GENERALIZED ELASTICA IN $SO(3)$

GÖZDE ÖZKAN TÜKEL, TUNAHAN TURHAN, AND AHMET YÜCESAN

Abstract. In a Lie group G equipped with bi-invariant Riemannian metric, we characterize the generalized elastica by an Euler-Lagrange equation in terms of the Lie reduction V of a curve γ in G. We define a generalized elastic Lie quadratic in the Lie algebra of G. For a generalized elastic Lie quadratic, we construct the Lax equation that is crucial to the solution of a generalized elastica with regard to its generalized elastic Lie quadratic. Then we solve this equation for a null generalized elastic Lie quadratic with $\left\| \dot{V}(t) \right\| =$ constant when G Lie group is $SO(3)$.

1. INTRODUCTION

Elastica (or elastic curve) proposed by Bernoulli to Euler is an extremal of the bending energy functional $\int_{\gamma} \left(\kappa^2 + \lambda \right) ds$, where γ is a curve, κ is the curvature of γ and λ is a Lagrange multiplier depending on the length [15]. The history of elastica is quite old. So far it has been studied and developed by a lot of authors under various point of view including a generalization of elastica to Riemannian manifold [8–12]. The generalized elastica is defined as critical point of the functional $\mathcal{F}(\gamma) = \int_{\gamma} P(\kappa) ds$, under some boundary conditions, where $P(\kappa)$ is a differentiable function of κ. Existence, classification or stability of this variational problem have been investigated in an Euclidean space, a Riemannian manifold, etc. [1–4,6,7]. Extremals of the functional $\mathcal{F}$ correspond to geodesics (when $P(\kappa) = \kappa^r$, $r = 0$), classical elastica (when $P(\kappa) = \kappa^2 + \lambda$), free elastica (when $P(\kappa) = \kappa^2$), elastica circular at rest (when $P(\kappa) = (\kappa + \lambda)^2$), $r-$elastica (or free hyperelastic curves, when $P(\kappa) = \kappa^r$, $r > 2$), etc. [1,4].

Lie groups which lie at the intersection of algebra and geometry play an important role both of them. While algebraic properties of Lie groups come from the group axioms, their geometric properties come from the identification of group operations with points in a topological space. The Lie group structures derive from combining the algebraic and topological properties via differentiability requirements. So, the

elements of this group are the points in a manifold that are parametrized by continuous real variables [5]. In this paper, we study the problem of generalized elastica in a manifold which is a Lie group equipped with bi-invariant Riemannian metric. A curve defined in the Lie group corresponds to the Lie reduction in its Lie algebra. Popiel and Noakes (2007) gave a characterization of elastic curves in Lie groups with regard to corresponding Lie reduction. They define "elastic Lie quadratics" as solutions of the Euler-Lagrange equation in the Lie algebra [14]. They solve this variational problem in $SO(3)$ by quadratures. Motivated by [14], we survey the theory of finding extremal of the generalized curvature energy functional in Lie groups equipped with a bi-invariant Riemannian metric.

Now we remind the characterization of a generalized elastica in a Riemannian manifold. Let M be a $n-$dimensional Riemannian manifold with Riemannian metric $<, >$, Riemannian norm $\|.\|$, Levi-Civita connection $\bigtriangledown$ and Riemannian curvature tensor R. Let Ω be the space of C^∞ curves $\gamma : [0, \ell] \to M$ satisfying

$$\|\dot\gamma\| = \left\| \frac{d\gamma}{dt} \right\| = 1$$

$$\gamma(i\ell) = p_i, \dot\gamma(i\ell) = v_i$$

for $p_i \in M$ and $v_i \in T_{p_i} M, i = 0, 1$.

A generalized elastic curve (or P-elastica) is an extremal of the generalized Euler-Bernoulli energy functional

$$
\begin{aligned}
\mathcal{F} : \quad \Omega \quad &\to \quad \mathbb{R} \\
\gamma \quad &\to \quad \mathcal{F}(\gamma) = \int\limits_0^\ell P(\kappa)dt,
\end{aligned}
\tag{1}
$$

where $P(\kappa)$ is a C^∞ function and $\kappa = \left\| \bigtriangledown_{\frac{d}{dt}} \dot\gamma \right\|$ is the geodesic curvature of γ, acting on space curves in a Riemannian manifold satisfying given boundary conditions. Any critical point of the functional (1) satisfies the following Euler-Lagrange equation

$$\bigtriangledown^2_{\frac{d}{dt}} \left(\frac{P'(\kappa)}{\kappa} \bigtriangledown_{\frac{d}{dt}} \dot\gamma \right) + \frac{P'(\kappa)}{\kappa} R(\bigtriangledown_{\frac{d}{dt}} \dot\gamma, \dot\gamma)\dot\gamma + \bigtriangledown_{\frac{d}{dt}} \left[(2\kappa P'(\kappa) - P(\kappa)) \dot\gamma \right] = 0,
\tag{2}$$

where $P'(\kappa) = \frac{dP}{d\kappa}$, [3, 4, 6].

In the following, we present a theorem which characterizes generalized elastica as a differential equation with boundary conditions of a special form.

Theorem 1. *Any C^∞ curve $\gamma : I \subset \mathbb{R} \to M$ is a generalized elastica iff γ satisfies the Euler Lagrange equation (2) for all $t \in I$ and following equalities*

$$1 = \|\dot\gamma(t_0)\|,
\tag{3}$$

$$0 = < \bigtriangledown_{\frac{d}{dt}} \dot\gamma \Big|_{t=t_0}, \dot\gamma(t_0) >,
\tag{4}$$

$$0 = < \nabla^2_{\frac{d}{dt}} \dot{\gamma} \Big|_{t=t_0}, \dot{\gamma}(t_0) > + \left\| \nabla_{\frac{d}{dt}} \dot{\gamma} \Big|_{t=t_0} \right\|^2, \tag{5}$$

for some $t_0 \in I$.

Proof. Let $\gamma : I \to M$ be a generalized elastica. Then $\dot{\gamma}$ satisfies the Euler-Lagrange equation (2) and

$$\|\dot{\gamma}(t)\|^2 = 1 \tag{6}$$

for any $t \in I$. By taking the first and second derivative of (6), we obtain

$$< \nabla_{\frac{d}{dt}} \dot{\gamma} \Big|_t, \dot{\gamma}(t) > = 0,$$

$$< \nabla^2_{\frac{d}{dt}} \dot{\gamma} \Big|_t, \dot{\gamma}(t) > + \left\| \nabla_{\frac{d}{dt}} \dot{\gamma} \Big|_t \right\|^2 = 0.$$

In particular, (3), (4) and (5) holds for any $t_0 \in I$.

Now we need to suppose that γ satisfies Equations $(2-5)$. Then we show that $\|\dot{\gamma}(t)\|^2 = 1$ for all $t \in I$. If we consider $I = (s_1, s_2)$, we show that $\|\dot{\gamma}(t)\|^2 = 1$ for all $t \in [t_0, s_2)$ and $t \in (s_1, t_0]$. We write $\mathcal{L} = \{t \in [t_0, s_2) : \|\dot{\gamma}(t)\| = 1\}$ and $\aleph = \sup(\mathcal{L})$. We show that in fact $\|\dot{\gamma}(t)\|^2 = 1$ on some open interval containing $\aleph$; this contradicts $\aleph = \sup(\mathcal{L})$, so we get $\|\dot{\gamma}(t)\|^2 = 1$ on $[t_0, s_2)$, (the proof for $(s_1, t_0]$ is similar). Write $\gamma_1 = \gamma, \gamma_2 = \dot{\gamma}, \gamma_3 = \nabla_{\frac{d}{dt}} \dot{\gamma}$ and $\gamma_4 = \nabla^2_{\frac{d}{dt}} \dot{\gamma}$. Then (2) can be written as the following system:

$$\dot{\gamma}_1 = \gamma_2, \ \nabla_{\frac{d}{dt}} \gamma_2 = \gamma_3, \ \nabla_{\frac{d}{dt}} \gamma_3 = \gamma_4$$

$$\nabla_{\frac{d}{dt}} \gamma_4 = -\frac{\|\gamma_3\|}{P'(\|\gamma_3\|)} \nabla^2_{\frac{d}{dt}} \left(\frac{P'(\gamma_3)}{\|\gamma_3\|} \right) \gamma_3 - R(\gamma_3, \gamma_2)\gamma_2$$

$$-\frac{\|\gamma_3\|}{P'(\|\gamma_3\|)} \left(\left(\nabla_{\frac{d}{dt}} \left(2\|\gamma_3\| P'(\|\gamma_3\|) - P(\|\gamma_3\|) \right) \right) \gamma_2 \right.$$

$$\left. + \left(2\|\gamma_3\| P'(\|\gamma_3\|) - P(\|\gamma_3\|) \right) \gamma_3 \right).$$

The rest of the proof can be seen by the similar methodology in the proof of Theorem 1.2 in [14]. □

After we give Theorem 1 which characterizes generalized elastica in M, we organize the next part of the manuscript as follows. In Section 2 we study the $n-$dimensional manifold M to be a Lie group equipped with bi-invariant Riemannian metric. We briefly talk about the Lie group and the corresponding Lie algebraic structure and remind to the reader basic structures to be needed throughout the paper. Then we give the transitions between some covariant derivatives of γ and the Lie reduction of γ. We derive an Euler-Lagrange equation which characterizes the generalized elastica with regard to the Lie reduction of a curve γ in Lie group G. In

Section 3, we assume the Lie group G is $SO(3)$ which is known as the group of rotations in the Euclidean 3−space. We solve this equation for a null generalized elastic Lie quadratic in $\mathfrak{so}(3)$ which is the set of skew-symmetric 3×3 matrices.

2. GENERALIZED ELASTICA IN LIE GROUPS

A Lie group G is a C^{∞} manifold that is also a group with smooth group operations [5]. The identity element of G is denoted by e. The left and right multiplications by $g \in G$ are the maps $L_x, R_x : G \to G$ defined by $L_x(y) := xy$ and $R_x(y) := yx$, respectively [13]. If a Riemannian metric $<,>$ satisfies for all $x, y \in G$ and $u, v \in T_y G$

$$< u, v >_y = < d\left(L_x\right)_y (u), d\left(L_x\right)_y (v) >_{L_x(y)}, \qquad (7)$$

then the metric $<,>$ is called left-invariant. A Riemannian metric $<,>$ is known bi-invariant if it is invariant both left and right invariance. Throughout this paper, we consider that the manifold M is a Lie group G equipped with bi-invariant Riemannian metric $<,>$. Bi-invariance of a left-invariant metric $<,>$ for all $X, Y, Z \in \mathfrak{g}$ is equivalent

$$< [X, Y], Z > = < [Z, X], Y > \qquad (8)$$

and the following properties hold:

$$\nabla_X Y = \frac{1}{2}[X, Y], \; R(X, Y)Z = -\frac{1}{4}[[X, Y], Z]$$

where [,] is the Lie bracket [5, 12].

Now, we suppose that $\gamma : I \subset \mathbb{R} \to G$ be a curve on G. Then we define $V : I \to \mathfrak{g}$ by

$$V(t) = \left(dL_{\gamma(t)^{-1}}\right)_{\gamma(t)} \dot{\gamma}(t). \qquad (9)$$

The curve V is known the Lie reduction corresponding to γ. (9) is equal to the differential equation

$$\dot{\gamma}(t) = \left(dL_{\gamma(t)}\right)_e V(t)$$

[10]. For all $t \in I$, the Riemannian curvature tensor is given by

$$\left(dL_{\gamma(t)^{-1}}\right)_{\gamma(t)} R(\nabla_{\frac{d}{dt}} \dot{\gamma}(t), \dot{\gamma}(t))\dot{\gamma}(t) = -\frac{1}{4}[V(t), [V(t), \dot{V}(t)]]$$

in the Lie algebra $\mathfrak{g}$ [14].

Lemma 1. *Let $\gamma : I \to G$ be a differentiable curve. Suppose that the Lie reduction of γ is given by $V : I \to \mathfrak{g}$. Then we have for all $t \in I$ in the following equations:*

$$i) \left(dL_{\gamma(t)^{-1}}\right)_{\gamma(t)} \nabla_{\frac{d}{dt}} \dot{\gamma}(t) = \dot{V}(t),$$

$$ii) \left(dL_{\gamma(t)^{-1}}\right)_{\gamma(t)} \nabla_{\frac{d}{dt}}^2 \left(\frac{P'(\kappa)}{\kappa} \nabla_{\frac{d}{dt}} \dot{\gamma}(t)\right) = \frac{d^2}{dt^2}\left(\frac{P'(\|\dot{V}(t)\|)}{\|\dot{V}(t)\|}\right) \dot{V}(t)$$

$$+ 2\frac{d}{dt}\left(\frac{P'(\|\dot{V}(t)\|)}{\|\dot{V}(t)\|}\right)\left(\ddot{V}(t) + \frac{1}{2}[V(t), \dot{V}(t)]\right)$$

$$+ \frac{P'(\|\dot{V}(t)\|)}{\|\dot{V}(t)\|}\left(\frac{d^3 V(t)}{dt^3} + [V(t), \ddot{V}(t)] + \frac{1}{4}[V(t), [V(t), \dot{V}(t)]]\right),$$

$$iii) \left(dL_{\gamma(t)^{-1}}\right)_{\gamma(t)} \frac{P'(\kappa)}{\kappa} R(\nabla_{\frac{d}{dt}} \dot{\gamma}(t), \dot{\gamma}(t))\dot{\gamma}(t) = -\frac{1}{4}\frac{P'(\|\dot{V}(t)\|)}{\|\dot{V}(t)\|}[V(t), [V(t), \dot{V}(t)]],$$

$$iv) \left(dL_{\gamma(t)^{-1}}\right)_{\gamma(t)} \nabla_{\frac{d}{dt}}[(2\kappa P'(\kappa) - P(\kappa))\dot{\gamma}] = \frac{d}{dt}\left(2\|\dot{V}(t)\| P'(\|\dot{V}(t)\|)\right)$$
$$- P(\|\dot{V}(t)\|)) V(t) + \left(2\|\dot{V}(t)\| P'(\|\dot{V}(t)\|) - P(\|\dot{V}(t)\|)\right) \dot{V}(t)$$

Proof. Let $\{E_1(t), E_2(t), \ldots, E_n(t)\}$ be an orthonormal frame of the Lie algebra $\mathfrak{g}$. Thus we have an orthonormal frame $\{\bar{E}_1(\gamma(t)), \bar{E}_2(\gamma(t)), \ldots, \bar{E}_n(\gamma(t))\}$ for $T_{\gamma(t)}G$, [11, 12]. By using the fact that

$$\nabla_{\bar{E}_i(\gamma(t))} \bar{E}_j(\gamma(t)) = \frac{1}{2}[\bar{E}_i(\gamma(t)), \bar{E}_j(\gamma(t))]$$

and the left invariance for vector fields of G, we can write

$$\dot{\gamma}(t) = \sum_i v_i \bar{E}_i(\gamma(t)).$$

One can found the proof of (i) of Lemma 1 in [11].

By using (7) and (i) of Lemma 1, we get

$$\kappa = \left\|\nabla_{\frac{d}{dt}} \dot{\gamma}(t)\right\| = \left\|\left(dL_{\gamma(t)^{-1}}\right)_{\gamma(t)} \nabla_{\frac{d}{dt}} \dot{\gamma}(t)\right\| = \|\dot{V}(t)\|.$$

So we can make the following calculations:

$$\nabla_{\frac{d}{dt}}\left(\frac{P'(\kappa)}{\kappa}\nabla_{\frac{d}{dt}}\dot\gamma(t)\right) = \frac{d}{dt}\left(\frac{P'(\kappa)}{\kappa}\right)(dL_{\gamma(t)})_e\sum_i\frac{dv_i}{dt}E_i(t)$$

$$+\frac{P'(\kappa)}{\kappa}(dL_{\gamma(t)})_e\left[\sum_i\frac{d^2v_i}{dt^2}E_i(t)+\frac{dv_i}{dt}\left(v_j\,\nabla_{\bar E_i(\gamma(t))}\bar E_j(\gamma(t))\right)\right]$$

$$=(dL_{\gamma(t)})_e\left[\frac{d}{dt}\left(\frac{P'(\|\dot V(t)\|)}{\|\dot V(t)\|}\right)\dot V(t)+\frac{P'(\|\dot V(t)\|)}{\|\dot V(t)\|}(\ddot V(t)\frac{1}{2}[V(t),\dot V(t)])\right]$$

and

$$\nabla^2_{\frac{d}{dt}}\left(\frac{P'(\kappa)}{\kappa}\nabla_{\frac{d}{dt}}\dot\gamma(t)\right)=(dL_{\gamma(t)})_e\left[\frac{d^2}{dt^2}\left(\frac{P'(\|\dot V(t)\|)}{\|\dot V(t)\|}\right)\dot V(t)\right.$$

$$+2\frac{d}{dt}\frac{P'(\|\dot V(t)\|)}{\|\dot V(t)\|}(\ddot V(t)\frac{1}{2}[V(t),\dot V(t)])$$

$$\left.+\frac{P'(\|\dot V(t)\|)}{\|\dot V(t)\|}\left(\frac{d^3V}{dt^3}+[V(t),\ddot V(t)]+\frac{1}{4}[V(t),[V(t),\ddot V(t)]]\right)\right]$$

$ii)$

$$\frac{P'(\kappa)}{\kappa}R(\nabla_{\frac{d}{dt}}\dot\gamma(t),\dot\gamma(t))\dot\gamma(t)\;=\;\frac{P'(\kappa)}{\kappa}(dL_{\gamma(t)})_e\left[-\frac{1}{4}[V(t),[V(t),\dot V(t)]]\right]$$

$$=\;(dL_{\gamma(t)})_e\left[-\frac{1}{4}\frac{P'(\|\dot V(t)\|)}{\|\dot V(t)\|}[V(t),[V(t),\dot V(t)]]\right].$$

The proof of (iv) of Lemma 1 is a result of (i) and (ii) of Lemma 1. $\square$

The following theorem gives the characterization of a generalized elastica in G.

Theorem 2. *Any differentiable curve $\gamma: I \to G$ in the Lie group G is a generalized elastica iff the curve $V: I \to \mathfrak{g}$ defined by (9) satisfies*

$$\|V(t)\|^2 = 1, \tag{10}$$

$$\frac{d}{dt}\left(\frac{P'(\|\dot V(t)\|)}{\|\dot V(t)\|}\dot V(t)\right)+\frac{P'(\|\dot V(t)\|)}{\|\dot V(t)\|}[V(t),\dot V(t)]$$
$$+\left(P(\|\dot V(t)\|)-2<C,V(t)>\right)V(t)+C=0 \tag{11}$$

for some constant $C \in \mathfrak{g}$ and all $t \in I$.

Proof. Assume that $\gamma : I \to G$ is a generalized elastica in G. Then we have from the left invariance of Riemannian metric and Eq. (9), we get

$$1 = \left\| \dot{\gamma}(t) \right\|^2 = \left\| \left(dL_{\gamma(t)^{-1}} \right)_{\gamma(t)} \dot{\gamma}(t) \right\|^2 = \| V(t) \|^2 . \tag{12}$$

If γ is a generalized elastica, then γ satisfies the Euler-Lagrange equation (2). Applying $\left(dL_{\gamma(t)^{-1}} \right)_{\gamma(t)}$ to (2) and using Lemma 1, we obtain

$$\frac{d}{dt} \left(\frac{d}{dt} \left(\frac{P'(\|\dot{V}(t)\|)}{\|\dot{V}(t)\|} \right) \dot{V}(t) \right) + \frac{d}{dt} \left(\left(\frac{P'(\|\dot{V}(t)\|)}{\|\dot{V}(t)\|} \right) [V(t), \dot{V}(t)] \right)$$
$$+ \frac{d}{dt} \left(\frac{P'(\|\dot{V}(t)\|)}{\|\dot{V}(t)\|} \ddot{V}(t) \right) + \frac{d}{dt} \left((2 \|\dot{V}(t)\| P'(\|\dot{V}(t)\|) - P(\|\dot{V}(t)\|)) V(t) \right) = 0.$$

Integrating once, we have,

$$\frac{d}{dt} \left(\frac{P'(\|\dot{V}(t)\|)}{\|\dot{V}(t)\|} \dot{V}(t) \right) + \left(\frac{P'(\|\dot{V}(t)\|)}{\|\dot{V}(t)\|} \right) [V(t), \dot{V}(t)]$$
$$+ \left(2 \|\dot{V}(t)\| P'(\|\dot{V}(t)\|) - P(\|\dot{V}(t)\|) \right) V(t) + C = 0 \tag{13}$$

where $C \in \mathfrak{g}$ is a constant. The first and second derivative of (12) are found as follows

$$< \dot{V}(t), V(t) > = 0, \tag{14}$$

$$< \ddot{V}(t), V(t) > + \left\| \dot{V}(t) \right\|^2 = 0. \tag{15}$$

Taking inner product of (13) with $\dot{V}(t)$ and applying (14), we have

$$\frac{d}{dt} \left(\frac{P'(\|\dot{V}(t)\|)}{\|\dot{V}(t)\|} < \dot{V}(t), \dot{V}(t) > \right)$$
$$- \frac{P'(\|\dot{V}(t)\|)}{\|\dot{V}(t)\|} < \ddot{V}(t), \dot{V}(t) > + < C, \dot{V}(t) > = 0. \tag{16}$$

Integrating (16), we obtain

$$P(\|\dot{V}(t)\|) = P'(\|\dot{V}(t)\|) \|\dot{V}(t)\| + < C, V(t) > + b. \tag{17}$$

for some constant $b \in \mathbb{R}$. If we take inner product of (13) with $V(t)$ and using (14) and (15), we have

$$P(\|\dot{V}(t)\|) = \|\dot{V}(t)\| P'(\|\dot{V}(t)\|) + < V(t), C > . \tag{18}$$

Combining (17) and (18), we obtain $b = 0$. Substituting (18) into (13), we have (11).

Conversely, let $V : I \to \mathfrak{g}$ corresponds the Lie reduction of a curve $\gamma : I \to G$. Suppose (10) and (11) are satisfied. From the left invariance of Riemannian metric, we have

$$\|V(t)\|^2 = \left\| \left(dL_{\gamma(t)^{-1}} \right)_{\gamma(t)} \dot{\gamma}(t) \right\|^2 = \left\| \dot{\gamma}(t) \right\|^2 = 1.$$

Then it remains the show that γ satisfies (2). Writing

$$< C, V(t) >= P(\|\dot{V}(t)\|) - \|\dot{V}(t)\| P'(\|\dot{V}(t)\|)$$

and differentiating (11), we get

$$\nabla^2_{\frac{d}{dt}} \left(\frac{P'(\|\dot{V}(t)\|)}{\|\dot{V}(t)\|} \dot{V}(t) \right) + \nabla_{\frac{d}{dt}} \left(\frac{P'(\|\dot{V}(t)\|)}{\|\dot{V}(t)\|} [V(t), \dot{V}(t)] \right)$$
$$+ \nabla_{\frac{d}{dt}} \left(2 \|\dot{V}(t)\| P'(\|\dot{V}(t)\|) - P(\|\dot{V}(t)\|) V(t) \right) = 0$$

Applying (9) and using Lemma 1, we obtain

$$\left(dL_{\gamma(t)^{-1}} \right)_{\gamma(t)} (\nabla^2_{\frac{d}{dt}} \left(\frac{P'(\|\nabla_{\frac{d}{dt}} \dot{\gamma}(t)\|)}{\|\nabla_{\frac{d}{dt}} \dot{\gamma}(t)\|} \nabla_{\frac{d}{dt}} \dot{\gamma}(t) \right) + \frac{P'(\|\nabla_{\frac{d}{dt}} \dot{\gamma}(t)\|)}{\|\nabla_{\frac{d}{dt}} \dot{\gamma}\|} R(\nabla_{\frac{d}{dt}} \dot{\gamma}(t), \dot{\gamma}(t)) \dot{\gamma}(t)$$
$$+ \nabla_{\frac{d}{dt}} (2 \|\nabla_{\frac{d}{dt}} \dot{\gamma}(t)\| P'(\|\nabla_{\frac{d}{dt}} \dot{\gamma}(t)\|) - P(\|\nabla_{\frac{d}{dt}} \dot{\gamma}(t)\|) \dot{\gamma}(t))) = 0.$$

for $\forall t \in I$. Since $\left(dL_{\gamma(t)^{-1}} \right)_{\gamma(t)}$ is an isomorphism, γ satisfies (2). $\square$

Definition 1. Any curve $V : I \to \mathfrak{g}$ satisfying (10) and (11) for some $C \in \mathfrak{g}$ and $\forall t \in I$ is called a generalized elastic Lie quadratic with constant C. Also, V defined by (9) is called a generalized elastic Lie quadratic associated with γ, if $\gamma : I \to G$ is a generalized elastica.

Corollary 1. *Let $V : I \to \mathfrak{g}$ be a generalized elastic Lie quadratic. We define $W : I \to \mathfrak{g}$ by*

$$W(t) = \frac{d}{dt} \left(\frac{P'(\|\dot{V}(t)\|)}{\|\dot{V}(t)\|} \dot{V}(t) \right) + (P(\|\dot{V}(t)\|) - 2 < C, V(t) >) V(t). \qquad (19)$$

Then we have

$$\dot{W}(t) = [W(t), V(t)] \qquad (20)$$

for all $t \in I$, and $\|W(t)\|$ is a constant.

Proof. Substituting (19) in (11), we have

$$W(t) = \frac{P'(\|\dot{V}(t)\|)}{\|\dot{V}(t)\|} [\dot{V}(t), V(t)] - C. \qquad (21)$$

Differentiating (21), we obtain

$$\dot{W}(t) = [\frac{d}{dt}\left(\frac{P'(\|\dot{V}(t)\|)}{\|\dot{V}(t)\|}\dot{V}(t)\right), V(t)]. \tag{22}$$

Combining (19) and (22), we obtain desired Eq. (20). On the other hand from (20), we have

$$\frac{d}{dt}\|W(t)\|^2 = \frac{d}{dt} < W(t), W(t) >= 2 < \dot{W}(t), W(t) >$$
$$= 2 < [W(t), V(t)], W(t) >= 0.$$

Therefore $\|W(t)\|$ is found a constant. □

The differential equation (20) known as Lax equation is an extremal to solution of (9) or equivalently $\dot{\gamma}(t) = (dL_{\gamma(t)})_e V(t)$ for a generalized elastic Lie quadratic V. Popiel and Noakes prove that the differential equation that gives the elastic curve can expand the whole real axis by Picard's theorem and Lax equations[14]. Then by Theorem 3.1 in [14] and Theorem 1, all generalized elastica in G extend uniquely to $\mathbb{R}$ when G is compact.

3. GENERALIZED ELASTICA IN $SO(3)$

In this section we suppose $G = SO(3)$ which is the group of rotations of Euclidean $3-$space. Then the Lie algebra of G is $\mathfrak{g} = \mathfrak{so}(3)$ which is the set off all skew-symmetric real 3×3 matrices. Recall that $\mathfrak{so}(3)$ is a Lie algebra with the Lie bracket $[A, B] = AB - BA$, for $A, B \in \mathfrak{so}(3)$ and E^3 is a Lie algebra with the Lie bracket the cross product $\times$. The Euclidean inner product and norm associated with the inner product are denoted by $<, >$ and $\|.\|$. $B : E^3 \to \mathfrak{so}(3)$ is a Lie algebra isomorphism given by

$$B(v)w = v \times w.$$

The unique dot product on E^3 satisfing (8) because dot product is up to a positive multiple. We may assume that B is an isometry without loss of generality.

Now we consider $\gamma : \mathbb{R} \to SO(3)$ is a generalized elastica and $\tilde{V} : \mathbb{R} \to \mathfrak{so}(3)$ is the associated generalized elastic Lie quadratic with the constant $\tilde{C}$. The inverse function is defined as follows:

$$V = B^{-1}(\tilde{V}):\mathbb{R} \to E^3 \tag{23}$$

and $C = B^{-1}(\tilde{C})$ for convenience. V satisfies for all $t \in I$

$$\|V(t)\|^2 = \left\|B^{-1}(\tilde{V})\right\|^2 = \left\|\tilde{V}\right\|^2 = 1 \tag{24}$$

because B is a Lie algebra isomorphism and isometry. By using Eq. (11)

$$\frac{d}{dt}\left(\frac{P'(\|\dot{V}(t)\|)}{\|\dot{V}(t)\|}\dot{V}(t)\right) + \frac{P'(\|\dot{V}(t)\|)}{\|\dot{V}(t)\|}V(t) \times \dot{V}(t) + (P(\|\dot{V}(t)\|)$$
$$-2 < C, V(t) >)V(t) + C = 0. \tag{25}$$

This implies V is a generalized elastic Lie quadratic with constant C in the Lie algebra $(E^3, \times)$. We study with V rather than $\tilde{V}$, solving (25) with (24). So, we can say that for any $A \in SO(3)$ and $t_0 \in \mathbb{R}$, $t \to A(V(t))$ is a generalized elastic Lie quadratic in E^3 with constant $A(C)$, and $t \to V(t - t_0)$ is a generalized elastic Lie quadratic in E^3 with constant C by local uniqueness in Picard theorem.

Now, we may suppose without loss of generality that

$$C = \begin{bmatrix} 0 & 0 & c \end{bmatrix}^T \text{ for some } c \in \mathbb{R}, \ V_1(0) = 0 \text{ for } V(t) = \begin{bmatrix} V_1(t) & V_2(t) & V_3(t) \end{bmatrix}^T. \tag{26}$$

If V is a generalized elastic Lie quadratic in E^3 with constant $C = 0$, then we call that V is a null generalized elastic Lie quadratic (see [10] and [14]). Then Eq. (25) reduces to

$$\frac{d}{dt} \left(\frac{P'(\|\dot{V}(t)\|)}{\|\dot{V}(t)\|} \dot{V}(t) \right) + \frac{P'(\|\dot{V}(t)\|)}{\|\dot{V}(t)\|} V(t) \times \dot{V}(t) + P(\|\dot{V}(t)\|) V(t) = 0. \tag{27}$$

Now, we suppose that $\|\dot{V}(t)\| = const.$ in the next part of the paper. Then we have from the first derivative of $\|\dot{V}(t)\|$

$$< \ddot{V}(t), \dot{V}(t) >= 0.$$

This implies that

$$\frac{d}{dt} \left(P\left(\|\dot{V}(t)\|\right) \right) = \frac{P'\left(\|\dot{V}(t)\|\right)}{\|\dot{V}(t)\|} < \ddot{V}(t), \dot{V}(t) >= 0. \tag{28}$$

From (18) and (28), (27) reduces to

$$\ddot{V}(t) = \dot{V}(t) \times V(t) - \|\dot{V}(t)\|^2 V(t).$$

Then we can give the following proposition;

Proposition 1. If V is a null generalized elastic Lie quadratic with $\|\dot{V}(t)\| = const.$ and satisfies (26), then we have

$$V(t) = \begin{bmatrix} a\sin(wt) & a\cos(wt) & \sqrt{1-a^2} \end{bmatrix}^T$$

for all $t \in \mathbb{R}$ and $w = 0$ or $w = 1/\sqrt{1-a^2}$, $a \in (-1, 1)$.

REFERENCES

[1] J. Arroyo, O. Garay, and J. Mencia, "Closed generalized elastic curves in $S^2(1)$." *Journal of Geometry and Physics*, vol. 48, no. 2-3, pp. 339–353, 2003, doi: 10.1016/S0393-0440(03)00047-0.

[2] J. Arroyo, O. Garay, and J. Mencia, "Closed hyperelastic curves in real space forms." *Proceeding of the XII Fall Workshop on Geometry and Physics, Coimbra*, pp. 1–13, 2003.

[3] J. Arroyo, O. Garay, and J. Mencia, "Extremals of curvature energy actions on spherical closed curves." *Journal of Geometry and Physics*, vol. 51, no. 1, pp. 101–125, 2004, doi: 10.1016/j.geomphys.2003.10.011.

[4] O. Garay, "Riemannian submanifolds shaped by the bending energy and its allies," *Proceeding of the Sixteenth International Workshop on Diff. Geom.*, vol. 16, pp. 55–68, 2012.

[5] R. Gilmore, *Lie groups, physics and geometry*. New York: Cambridge University Press., 2008. doi: 0521884004.

[6] R. Huang, "A note on p-elastica in a constant sectional curvature manifold." *Journal of Geometry and Physics*, vol. 49, no. 3, pp. 343–349, 2004, doi: 10.1016/S0393-0440(03)00107-4.

[7] R. Huang, C. Liao, and D. Shang, "Generalized elastica in anti-de Sitter space H_1^3,." *Chin. Quart. J. of Math.*, vol. 26, no. 2, pp. 311–316, 2011.

[8] J. Jurdevic, "Non-Euclidean elastica." *American Journal of Mathematics*, vol. 117, no. 1, pp. 93–124, 1995, doi: 10.2307/2375037.

[9] J. Langer and D. Singer, "The total squared curvature of closed curves," *Journal of Differential Geometry*, vol. 20, pp. 1–22, 1984, doi: 10.4310/jdg/1214438990.

[10] L. Noakes, "Null cubics and Lie quadratures," *Journal of Mathematical Physics*, vol. 44, no. 3, pp. 1436–1448, 2003, doi: 10.1063/1.1537461.

[11] L. Noakes, G. Heinzinger, and B. Paden, "Cubic splines on curved spaces," *IMA Journal of Mathematical Control and Information*, vol. 6, pp. 465–473, 1989, doi: 10.1093/imamci/6.4.465.

[12] G. Ozkan Tükel, T. Turhan, and A. Yücesan, "Hyperelastic Lie Quadratics," *Honam Mathematical Journal*, vol. 41, no. 2, pp. 369–380, 2019, doi: 10.5831/HMJ.2019.41.2.369.

[13] T. Popiel, *Geometrically-defined curves in Riemannian manifolds*. Australia: The University of Western Australia, School of Mathematics and Statistics, 2007.

[14] T. Popiel and L. Noakes, "Elastica in $SO(3)$,," *J. Aust. Math. Soc.*, vol. 83, no. 1, pp. 105–124, 2007, doi: 10.1017/S1446788700036417.

[15] D. Singer, "Lectures on elastic curves and rods," *AIP Conf. Proc., Amer. Inst. Phys., Melville, NY*, vol. 1002, 2008.

Authors' addresses

Gözde Özkan Tükel
Isparta University of Applied Sciences, 32200 Isparta, Turkey
E-mail address: gozdetukel@isparta.edu.tr

Tunahan Turhan
Isparta University of Applied Sciences, 32200 Isparta, Turkey
E-mail address: tunahanturhan@isparta.edu.tr

Ahmet Yücesan
Süleyman Demirel University, Department of Mathematics, 32200 Isparta, Turkey
E-mail address: ahmetyucesan@sdu.edu.tr