of $a \vee b$ in L and y is a supplement of $a \wedge (x \vee b)$ in $a/0$, then $x \vee y$ is a supplement of b in L. (See also [5]).*

Proof. Since x is a supplement of $a \vee b$ in L and y is a supplement of $a \wedge (x \vee b)$ in $a/0$, then $1 = a \vee b \vee x$, $(a \vee b) \wedge x \ll x/0$, $a = [a \wedge (x \vee b)] \vee y$ and $(x \vee b) \wedge y = a \wedge (x \vee b) \wedge y \ll y/0$. Here $1 = a \vee b \vee x = [a \wedge (x \vee b)] \vee y \vee b \vee x = b \vee x \vee y$. By Lemma 5, $(x \vee y) \wedge b \leq [(y \vee b) \wedge x] \vee [(x \vee b) \wedge y] \leq [(a \vee b) \wedge x] \vee [(x \vee b) \wedge y] \ll (x \vee y)/0$. Hence $x \vee y$ is a supplement of b in L. □

Lemma 8. *Let L be a lattice and $1 = \bigoplus_{i \in I} a_i$ with $a_i \in L$. If $a_i/0$ is cofinitely $\oplus-$supplemented for every $i \in I$, then L is also cofinitely $\oplus-$supplemented.*

Proof. Let x be any cofinite element of L. Since $1/x$ is compact and $1 = \bigvee_{i \in I} (x \vee a_i)$, there exists a finite subset $F = \{i_1, i_2, \ldots, i_n\}$ of I such that $1 = \bigvee_{t=1}^{n} (x \vee a_{i_t}) = x \vee \left(\bigvee_{t=1}^{n} a_{i_t} \right)$. Since x is a cofinite element of L, $x \vee \left(\bigvee_{t=1}^{n-1} a_{i_t} \right)$ is a cofinite element of L. Then by $\frac{1}{x \vee \left(\bigvee_{t=1}^{n-1} a_{i_t} \right)} = \frac{x \vee \left(\bigvee_{t=1}^{n-1} a_{i_t} \right) \vee a_{i_n}}{x \vee \left(\bigvee_{t=1}^{n-1} a_{i_t} \right)} \cong \frac{a_{i_n}}{a_{i_n} \wedge \left(x \vee \left(\bigvee_{t=1}^{n-1} a_{i_t} \right) \right)}$, $a_{i_n} \wedge \left(x \vee \left(\bigvee_{t=1}^{n-1} a_{i_t} \right) \right)$ is a cofinite element of $a_{i_n}/0$ and since $a_{i_n}/0$ is cofinitely $\oplus-$supplemented, $a_{i_n} \wedge \left(x \vee \left(\bigvee_{t=1}^{n-1} a_{i_t} \right) \right)$ has a supplement x_{i_n} that is a direct summand in

$a_{i_n}/0$. Since 0 is a supplement of $x \vee \left(\bigvee_{t=1}^{n} a_{i_t}\right)$ in L and x_{i_n} is a supplement of $a_{i_n} \wedge \left(x \vee \left(\bigvee_{t=1}^{n-1} a_{i_t}\right)\right)$ in $a_{i_n}/0$, by Lemma 7, $x_{i_n} = x_{i_n} \vee 0$ is a supplement of $x \vee \left(\bigvee_{t=1}^{n-1} a_{i_t}\right)$ in L. Since x is a cofinite element of L, $x \vee \left(\bigvee_{t=1}^{n-2} a_{i_t}\right) \vee x_{i_n}$ is a cofinite element of L. Then by

$$\frac{1}{x \vee \left(\bigvee_{t=1}^{n-2} a_{i_t}\right) \vee x_{i_n}} = \frac{x \vee \left(\bigvee_{t=1}^{n-2} a_{i_t}\right) \vee x_{i_n} \vee a_{i_{n-1}}}{x \vee \left(\bigvee_{t=1}^{n-2} a_{i_t}\right) \vee x_{i_n}} \cong \frac{a_{i_{n-1}}}{a_{i_{n-1}} \wedge \left(x \vee \left(\bigvee_{t=1}^{n-2} a_{i_t}\right) \vee x_{i_n}\right)},$$

$a_{i_{n-1}} \wedge \left(x \vee \left(\bigvee_{t=1}^{n-2} a_{i_t}\right) \vee x_{i_n}\right)$ is a cofinite element of $a_{i_{n-1}}/0$ and since $a_{i_{n-1}}/0$ is cofinitely $\oplus-$supplemented, $a_{i_{n-1}} \wedge \left(x \vee \left(\bigvee_{t=1}^{n-2} a_{i_t}\right) \vee x_{i_n}\right)$ has a supplement $x_{i_{n-1}}$ that is a direct summand in $a_{i_{n-1}}/0$. Since x_{i_n} is a supplement of $x \vee \left(\bigvee_{t=1}^{n-1} a_{i_t}\right)$ in L and $x_{i_{n-1}}$ is a supplement of $a_{i_{n-1}} \wedge \left(x \vee \left(\bigvee_{t=1}^{n-2} a_{i_t}\right) \vee x_{i_n}\right)$ in $a_{i_{n-1}}/0$, by Lemma 7, $x_{i_{n-1}} \vee x_{i_n}$ is a supplement of $x \vee \left(\bigvee_{t=1}^{n-2} a_{i_t}\right)$ in L. If so, x has a supplement $\bigvee_{t=1}^{n} x_{i_t}$ in L where x_{i_t} is a direct summand of $a_{i_t}/0$ for every $t = 1, 2, \ldots, n$. Since x_{i_t} is a direct summand of $a_{i_t}/0$ for every $t = 1, 2, \ldots, n$ and $1 = \bigoplus_{i \in I} a_i$, $\bigvee_{t=1}^{n} x_{i_t}$ is a direct summand of L. Hence L is cofinitely $\oplus-$supplemented. □

Corollary 1. *Let L be a lattice, $a_1, a_2, \ldots, a_n \in L$ and $1 = a_1 \oplus a_2 \oplus \ldots \oplus a_n$. If $a_i/0$ is cofinitely $\oplus-$supplemented for every $i = 1, 2, \ldots, n$, then L is cofinitely $\oplus-$supplemented.*

Proof. Clear from Lemma 8. □

Lemma 9. *Let L be a lattice, $a \in L$ and $a = (a \wedge a_1) \oplus (a \wedge a_2)$ for every $a_1, a_2 \in L$ with $1 = a_1 \oplus a_2$. If L is cofinitely $\oplus-$supplemented, then $1/a$ is also cofinitely $\oplus-$supplemented.*

Proof. Let x be a cofinite element of $1/a$. Then $1/x$ is compact and x is a cofinite element of L. Since L is cofinitely $\oplus-$supplemented, there exist $y, z \in L$ such that $1 = x \vee y$, $x \wedge y \ll y/0$ and $1 = y \oplus z$. Since y is a supplement of x in L and $a \leq x$, by Lemma 1, $a \vee y$ is a supplement of x in $1/a$. Since $1 = y \oplus z$, by hypothesis, $a = (a \wedge y) \oplus (a \wedge z)$. Then $(a \vee y) \wedge (a \vee z) = [(a \wedge y) \vee (a \wedge z) \vee y] \wedge [(a \wedge y) \vee (a \wedge z) \vee z] = [y \vee (a \wedge z)] \wedge [(a \wedge y) \vee z] = (a \wedge y) \vee [(y \vee (a \wedge z)) \wedge z] = (a \wedge y) \vee [(y \wedge$

$z) \vee (a \wedge z)] = (a \wedge y) \vee (0 \vee (a \wedge z)) = (a \wedge y) \vee (a \wedge z) = a$. Hence $1/a$ is cofinitely $\oplus-$supplemented. □

Corollary 2. *Let L be a distributive lattice. If L is cofinitely $\oplus-$supplemented, then $1/a$ is also cofinitely $\oplus-$supplemented for every $a \in L$.*

Proof. Clear from Lemma 9. □

Proposition 3. *Let L be a cofinitely $\oplus-$supplemented lattice and $r(L)$ be a cofinite element of L. Then there exist $a_1, a_2 \in L$ such that $1 = a_1 \oplus a_2$, $r(a_1/0) \ll a_1/0$ and $r(a_2/0) = a_2$.*

Proof. Since L is cofinitely $\oplus-$supplemented and $r(L)$ is a cofinite element of L, there exist $a_1, a_2 \in L$ such that $1 = r(L) \vee a_1 = a_1 \oplus a_2$ and $r(L) \wedge a_1 \ll a_1/0$. Then by Lemma 4, $r(a_1/0) \leq r(L) \wedge a_1 \ll a_1/0$.

Assume x be a maximal $(\neq a_2)$ element of $a_2/0$. Since $1/(a_1 \vee x) = (a_1 \oplus a_2)/(a_1 \vee x) = (a_1 \vee x \vee a_2)/(a_1 \vee x) \cong a_2/[a_2 \wedge (a_1 \vee x)] = a_2/[(a_2 \wedge a_1) \vee x] = a_2/x$, $a_1 \vee x$ is a maximal element $(\neq 1)$ of L and since $1 = r(L) \vee a_1 \leq a_1 \vee x$, this is a contradiction. Hence $r(a_2/0) = a_2$. □

Definition 2. Let L be a cofinitely supplemented lattice. L is called a strongly cofinitely $\oplus-$supplemented lattice if every supplement element of any cofinite element in L is a direct summand of L.

Clearly we can see that every strongly cofinitely $\oplus-$supplemented lattice is cofinitely $\oplus-$supplemented and every strongly $\oplus-$supplemented lattice is strongly cofinitely $\oplus-$supplemented.

Lemma 10 (See also [5])**.** *Let a be a supplement of b in L and $x, y \in a/0$. Then y is a supplement of x in $a/0$ if and only if y is a supplement of $b \vee x$ in L.*

Proof. $(\Longrightarrow)$ Let y be a supplement of x in $a/0$ and $b \vee x \vee z = 1$ with $z \leq y$. Because of $x, y \in a/0$ and $z \leq y$, $x \vee z \leq a$. Since a is a supplement of b in L, $a = x \vee z$. Since and y is a supplement of x in $a/0$, $z = y$. Hence y is a supplement of $b \vee x$ in L.

$(\Longleftarrow)$Let y be a supplement of $b \vee x$ in L. So, $b \vee x \vee y = 1$ and $(b \vee x) \wedge y \ll y/0$. Since $x \vee y \leq a$ and a is a supplement of b in L, $x \vee y = a$ and $x \wedge y \leq (b \vee x) \wedge y \ll y/0$. Hence y is a supplement of x in $a/0$. □

Proposition 4. *Let L be a strongly cofinitely $\oplus-$supplemented lattice. Then for every direct summand a of L, the quotient sublattice $a/0$ is strongly cofinitely $\oplus-$supplemented.*

Proof. Since a is a direct summand of L, there exists $b \in L$ such that $1 = a \oplus b$. Since L is cofinitely supplemented, we can see that $1/b$ is cofinitely supplemented. Then by $1/b = (a \vee b)/b \cong a/(a \wedge b) = a/0$, $a/0$ is cofinitely supplemented. Let x be a cofinite element of $a/0$ and y be supplement of x in $a/0$. By Lemma 10, y is a supplement of $b \vee x$ in L. By $\frac{1}{b \vee x} = \frac{a \vee b \vee x}{b \vee x} \cong \frac{a}{a \wedge (b \vee x)} = \frac{a}{(a \wedge b) \vee x} = \frac{a}{x}$, $b \vee x$

is a cofinite element of L. Since L is strongly cofinitely $\oplus-$supplemented, y is a direct summand of L. Here there exists $z \in L$ such that $1 = y \oplus z$. By modularity, $a = a \wedge 1 = a \wedge (y \oplus z) = y \oplus (a \wedge z)$. Thus y is a direct summand of $a/0$. Hence $a/0$ is strongly cofinitely $\oplus-$supplemented. □

Lemma 11. *Let L be a distributive lattice and $a_1, a_2 \in L$ with $1 = a_1 \oplus a_2$. If $a_1/0$ and $a_2/0$ are strongly cofinitely $\oplus-$supplemented, then L is also strongly cofinitely $\oplus-$supplemented.*

Proof. Let b be a cofinite element of L and a be a supplement of b in L. Since L is distributive, $a = a \wedge 1 = a \wedge (a_1 \oplus a_2) = (a \wedge a_1) \oplus (a \wedge a_2)$ holds. By Lemma 10, $a \wedge a_1$ is a supplement of $(a \wedge a_2) \vee b$ in L. Then we can see that $a \wedge a_1$ is a supplement of $a_1 \wedge ((a \wedge a_2) \vee b)$ in $a_1/0$. Since b is a cofinite element of L, we can see that $a_1 \wedge ((a \wedge a_2) \vee b)$ is a cofinite element of $a_1/0$. Since $a_1/0$ is strongly cofinitely $\oplus-$supplemented, $a \wedge a_1$ is a direct summand of $a_1/0$. Similarly we can see that $a \wedge a_2$ is a direct summand of $a_2/0$. Since $1 = a_1 \oplus a_2$ and $a = (a \wedge a_1) \oplus (a \wedge a_2)$, a is a direct summand of L. Hence L is strongly cofinitely $\oplus-$supplemented. □

Corollary 3. *Let L be a distributive lattice, $a_1, a_2, ..., a_n \in L$ and $1 = a_1 \oplus a_2 \oplus ... \oplus a_n$. If $a_i/0$ is strongly cofinitely $\oplus-$supplemented for every i$= 1, 2, \ldots, n$, then L is strongly cofinitely $\oplus-$supplemented.*

Proof. Clear from Lemma 11. □

Proposition 5. *Let L be a cofinitely supplemented lattice. The following statements are equivalent.*
(i) L is strongly cofinitely $\oplus-$supplemented.
(ii) Every supplement element of a cofinite element of L lies above a direct summand in L.
(iii) (a) For every nonzero supplement element a which is a supplement of a cofinite element of L, $a/0$ contains a nonzero direct summand of L.
(b) For every nonzero supplement element a which is a supplement of a cofinite element of L, $a/0$ contains a maximal direct summand of L.

Proof. $(i) \Longrightarrow (ii)$ Clear, since every element of L lies above itself.
$(ii) \Longrightarrow (iii)$ Let a be a nonzero supplement element which is a supplement of a cofinite element of L. Assume a is a supplement of a cofinite element b of L. By hypothesis, there exists a direct summand x of L such that a lies above x in L. By Lemma 6, $a\beta_* x$ and since $a \vee b = 1$, $x \vee b = 1$. Since a is a supplement of b in L and $x \leq a$, $a = x$ and a is a nonzero direct summand of L.
$(iii) \Longrightarrow (i)$ Let a be a supplement of a cofinite element b of L and x be a maximal direct summand of L with $x \leq a$. Assume $1 = x \oplus y$ with $y \in L$. Then $a = a \wedge 1 = a \wedge (x \oplus y) = x \oplus (a \wedge y)$ and by Lemma 10, $a \wedge y$ is a supplement of $b \vee x$ in L. If $a \wedge y$ is not zero, then by hypothesis, $(a \wedge y)/0$ contains a nonzero direct summand c of L. Here $x \oplus c$ is a direct summand of L and $x \oplus c \leq a$. This contradicts the choice

of x. Hence $a \wedge y = 0$ and $a = x$. Thus a is a direct summand of L and L is strongly cofinitely $\oplus-$supplemented. □

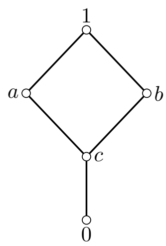
Lemma 12. *Let L be a lattice. If every cofinite element of L is β_* equivalent to a direct summand in L, then L is cofinitely $\oplus-$supplemented.*

Proof. Let a be a cofinite element of L. By hypothesis, there exist $x, y \in L$ with $x \oplus y = 1$ and $a\beta_* x$. Then $a \vee y = 1$. Let $a \vee t = 1$ with $t \leq y$. Since $a\beta_* x$, $x \vee t = 1$ and since $x \oplus y = 1$, $t = y$. Hence y is a supplement of a in L and L is cofinitely $\oplus-$supplemented. □

Corollary 4. *Let L be a lattice. If every cofinite element of L lies above a direct summand in L, then L is cofinitely $\oplus-$supplemented.*

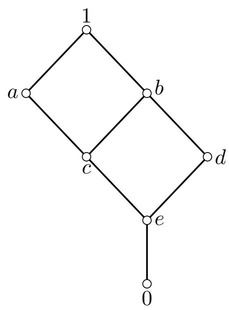
Proof. Clear from Lemma 6 and Lemma 12. □

Example 1. Consider the lattice $L = \{0, a, b, c, 1\}$ given by the following diagram.



Then L is cofinitely supplemented but not cofinitely $\oplus-$supplemented.

Example 2. Consider the lattice $L = \{0, a, b, c, d, e, 1\}$ given by the following diagram.



Then L is cofinitely supplemented but not cofinitely $\oplus-$supplemented.

Example 3. Consider the interval $[0,1]$ with natural topology. Let P be the set of all closed subsets of $[0,1]$. P is complete modular lattice by the inclusion (See [1, Example 2.10]). Here $\bigwedge_{i\in I} C_i = \bigcap_{i\in I} C_i$ and $\bigvee_{i\in I} C_i = \overline{\bigcup_{i\in I} C_i}$ for every $C_i \in P$ $(i \in I)$ $\left(\overline{\bigcup_{i\in I} C_i} \text{ is the closure of } \bigcup_{i\in I} C_i\right)$. Let $X \in P$ and $X \vee Y = [0,1]$ with $Y \in P$. Then

$[0,1]-X \subset Y$ and since Y is closed $\overline{[0,1]-X} \subset Y$. Let $X' = \overline{[0,1]-X}$. Then $X' \in P$, $X \vee X' = X \cup X' = [0,1]$ and $X' \subset Y$ for every $Y \in P$ with $X \vee Y = [0,1]$. Hence X has ample supplements in P (here $X' = \overline{[0,1]-X}$ is the only supplement of X in P) and P is amply supplemented. Let $A = [0,a] \in P$ with $0 < a < 1$. Here $A' = \overline{[0,1]-A} = [a,1]$ is the only supplement of A in P. Let $A' \vee B = A' \cup B = [0,1]$ with $B \in P$. Since $A' \cup B = [0,1]$, $[0,a) = [0,1]-A' \subset B$ and since B is closed, $[0,a] \subset B$. This case $a \in B$ and since $a \in A'$, $A' \wedge B = A' \cap B \neq \varnothing$. Hence A' is not a direct summand of P and P is not $\oplus-$supplemented (See also [5]). We can see that $[0,1]$ is only a cofinite element of L. Hence P is strongly cofinitely $\oplus-$supplemented.

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