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ANALYSIS OF HIGHER ORDER DIFFERENCE METHOD FOR A PSEUDO-PARABOLIC EQUATION WITH DELAY

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Abstract. In this paper, the author considers the one dimensional initial-boundary problem for a pseudo-parabolic equation with time delay in second spatial derivative. To solve this problem numerically, the author constructs higher order difference method and obtain the error estimate for its solution. Based on the method of energy estimates the fully discrete scheme is shown to be convergent of order four in space and of order two in time. Some numerical examples illustrate the convergence and effectiveness of the numerical method.

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1. INTRODUCTION

In the domain $\overline{Q} = \overline{\Omega} \times [0,T]$; $\overline{\Omega} = [0,l]$, $Q = \Omega \times (0,T]$, $\Omega = (0,l)$, we consider the following pseudo-parabolic equation with delay (DPPEs)

$$\frac{\partial u(x,t)}{\partial t} - a(t)\frac{\partial^3 u(x,t)}{\partial t \partial x^2} = b(t)\frac{\partial^2 u(x,t)}{\partial x^2} + c(t)\frac{\partial^2 u(x,t-r)}{\partial x^2} + d(t)u(x,t) + f(x,t), (x,t) \in Q, \tag{1.1}$$

$$u(x,t) = \phi(x,t), (x,t) \in \overline{\Omega} \times [-r,0], \tag{1.2}$$

$$u(0,t) = u(l,t) = 0, t \in (0,T], \tag{1.3}$$

where $r > 0$ represents the delay parameter, $a \geqslant \alpha > 0$, b, c, d, f and ϕ are given sufficiently smooth functions satisfying certain regularity conditions to be specified.

Pseudo-parabolic or Sobolev-type differential equations appears in a variety of physical problems such as flow of fluid through fissured rocks, thermodynamics and propagation of long waves of small amplitude (see, e.g. [9, 24, 25]). The significant characteristic of these equations is that they state the conservation of a certain quantity (mass, momentum, heat, etc.) in any sub-domain. Such problems are interesting not only because they are generalizations of a standard parabolic problem, but also because they arise naturally in a large variety of applications. Various numerical schemes have been constructed to treat PPEs in [2, 3, 5, 6, 10, 12, 14, 15, 23] (see also the references cited in them). Not only the existence, uniqueness and nonexistence results for pseudo-parabolic equations were obtained, but also the asymptotic behavior, regularity and others properties of solutions were investigated. For example, in [6] the initial-boundary value problem for a linear PPEs with boundary layers is considered. They developed an exponentially fitted difference scheme and get its discrete energy estimation. The difference scheme is constructed by the method of integral identities with the use of exponential basis functions and interpolating quadrature rules with the weight and remainder terms in integral form. Both explicit and implicit in time discretization schemes have been developed in [10] which were based on the piecewise linear finite elements for the solution a pseudo parabolic Burgers equation. In [12] a Crank-Nicolson-Galerkin approximation with extrapolated coefficients is presented for three cases for the nonlinear PPEs along with a conjugate gradient iterative procedure which can be used efficiently to solve the different linear systems of algebraic equations arising at each step from the Galerkin method. In [23] authors presented two different schemes with respect to artificial diffusion parameter using extension of the finite difference streamline diffusion method for linear Sobolev equations with convection-dominated term. Further in [15] two difference approximation schemes to a nonlinear pseudo-parabolic equation are developed. Each of these schemes possesses a unique solution which can be obtained by an iterative procedure. The equivalence of the three different formulations for the PPEs and different time discrete (implicit or semi-implicit) numerical schemes has been discussed in [14]. The one-dimensional initial-boundary value problem for a linear PPEs with initial jump is studied in [5]. They developed a numerical method which combines a finite difference spatial discretization on uniform mesh and the implicit rule on Shishkin mesh (S-mesh) for the time variable. For a discussion of existence and uniqueness results of PPEs see [8, 13, 18]. The above mentioned papers, related with PPEs were only concerned with the cases without delay. Also delay pseudo-parabolic equations (DPPEs) frequently arise in many scientific applications. For works on existence and uniqueness results and for applications of DPPEs, see [11, 16]. In [4] for solving one dimensional initial-boundary delay PPE numerically, authors constructed high-order finite difference technique to the considered problem and obtain the error estimate for its solution. In [17] authors gave fourth order differential-difference scheme for solving one dimensional initial-boundary DPPE and obtain the error estimate for its solution. Further, the fourth order accurate Runge-Kutta method was used for the realization of acquired differential-difference problem. In [1] authors considered the explicit finite difference method for quasilinear DPPEs and proved that the fully discrete scheme is absolutely stable and convergent of order two in space and of order one in time variable. In [22] a super accurate numerical scheme to solve the one-dimensional Sobolev type partial differential equation with an initial and two nonlocal integral boundary conditions is considered. This methods are based on the shifted Standard and shifted Chebyshev Tau method. In [7] the abstract quasilinear evolution equations of Sobolev type in a Hilbert setting are considered. Authors proposed two fully discrete schemes and proved some error estimates under minimal assumptions. In [20, 21] the authors present a regularity result for solutions of parabolic equations in the framework of mixed Morrey spaces. [19] aims at defining new spaces and to study some embeddings between them. These spaces generalize Morrey spaces and give a refinement of Lebesgue spaces. Some embeddings between these new classes are also proved. The authors apply these classes of functions to obtain regularity results for solutions of partial differential equations of parabolic type in nondivergence form.

The present study is concerned with the one dimensional pseudo-parabolic equation containing time delay in second-order spatial derivative. Our aim is to construct higher order difference method for approximation to the considered problem when the coefficients are independent of spatial variable. Based on the method of energy estimates and difference analogue of the Gronwall's inequality with delay, the fully discrete scheme is shown to be convergent of order four in space and of order two in time. Numerical example on the performance of the method is presented.

2. The mesh and difference scheme

2.1. *Notation*

Let a set of nodes that discretises Q be given by $\omega = \omega_N \times \omega_{N_0}$ with

$$\omega_N = \{x_i = ih,\, i = 1,2,...,N-1,\, h = l/N\},$$

$$\omega_{N+} = \omega_N \cup \{x_N = l\},\; \bar{\omega}_N = \omega_N \cup \{x_0 = 0,\; x_N = l\},$$

$$\omega_{N_0} = \left\{t_j = j\tau,\;\; j = 1,2,...,N_0,\, \tau = T/N_0 = r/n_0\right\},$$

$$\bar{\omega}_{N_0} = \omega_{N_0} \cup \{t_0 = 0\},\; \bar{\omega} = \bar{\omega}_N \times \bar{\omega}_{N_0},$$

$$\omega_{n_0} = \left\{t_j = j\tau,\;\; j = 1,2,...,n_0,\;\; \tau = r/n_0\right\},$$

$$\omega_{n_0^-} = \left\{t_j = j\tau,\;\; j = -n_0,...,0,\;\; \tau = r/n_0\right\}$$

and define the following finite differences and notation

$$v_{\bar{x},i}^j = \frac{v_i^j - v_{i-1}^j}{h}, v_{\bar{x}x,i}^j = \frac{v_{i+1}^j - 2v_i^j + v_{i-1}^j}{h^2}, v_{\bar{t},i}^j = \frac{v_i^j - v_i^{j-1}}{\tau},$$

$$v_{\bar{t}t,i}^j = \frac{v_i^{j+1} - 2v_i^j + v_i^{j-1}}{\tau^2}, v_i^{(0.5)j} = \frac{v_i^j + v_i^{j-1}}{2}, v_i^{j-0.5} = v(x_i, t_j - \frac{\tau}{2})$$

for any mesh function $v_i^j = v(x_i, t_j)$ given on $\bar{\omega}$.

Introduce the following inner product and norm for the mesh functions v_i and w_i

$$(v,w) \equiv (v,w)_{\omega_N} = \sum_{i=1}^{N-1} h v_i w_i,\; (v,w)_{\omega_{N+}} = (v,w)_{\omega_N} \sum_{i=1}^{N} h v_i w_i,$$

$$\|v\|^2=(v,v)\,,\quad \|v_{\bar{x}}\|^2=(v_{\bar{x}},v_{\bar{x}})_{\omega_{N+}}\,,\ (v_0=v_N=0).$$

2.2. *Difference scheme*

To construct the difference scheme, we will use the following relation which is valid for any $g(x)\in C^6\left(\overline{\Omega}\right)$

$$\frac{1}{12}\left[g''(x_{i+1})+10g''(x_i)+g''(x_{i-1})\right]=g_{\bar{x}x,i}+\bar{R}_i, \tag{2.1}$$

where

$$\bar{R}_i=h^{-1}\int\limits_{x_{i-1}}^{x_{i+1}}\frac{\partial^6 g}{\partial x^6}(\xi_i)\,\Lambda(\xi)d\xi=\frac{h^4}{240}\frac{\partial^6 g}{\partial x^6}(\xi_i)\,,$$

$$\Lambda(\xi)=\begin{cases}\frac{h}{72}(x_{i+1}-\xi)^3-\frac{h^{-1}}{120}(x_{i+1}-\xi)^5\,,\xi>x_i\\ \frac{h}{72}(\xi-x_{i-1})^3-\frac{h^{-1}}{120}(\xi-x_{i-1})^5\,,\xi<x_i\end{cases}\,,\xi_i\in(x_{i-1},x_{i+1})\,.$$

Using formula (2.1) we get

$$\frac{1}{12}[\frac{\partial^3 u(x_{i+1},t)}{\partial t\partial x^2}+10\frac{\partial^3 u(x_i,t)}{\partial t\partial x^2}+\frac{\partial^3 u(x_{i-1},t)}{\partial t\partial x^2}]=u_{\bar{x}x,i}(t)+\frac{h^4}{240}\frac{\partial^7 u(\xi_i,t)}{\partial t\partial x^6},$$

$$\frac{1}{12}[\frac{\partial^2 u(x_{i+1},t)}{\partial x^2}+10\frac{\partial^2 u(x_i,t)}{\partial x^2}+\frac{\partial^2 u(x_{i-1},t)}{\partial x^2}]$$

$$=u_{\bar{x}x,i}(t)+\frac{h^4}{240}\frac{\partial^6 u(\xi_i,t)}{\partial x^6},$$

$$\frac{1}{12}[\frac{\partial^2 u(x_{i+1},t-r)}{\partial x^2}+10\frac{\partial^2 u(x_i,t-r)}{\partial x^2}+\frac{\partial^2 u(x_{i-1},t-r)}{\partial x^2}]$$

$$=u_{\bar{x}x,i}(t-r)+\frac{h^4}{240}\frac{\partial^6 u(\xi_i,t-r)}{\partial x^6}.$$

Note also that

$$\frac{1}{12}[\frac{\partial^3 u(x_{i+1},t)}{\partial t\partial x^2}+10\frac{\partial^3 u(x_i,t)}{\partial t\partial x^2}+\frac{\partial^3 u(x_{i-1},t)}{\partial t\partial x^2}]=u_i'(t)+\frac{h^2}{12}u'_{\bar{x}x,i}(t)$$

and

$$\frac{1}{12}[u(x_{i+1},t)+10u(x_i,t)+u(x_{i-1},t)]=u_i(t)+\frac{h^2}{12}u_{\bar{x}x,i}(t),$$

then we obtain the semi-discrete relation on $\omega_N\times[0,T]$ for equation (1.1)

$$u_i'(t)-(a(t)-\frac{h^2}{12})u'_{\bar{x}x,i}(t)=(b(t)+d(t)\frac{h^2}{12})u_{\bar{x}x,i}(t)+c(t)u_{\bar{x}x,i}(t-r)+d(t)u_i(t)$$
$$+\bar{f}_i(t)+R_i{}^{(0)}(t),\ i=1,2,...,N-1,\ t\in(0,T], \tag{2.2}$$

with

$$\bar{f}_i(t)=\frac{1}{12}[f_{i+1}(t)+10f_i(t)+f_{i-1}(t)],$$

$$R_i{}^{(0)}(t)=a(t)\frac{h^4}{240}\frac{\partial^7 u(\xi_i,t)}{\partial t\partial x^6}+b(t)\frac{h^4}{240}\frac{\partial^6 u(\xi_i,t)}{\partial x^6}+c(t)\frac{h^4}{240}\frac{\partial^6 u(\xi_i,t-r)}{\partial x^6},$$
$$\xi_i\in(x_{i-1},x_{i+1}).$$

Setting $t=t_{j-0.5}=t_j-\frac{\tau}{2}$ in (2.2) and taking into account there the relations

$$u_i'(t_{j-0.5})=u^j_{\bar{t},i}-\frac{\tau^2}{24}\frac{\partial^3 u_i(x_i,\eta_j^{(1)})}{\partial t^3},$$

$$u'_{\bar{x}x,i}(t_{j-0.5})=u^j_{\bar{t}\bar{x}x,i}-\frac{\tau^2}{24}\frac{\partial^5 u_i(x_i,\eta_j^{(2)})}{\partial t^3\partial x^2},$$

$$u_i(t_{j-0.5})=\frac{u_i^j+u_i^{j-1}}{2}-\frac{\tau^2}{8}\frac{\partial^2 u_i(x_i,\eta_j^{(3)})}{\partial t^2},$$

$$u_{\bar{x}x,i}(t_{j-0.5})=\frac{u^j_{\bar{x}x,i}+u^{j-1}_{\bar{x}x,i}}{2}-\frac{\tau^2}{8}\frac{\partial^4 u_i(x_i,\eta_j^{(3)})}{\partial t^2\partial x^2},$$

$$u_i(t_{j-0.5}-r)=\frac{u_i^{j-n_0}+u_i^{j-n_0-1}}{2}-\frac{\tau^2}{8}\frac{\partial^2 u_i(x_i,\bar{\eta}_j^{(1)})}{\partial t^2},$$

$$u_{\bar{x}x,i}(t_{j-0.5}-r)=\frac{u^{j-n_0}_{\bar{x}x,i}+u^{j-n_0-1}_{\bar{x}x,i}}{2}-\frac{\tau^2}{8}\frac{\partial^4 u_i(x_i,\bar{\eta}_j^{(2)})}{\partial t^2\partial x^2},$$
$$t_{j-1}<\eta_j^{(k)}<t_j,k=1,2,3,4;t_{j-n_0-1}<\bar{\eta}_j^{(k)}<t_{j-n_0},k=1,2,$$

we get

$$u^j_{\bar{t},i}-(a(t_{j-0.5})-\frac{h^2}{12})u^j_{\bar{t}\bar{x}x,i}=(b(t_{j-0.5})+d(t_{j-0.5})\frac{h^2}{12})u^{(0.5)j}_{\bar{x}x,i}$$
$$+c(t_{j-0.5})u^{(0.5)(j-n_0)}_{\bar{x}x,i}+d(t_{j-0.5})u_i^{(0.5)j}$$
$$+\bar{f}_i^j+R_i^j,i=1,2,...,N-1;j=1,2,...,N_0, \tag{2.3}$$

where

$$R_i^j=R_i^{(0)j}+R_i^{(1)j},$$

$$R_i^{(1)j}=-\frac{\tau^2}{24}(\frac{\partial^3 u_i(x_i,\eta_j^{(1)})}{\partial t^3}+\frac{\partial^5 u_i(x_i^{(1)},\eta_j^{(2)})}{\partial t^3\partial x^2})$$
$$-\frac{\tau^2}{8}(\frac{\partial^2 u_i(x_i,\eta_j^{(3)})}{\partial t^2}+\frac{\partial^4 u_i(x_i^{(2)},\eta_j^{(4)})}{\partial t^2\partial x^2})$$
$$+\frac{\tau^2}{8}(\frac{\partial^2 u_i(x_i,\bar{\eta}_j^{(1)})}{\partial t^2}+\frac{\partial^4 u_i(x_i,\bar{\eta}_j^{(2)})}{\partial t^2\partial x^2}).$$

Since

$$v^{(0.5)j}=v^{j-1}+\frac{\tau}{2}v_{\bar{t}}^j,$$

we then obtain

$$(1-d_{j-0.5}\frac{\tau}{2})u_{\bar{t},i}^j-(a_{j-0.5}-\frac{h^2}{12}+\frac{\tau}{2}(b_{j-0.5}+c_{j-0.5}\frac{h^2}{12}))u_{\bar{t}\bar{x}x,i}^j$$
$$=(b_{j-0.5}+d_{j-0.5}\frac{h^2}{12})u_{\bar{x}x,i}^{j-1}+c_{j-0.5}u_{\bar{x}x,i}^{(0.5)(j-n_0)}+d_{j-0.5}u_i^{j-1}$$
$$+\bar{f}_{\hat{i}}^j+R_i^j, i=1,2,...,N-1; j=1,2,...,N_0. \tag{2.4}$$
$$u_i^j=\phi_i^j, i=1,2,...,N-1; j=-n_0,-n_0+1,...,0, \tag{2.5}$$
$$u_0^j=u_N^j=0, j=1,2,...,N_0. \tag{2.6}$$

Neglecting the remainder term R_i^j in (2.4), we propose the following difference scheme for approximating (1.1)-(1.3):

$$E^j y_{\bar{t},i}^j-A^j y_{\bar{t}\bar{x}x,i}^j=B^j y_{\bar{x}x,i}^{j-1}+C^j y_{\bar{x}x,i}^{(0.5)(j-n_0)}$$
$$+\bar{f}_i^j, i=1,2,...,N-1, j=1,2,...,N_0, \tag{2.7}$$
$$y_i^j=\phi_i^j, i=1,2,...,N-1, j=-n_0,-n_0+1,...,0, \tag{2.8}$$
$$y_0^j=y_N^j=0, j=1,2,...,N_0, \tag{2.9}$$

where

$$E^j=(1-d_{j-0.5}\frac{\tau}{2}), A^j=a_{j-0.5}-\frac{h^2}{12}+\frac{\tau}{2}(b_{j-0.5}+c_{j-0.5}\frac{h^2}{12}),$$
$$B^j=b_{j-0.5}+d_{j-0.5}\frac{h^2}{12}, C^j=c_{j-0.5}.$$

For the error function $z=y-u$, from the relations (2.4)-(2.6) and (2.7)-(2.9), we have the following difference problem

$$E^j z_{\bar{t},i}^j-A^j z_{\bar{t}\bar{x}x,i}^j=B^j z_{\bar{x}x,i}^{j-1}+C^j z_{\bar{x}x,i}^{(0.5)(j-n_0)}$$
$$+D^j z_i^{j-1}+R_i^j, i=1,2,...,N-1, j=1,2,...,N_0, \tag{2.10}$$

$$z_i^j = 0, i = 1,2,...,N-1, j = -n_0, -n_0+1,...,0, \tag{2.11}$$

$$z_0^j = z_N^j = 0, j = 1,2,...,N_0. \tag{2.12}$$

The following lemma is used to discuss the stability and convergence properties of our discrete problem (2.7)-(2.9).

Lemma 1 ([4]). *Let the mesh function $\delta \geqslant 0$, defined on ω_{N_0}, satisfies*

$$\delta_j \leqslant \alpha + \tau \sum_{k=1}^{j} \{a\delta_k + b\delta_{k-1} + c\delta_{k-N} + d\delta_{k-N-1} + f_k\}, j \geqslant 1 \tag{2.13}$$

$$\delta_j \leqslant \phi_j, -N \leqslant j \leqslant 0, \delta_0 \leqslant \alpha,$$

where $\alpha, a, b, c, d, f_j \geqslant 0$, ϕ_j given, $N \geqslant 0$ integer, $1-\tau a > 0$.

Then

$$\delta_j \leqslant \tilde{\alpha} e^{\gamma t_j} + \frac{\tau}{1-\tau a} \sum_{k=1}^{j} f_k e^{\gamma t_{j-k}}, \tag{2.14}$$

where

$$\tilde{\alpha} = \alpha + (c+d)\|\phi\|_1, \gamma = \frac{a+b+c+d}{1-\tau a}, \|\phi\|_1 = \sum_{j=-N}^{0} \tau \mid \phi_j \mid.$$

3. Error analysis and convergence

Now we give the main result of this paper.

Theorem 1. *Let the derivatives $\frac{\partial^7 u}{\partial t \partial x^6}, \frac{\partial^6 u}{\partial x^6}, \frac{\partial^2 u}{\partial t^2}, \frac{\partial^4 u}{\partial t^2 \partial x^2}$ are bounded on the $\overline{Q}$ and $E^j \geqslant \beta_* > 0$, $A^j \geqslant \alpha_* > 0$.*

Then the error of the problem (2.7)-(2.9) satisfies

$$\|y-u\| + \|y_{\bar{x}} - u_{\bar{x}}\| \leqslant C(h^4+\tau^2), \tag{3.1}$$

where C is a constant which is independent of h and τ.

Proof. Consider the following identity

$$(E^j z_{\bar{t}}^j, z_{\bar{t}}^j) - (A^j z_{\bar{t}\bar{x}x}^j, z_{\bar{t}}^j) = (B^j z_{\bar{x}x}^{j-1}, z_{\bar{t}}^j)$$
$$+ (C^j z_{\bar{x}x}^{(0.5)(j-n_0)}, z_{\bar{t}}^j) + (D^j z^{j-1}, z_{\bar{t}}^j) + (R^j, z_{\bar{t}}^j).$$

After some manipulations, we get

$$\beta_* \left\| z_{\bar{t}}^j \right\|^2 + \alpha_* \left\| z_{\bar{t}\bar{x}}^j \right\|^2 \leq B^* \left\| z_{\bar{t}\bar{x}}^j \right\| \left\| z_{\bar{x}}^{j-1} \right\| + C^* \left\| z_{\bar{t}\bar{x}}^j \right\| \left\| z_{\bar{x}}^{(0.5)(j-n_0)} \right\|$$
$$+ D_* \left\| z_{\bar{t}}^j \right\| \left\| z^{j-1} \right\| + \left\| z_{\bar{t}}^j \right\| \left\| R^j \right\|,$$

where $\mid B^j \mid \leq B^*$, $\mid C^j \mid \leq C^*$ and $\mid D^j \mid \leq D^*$.

From here, using the inequality $| ab |\leq \mu a^2+(1/4\mu)b^2$ $(\mu>0)$ we have

$$(\beta_*-\mu_1-\mu_2)\left\|z_{\bar{t}}^{j}\right\|^2+(\alpha_*-\mu_3-\mu_4)\left\|z_{\bar{t}\bar{x}}^{j}\right\|^2\leq\frac{1}{4\mu_3}(B^*)^2\left\|z_{\bar{x}}^{j-1}\right\|^2$$

$$+\frac{1}{4\mu_4}(C^*)^2\left\|z_{\bar{x}}^{(0.5)(j-n_0)}\right\|^2+\frac{1}{4\mu_1}(D^*)^2\left\|z^{j-1}\right\|^2+\frac{1}{4\mu_2}\left\|R^j\right\|^2.$$

After choosing, $\mu_1=\mu_2=\frac{\beta_*}{4}$ and $\mu_3=\mu_4=\frac{\alpha_*}{2}$ this inequality reduces to

$$\beta_*\left\|z_{\bar{t}}^{j}\right\|^2+\alpha_*\left\|z_{\bar{t}\bar{x}}^{j}\right\|^2\leq\frac{(B^*)^2}{\alpha_*}\left\|z_{\bar{x}}^{j-1}\right\|^2+\frac{(C^*)^2}{\alpha_*}\left\|z_{\bar{x}}^{(0.5)(j-n_0)}\right\|^2$$

$$+\frac{(D^*)^2}{\beta_*}\left\|z^{j-1}\right\|^2+\frac{1}{\beta_*}\left\|R^j\right\|^2.$$

Multiplying this inequality by $T\tau$ and summing it up from $k=1$ to $k=j$, using the inequality

$$v_j^2\leqslant t_j\tau\sum_{k=1}^{j}v_{\bar{t},k}^2\leqslant T\tau\sum_{k=1}^{j}v_{\bar{t},k}^2,(v_0=0)$$

we obtain

$$\beta_*\left\|z^j\right\|^2+\alpha_*\left\|z_{\bar{x}}^{j}\right\|^2\leqslant T\tau\sum_{k=1}^{j}\left\{(B^*)^2\alpha_*^{-1}\left\|z_{\bar{x}}^{j-1}\right\|^2+(C^*)^2\alpha_*^{-1}\left\|z_{\bar{x}}^{(0.5)(j-n_0)}\right\|^2\right.$$

$$\left.+(D^*)^2\beta_*^{-1}\left\|z^{j-1}\right\|^2+\beta_*^{-1}\left\|R^j\right\|^2\right\}.$$

Denoting now

$$\delta_j=\beta_*\left\|z^j\right\|^2+\alpha_*\left\|z_{\bar{x}}^{j}\right\|^2,$$

we have

$$\delta_j\leqslant\sum_{k=1}^{j}\tau\left\{c_1\delta_{k-1}+c_2\delta_{k-n_0}+c_2\delta_{k-n_0-1}+\rho_k\right\},j\geqslant 1,$$

with

$$c_1=T\max\left\{(B^*)^2\alpha_*^{-2},(D^*)^2\beta_*^{-2}\right\},$$

$$c_2=T(C^*)^2\alpha_*^{-2}.$$

$$\rho_k=T\beta_*^{-1}\left\|R^k\right\|^2.$$

Applying now Lemma 1 we obtain

$$\beta_* \left\| z^j \right\|^2 + \alpha_* \left\| z_{\bar{x}}^j \right\|^2 \leqslant T\beta_*^{-1}\tau \sum_{k=1}^{j} e^{(c_1+c_2)t_{j-k}} \left\| R^k \right\|^2. \tag{3.2}$$

It is not difficult to see that, under the assumed smoothness

$$(\tau \sum_{k=1}^{N_0} \left\| R^k \right\|^2)^{1/2} = O(h^4 + \tau^2),$$

which together with (3.2), completes the proof of the theorem. □

4. Numerical results

In this section we present some numerical results for the scheme discussed in this paper. Consider the problem

$$\frac{\partial u(x,t)}{\partial t} - \frac{\partial^3 u(x,t)}{\partial t \partial x^2} = \frac{\partial^2 u(x,t)}{\partial x^2} + 2\frac{\partial^2 u(x,t-1)}{\partial x^2} - u(x,t) = 50e^{1-t} sinh(x).$$

$$u(x,t) = e^{-t}(x\sinh(1) - \sinh(x)), (x,t) \in [0,1]\times[-1,0],$$

$$u(0,t) = u(1,t) = 0, t \in (0,2].$$

The exact solution is given by

$$u(x,t) = 25e^{-t}(x\sinh(1) - \sinh(x)).$$

The computational results are presented in Table 1 and Table 2.

TABLE 1. The numerical results on $(0,1)\times(0,1)$

Nodes (x,t)	Exact Solution	Numerical Solution $h = 0.1, \tau = 0.05$	Pointwise Error $\lvert y-u \rvert$
(0.1,0.1)	0.392565	0.392566	0.47440E-05
(0.2,0.2)	0.689985	0.690003	0.74700E-04
(0.3,0.3)	0.889722	0.889745	0.94742E-04
(0.4,0.4)	0.994252	0.994276	0.99745E-04
(0.5,0.5)	1.008508	1.008538	0.12070E-03
(0.6,0.6)	0.939428	0.939452	0.96937E-04
(0.7,0.7)	0.795281	0.795303	0.90725E-04
(0.8,0.8)	0.584801	0.584818	0.69740E-04
(0.9,0.9)	0.316850	0.316851	0.46582E-05

We see from the above tables that these results display a well agreement with our theoretical analysis.

TABLE 2. The numerical results on $(0,1)\times(1,2)$

Nodes (x,t)	Exact Solution	Numerical Solution $h=0.1, \tau=0.05$	Pointwise Error $\lvert y-u\rvert$
(0.1,1.1)	0.144415	0.144418	0.14730E-04
(0.2,1.2)	0.253786	0.253794	0.32855E-04
(0.3,1.3)	0.327310	0.327327	0.70685E-04
(0.4,1.4)	0.365754	0.365770	0.64967E-04
(0.5,1.5)	0.371012	0.371033	0.86997E-04
(0.6,1.6)	0.345599	0.345623	0.98410E-04
(0.7,1.7)	0.292567	0.292588	0.85730E-04
(0.8,1.8)	0.215136	0.215154	0.73182E-04
(0.9,1.9)	0.116528	0.116535	0.28105E-04

5. CONCLUSIONS

In this paper, the higher order difference method is applied to the problem (1.1)-(1.3). Based on the method of energy estimates, the fully discrete scheme was shown to be convergent of order four in space and of order two in time. To demonstrate the accuracy and usefulness of this method, numerical example has been presented.

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