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ENDPOINT BOUNDEDNESS FOR MULTILINEAR COMMUTATORS OF LITTLEWOOD-PALEY OPERATORS

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Abstract. In this paper, we prove endpoint boundedness of multilinear commutators of Littlewood-Paley operators.

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1. INTRODUCTION AND NOTATIONS

Let $b \in BMO(R^n)$ and let T be the Calderón-Zygmund operator. The commutator $[b,T]$ of b and T is defined by

$$[b,T](f)(x) = b(x)T(f)(x) - T(bf)(x).$$

A classical result of Coifman, Rochberb and Weiss (see [3]) states that the commutator $[b,T]$ is bounded on $L^p(R^n)$, $(1 < p < \infty)$. In [2] and [5], boundedness properties of the commutators for the extreme values of p are obtained. In this paper, we will introduce the multilinear commutators of Littlewood-Paley operators and prove boundedness properties of the operators in the extreme cases.

First let us introduce some notations (see [1],[4],[8],[9],[10]). Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For a cube Q and for a locally integrable function f, let $f_Q = |Q|^{-1}\int_Q f(x)dx$ and $f^{\#}(x) = \sup_{x\in Q} |Q|^{-1}\int_Q |f(y) - f_Q|dy$. Moreover, f is said to belong to $BMO(R^n)$ if $f^{\#} \in L^\infty$ and define $||f||_{BMO} = ||f^{\#}||_{L^\infty}$. We also define the central BMO space by $CMO(R^n)$, which is the space of those functions $f \in L_{loc}(R^n)$ such that

$$||f||_{CMO} = \sup_{r>1} |Q(0,r)|^{-1} \int_Q |f(y) - f_Q|dy < \infty.$$

It has been known that(see [9])

$$||f||_{CMO} = \sup_{r>1} |Q(0,r)|^{-1} \int_Q |f(y) - f_Q|dy < \infty.$$

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Also, we give the concepts of the atom and H^1 space. A function a is called as a H^1 atom if there exists a cube Q such that a is supported on Q, $||a||_{L^\infty} \le |Q|^{-1}$ and $\int a(x)dx = 0$. It is well known that the Hardy space $H^1(R^n)$ can be characterized in terms of the atomic decomposition (see [4], [9]).

Definition 1. Let $0 < \delta < n$ and $1 < p < n/\delta$. We shall call $B_p^\delta(R^n)$ the space of those functions f on R^n such that

$$||f||_{B_p^\delta} = \sup_{r>1} r^{-n(1/p-\delta/n)} ||f\chi_{Q(0,r)}||_{L^p} < \infty.$$

Definition 2. Let $\varepsilon > 0$, $n > \delta > 0$ and let ψ be a fixed function that satisfies the following properties:

1) $\int \psi(x)dx = 0$,

2) $|\psi(x)| \le C(1+|x|)^{-(n+1-\delta)}$,

3) $|\psi(x+y) - \psi(x)| \le C|y|^\varepsilon(1+|x|)^{-(n+1+\varepsilon-\delta)}$ when $2|y| < |x|$;

We denote $\Gamma(x) = \{(y,t) \in R_+^{n+1} : |x-y| < t\}$ and the characteristic function of $\Gamma(x)$ is written as $\chi_{\Gamma(x)}$. Let $\vec{b} = (b_1, \cdots, b_m)$ with $b_j \in BMO(R^n)$ for $1 \le j \le m$. Then the multilinear commutator of our Littlewood-Paley operator is defined by

$$S_\delta^{\vec{b}}(f)(x) = \left[\int\int_{\Gamma(x)} |F_t^{\vec{b}}(f)(x,y)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2},$$

where

$$F_t^{\vec{b}}(f)(x,y) = \int_{R^n} \prod_{j=1}^m (b_j(x) - b_j(z))\psi_t(y-z)f(z)dz;$$

When $m = 1$, set

$$S_\delta^b(f)(x) = \left[\int\int_{\Gamma(x)} |F_t^b(f)(x,y)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2},$$

where

$$F_t^b(f)(x,y) = \int_{R^n} (b(x) - b(z))\psi_t(y-z)f(z)dz$$

and $\psi_t(x) = t^{-n+\delta}\psi(x/t)$ for $t > 0$. Set $F_t(f)(x) = f * \psi_t(x)$, we also define

$$S_\delta(f)(x) = \left[\int\int_{\Gamma(x)} |f * \psi_t(x)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2},$$

which is the Littlewood-Paley operator (see [1][6,7][10]).

Let H be the Hilbert space $H = \{h : ||h|| = (\int\int_{R_+^{n+1}} |h(y,t)|^2 dydt/t^{n+1})^{1/2} < \infty\}$. Then for each fixed $x \in R^n$, $F_t(f)(x,y)$ may be viewed as a mapping from $[0, +\infty)$ to H. It is clear that

$$S_\delta(f)(x) = ||\chi_{\Gamma(x)} F_t(f)(y)|| \quad \text{and} \quad S_\delta^{\vec{b}}(f)(x) = ||\chi_{\Gamma(x)} F_t^{\vec{b}}(f)(y)||.$$

Given a positive integer m and $1 \le j \le m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$ of j different elements. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \cdots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \cdots, b_m)$ and $\sigma = \{\sigma(1), \cdots, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and set $||\vec{b}_\sigma||_{BMO} = ||b_{\sigma(1)}||_{BMO} \cdots ||b_{\sigma(j)}||_{BMO}$.

2. THEOREMS AND PROOFS

We begin with a few preliminary lemmas.

Lemma 1. *Let* $1 < r < \infty$, $b_j \in BMO(R^n)$ *for* $j = 1, \cdots, k$ *and* $k \in N$. *Then we have*

$$\frac{1}{|Q|} \int_Q \prod_{j=1}^k |b_j(y) - (b_j)_Q| dy \le C \prod_{j=1}^k ||b_j||_{BMO}$$

and

$$\left(\frac{1}{|Q|} \int_Q \prod_{j=1}^k |b_j(y) - (b_j)_Q|^r dy \right)^{1/r} \le C \prod_{j=1}^k ||b_j||_{BMO}.$$

Lemma 2. *Let* $0 < \delta < n$, $1 < p < n/\delta$ *and* $1/q = 1/p - \delta/n$. *Then* S_δ *is bounded from* $L^p(R^n)$ *to* $L^q(R^n)$.

Theorem 1. *Let* $0 < \delta < n$ *and* $\vec{b} = (b_1, \cdots, b_m)$ *with* $b_j \in BMO(R^n)$ *for* $1 \le j \le m$. *Then* $S_\delta^{\vec{b}}$ *is bounded from* $L^{n/\delta}(R^n)$ *to* $BMO(R^n)$.

Proof of Theorem 1. It is necessary only to prove that there exist a constant C_Q such that

$$\frac{1}{|Q|} \int_Q |S_\delta^{\vec{b}}(f)(x) - C_Q| dx \le C ||f||_{L^{n/\delta}}.$$

Fix a cube $Q = Q(x_0, r)$, we decompose f into $f = f_1 + f_2$ with $f_1 = f\chi_Q$, $f_2 = f\chi_{(R^n \setminus Q)}$.

When $m = 1$, set $(b_1)_Q = |Q|^{-1} \int_Q b_1(y) dy$, we have

$$F_t^{b_1}(f)(x,y)$$
$$= (b_1(x) - (b_1)_Q) F_t(f)(y) - F_t((b_1 - (b_1)_Q) f_1)(y) - F_t((b_1 - (b_1)_Q) f_2)(y),$$

so

$$|S_\delta^{b_1}(f)(x) - S_\delta(((b_1)_Q - b_1) f_2)(x_0)|$$
$$= \Big| ||\chi_{\Gamma(x)} F_t^{b_1}(f)(x,y)|| - ||\chi_{\Gamma(x_0)} F_t(((b_1)_Q - b_1) f_2)(y)|| \Big|$$
$$\le ||\chi_{\Gamma(x)} F_t^{b_1}(f)(x,y) - \chi_{\Gamma(x_0)} F_t(((b_1)_Q - b_1) f_2)(y)||$$
$$\le ||\chi_{\Gamma(x)} (b_1(x) - (b_1)_Q) F_t(f)(y)|| + ||\chi_{\Gamma(x)} F_t(((b_1)_Q - b_1) f_1)(y)||$$

$$+||\chi_{\Gamma(x)}F_t((b_1-(b_1)_Q)f_2)(y)-\chi_{\Gamma(x_0)}F_t((b_1-(b_1)_Q)f_2)(y)||$$
$$=A(x)+B(x)+C(x).$$

For $A(x)$, set $1<p<n/\delta$, $1/q=1/p-\delta/n$ and $1/q+1/q'=1$. By Hölder's inequality and Lemma 1,2, we have

$$\begin{aligned}\frac{1}{Q}\int_Q|A(x)|dx &\le \left(\frac{1}{|Q|}\int_Q|b_1(x)-(b_1)_Q|^{q'}dx\right)^{1/q'}\\ &\quad\left(\frac{1}{|Q|}\int_{R^n}|S_\delta(f)(x)|^q\chi_Q(x)dx\right)^{1/q}\\ &\le C||b_1||_{BMO}\frac{1}{|Q|^q}\left(\int_{R^n}|f(x)|^p\chi_Q(x)dx\right)^{1/p}\\ &\le C||b_1||_{BMO}\frac{1}{|Q|^q}||f||_{L^{n/\delta}}|Q|^{(1-\delta p/n)p}\\ &\le C||b_1||_{BMO}||f||_{L^{n/\delta}}.\end{aligned}$$

For $B(x)$, take $1<r<n/\delta$ and $1/s=1/r-\delta/n$, then by Hölder's inequality we have

$$\begin{aligned}\frac{1}{|Q|}\int_Q|B(x)|dx &\le \left(\frac{1}{|Q|}\int_{R^n}(S_\delta((b_1(x)-(b_1)_Q)f_1)(x))^s dx\right)^{1/s}\\ &\le C|Q|^{-1/s}||(b_1(x)-(b_1)_Q)f\chi_Q||_{L^r}\\ &\le C\left(\frac{1}{|Q|}\int_Q|b_1(x)-(b_1)_Q|^s dx\right)^{1/s}||f||_{L^{n/\delta}}\\ &\le C||b_1||_{BMO}||f||_{L^{n/\delta}}.\end{aligned}$$

For $C(x)$, we have

$$\begin{aligned}&C(x)\\ &\le\left[\int\int_{R^{n+1}_+}\left(\int_{Q^c}|\chi_{\Gamma(x)}-\chi_{\Gamma(x_0)}||b_1(z)-(b_1)_Q||\psi_t(y-z)||f(z)|dz\right)^2\frac{dydt}{t^{n+1}}\right]^{1/2}\\ &\le C\int_{Q^c}|b_1(z)-(b_1)_Q||f(z)||\int\int_{|x-y|\le t}\frac{t^{1-n}dydt}{(t+|y-z|)^{2n+2-2\delta}}\\ &\quad-\int\int_{|x_0-y|\le t}\frac{t^{1-n}dydt}{(t+|y-z|)^{2n+2-2\delta}}|^{1/2}dz\\ &\le C\int_{Q^c}|b_1(z)-(b_1)_Q||f(z)|\end{aligned}$$

$$\times\left(\int\int_{|y|\le t,|x+y-z|\le t}\left|\frac{1}{(t+|x+y-z|)^{2n+2-2\delta}}\right.\right.$$

$$\left.\left.-\frac{1}{(t+|x_0+y-z|)^{2n+2-2\delta}}\right|\frac{dydt}{t^{n-1}}\right)^{1/2}dz$$

$$\le\int_{Q^c}|b_1(z)-(b_1)_Q||f(z)|$$

$$\left(\int\int_{|y|\le t,|x+y-z|\le t}\frac{|x-x_0|t^{1-n}}{(t+|x+y-z|)^{2n+3-2\delta}}dydt\right)^{1/2}dz,$$

note that $2t+|x+y-z|\ge 2t+|x-z|-|y|\ge t+|x-z|$ when $|y|\le t$ and

$$\int_0^\infty\frac{t\,dt}{(t+|x-z|)^{2n+3-2\delta}}=C|x-z|^{-2n-1+2\delta}.$$

Then, for $x\in Q$,

$$C(x)$$

$$\le C\int_{Q^c}|b_1(z)-(b_1)_Q||f(z)||x-x_0|^{1/2}$$

$$\left(\int\int_{|y|\le t}\frac{t^{1-n}dydt}{(t+|x-z|)^{2n+3-2\delta}}\right)^{1/2}dz$$

$$\le C\int_{Q^c}|b_1(z)-(b_1)_Q||f(z)||x-x_0|^{1/2}$$

$$\left(\int_0^\infty\frac{t\,dt}{(t+|x-z|)^{2n+3-2\delta}}\right)^{1/2}dz$$

$$\le C\int_{Q^c}|b_1(z)-(b_1)_Q||f(z)|\frac{|x_0-x|^{1/2}}{|x_0-z|^{n+1/2-\delta}}dz$$

$$\le C\sum_{k=0}^\infty\int_{2^{k+1}Q\setminus 2^kQ}|b_1(z)-(b_1)_Q||f(z)|\frac{|x_0-x|^{1/2}}{|x_0-z|^{n+1/2-\delta}}dz$$

$$\le C\sum_{k=1}^\infty 2^{-k/2}\left(\frac{1}{|2^kQ|}\int_{2^kQ}|b_1(z)-(b_1)_Q|^{n/(n-\delta)}dz\right)^{(n-\delta)/n}$$

$$\left(\int_{2^kQ}|f(z)|^{n/\delta}dz\right)^{\delta/n}$$

$$\le C\sum_{k=1}^\infty k2^{-k/2}||b_1||_{BMO}||f||_{L^{n/\delta}}$$

$$\le C||b_1||_{BMO}||f||_{L^{n/\delta}}.$$

When $m > 1$, set $\vec{b}_Q = ((b_1)_Q, \cdots, (b_m)_Q) \in R^n$, where $(b_j)_Q = |Q|^{-1}\int_Q b_j(y)dy,\ 1 \le j \le m$. Then we have

$$\begin{aligned} F_t^{\vec{b}}(f)(x,y) &= (b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)F_t(f)(y) \\ &+ (-1)^m F_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f)(y) \\ &+ \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(\vec{b}(x)-\vec{b}_Q)_\sigma \int_{R^n}(\vec{b}(z)-\vec{b}_Q)_{\sigma^c}\psi_t(y-z)f(z)dz \\ &= (b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)F_t(f)(y) \\ &+ (-1)^m F_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_1)(y) \\ &+ (-1)^m F_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_2)(y) \\ &+ \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(\vec{b}(x)-\vec{b}_Q)_\sigma F_t((\vec{b}-\vec{b}_Q)_{\sigma^c}f)(x,y), \end{aligned}$$

thus

$$\begin{aligned} &|S_\delta^{\vec{b}}(f)(x) - S_\delta(((b_1)_Q-b_1)\cdots((b_m)_Q-b_m)f_2)(x_0)| \\ &\le ||\chi_{\Gamma(x)}F_t^{\vec{b}}(f)(x,y) - \chi_{\Gamma(x_0)}F_t(((b_1)_Q-b_1)\cdots((b_m)_Q-b_m)f_2)(y)|| \\ &\le ||\chi_{\Gamma(x)}(b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)F_t(f)(y)|| \\ &+ \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\chi_{\Gamma(x)}(\vec{b}(x)-\vec{b}_Q)_\sigma F_t((\vec{b}-\vec{b}_Q)_{\sigma^c}f)(x,y)|| \\ &+ ||\chi_{\Gamma(x)}F_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_1)(y)|| \\ &+ ||\chi_{\Gamma(x)}F_t(\prod_{j=1}^m(b_j-(b_j)_Q)f_2)(y) - \chi_{\Gamma(x_0)}F_t(\prod_{j=1}^m(b_j-(b_j)_Q)f_2)(y)|| \\ &= I_1(x)+I_2(x)+I_3(x)+I_4(x). \end{aligned}$$

For $I_1(x)$, by taking $1 < p < n/\delta$ and $1/q = 1/p - \delta/n$, and by Hölder's inequality and Lemma 1,2, we have

$$\begin{aligned} &\frac{1}{|Q|}\int_Q I_1(x)dx \\ &\le \left(\frac{1}{|Q|}\int_Q|\prod_{j=1}^m(b_j(x)-(b_j)_Q)|^{q'}dx\right)^{1/q'}\left(\frac{1}{|Q|}\int_Q|S_\delta(f)(x)|^q dx\right)^{1/q} \\ &\le C||\vec{b}||_{BMO}|Q|^{-1/q}\left(\int_Q|f(x)|^p dx\right)^{1/p} \le C||\vec{b}||_{BMO}||f||_{L^{n/\delta}}. \end{aligned}$$

For $I_2(x)$, if we take $1 < p < n/\delta$ and $1/q = 1/p - \delta/n$, then

$$
\begin{aligned}
&\frac{1}{|Q|}\int_Q I_2(x)dx \\
&\le \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|}\int_Q |(b(\vec{x})-\vec{b}_Q)_\sigma||S_\delta((\vec{b}-\vec{b}_Q)_{\sigma^c}f)(x)|dx \\
&\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\left(\frac{1}{|Q|}\int_Q |(\vec{b}(x)-\vec{b}_Q)_\sigma|^{q'}dx\right)^{1/q'} \\
&\qquad\qquad \left(\frac{1}{|Q|}\int_Q |S_\delta((\vec{b}-\vec{b}_Q)_{\sigma^c})f)(x)|^q dx\right)^{1/q} \\
&\le C\sum_{j=1}^{m-1}||\vec{b}_\sigma||_{BMO}|Q|^{1/q}\left(\int_{R^n}|(b(\vec{x})-\vec{b}_Q)_{\sigma^c})f(x)|^p\chi_Q dx\right)^{1/q} \\
&\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\vec{b}_\sigma||_{BMO}\left(\frac{1}{|Q|}\int_Q |(\vec{b}(x)-\vec{b}_Q)_\sigma|^q dx\right)^{1/q}||f||_{L^{n/\delta}} \\
&\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\vec{b}_\sigma||_{BMO}||\vec{b}_{\sigma^c}||_{BMO}||f||_{L^{n/\delta}} \\
&\le C||\vec{b}||_{BMO}||f||_{L^{n/\delta}}.
\end{aligned}
$$

For $I_3(x)$, take $1 < p < n/\delta$ and $1/q = 1/p - \delta/n$, so we get

$$
\begin{aligned}
&\frac{1}{|Q|}\int_Q I_3(x)dx \\
&\le \left(\frac{1}{|Q|}\int_Q |S_\delta((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_1)(x)|^q dx\right)^{1/q} \\
&\le C|Q|^{-1/q}||((b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)f_1(x)||_{L^p} \\
&\le C\left(\frac{1}{|Q|}\int_Q |(b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)|^q dx\right)^{1/q}||f||_{L^{n/\delta}} \\
&\le C||\vec{b}||_{BMO}||f||_{L^{n/\delta}}.
\end{aligned}
$$

For $I_4(x)$, similary as in the proof of $C(x)$ in Case $m=1$, we have

$$
\begin{aligned}
I_4(x) &\le C\int_{Q^c}|x_0-x|^{1/2}|x_0-z|^{-(n+1/2-\delta)}\left|\prod_{j=1}^m(b_j(z)-(b_j)_Q)\right||f(z)|dz\\
&\le C\sum_{k=0}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}|x_0-x|^{1/2}|x_0-z|^{-(n+1/2-\delta)}\left|\prod_{j=1}^m(b_j(z)-(b_j)_Q)\right||f(z)|dz\\
&\le C\sum_{k=0}^{\infty}2^{-k/2}\frac{1}{|2^{k+1}Q|^{1-\delta/n}}\int_{2^{k+1}Q}\left|\prod_{j=1}^m(b_j(z)-(b_j)_Q)\right||f(z)|dz\\
&\le C\sum_{k=1}^{\infty}2^{-k/2}\left(\frac{1}{|2^kQ|}\int_{2^kQ}\left|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right|^{n/(n-\delta)}dy\right)^{(n-\delta)/n}||f||_{L^{n/\delta}}\\
&\le C||\vec{b}||_{BMO}||f||_{L^{n/\delta}}.
\end{aligned}
$$

This completes the proof of Theorem 1. □

Theorem 2. *Let* $0<\delta<n$, $1<p<n/\delta$ *and* $\vec{b}=(b_1,\cdots,b_m)$ *with* $b_j\in BMO(R^n)$ *for* $1\le j\le m$. *Then* $S_\delta^{\vec{b}}$ *is bounded from* $B_p^\delta(R^n)$ *to* $CMO(R^n)$.

Proof of Theorem 2. It suffices to prove that there exist a constant C_Q, such that

$$\frac{1}{|Q|}\int_Q|S_\delta^{\vec{b}}(f)(x)-C_Q|dx\le C||f||_{B_p^\delta}$$

holds for any cube $Q=Q(0,r)$ with $r>1$. Fix a cube $Q=Q(0,r)$ with $r>1$. Set $f_1=f\chi_Q$, $f_2=f\chi_{R^n\setminus Q}$ and $\vec{b}_Q=((b_1)_Q,\cdots,(b_m)_Q)\in R^n$, where $(b_j)_Q=|Q|^{-1}\int_Q|b_j(y)|dy$, $1\le j\le m$. Then we have

$$
\begin{aligned}
&|S_\delta^{\vec{b}}(f)(x)-S_\delta(((b_1)_Q-b_1)\cdots((b_m)_Q-b_m)f_2)(x_0)|\\
&\le||\chi_{\Gamma(x)}(b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)F_t(f)(y)||\\
&+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\chi_{\Gamma(x)}(\vec{b}(x)-\vec{b}_Q)_\sigma F_t((\vec{b}-\vec{b}_Q)_{\sigma^c}f)(x,y)||\\
&+||\chi_{\Gamma(x)}F_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_1)(y)||\\
&+||\chi_{\Gamma(x)}F_t(\prod_{j=1}^m(b_j-(b_j)_Q)f_2)(y)-\chi_{\Gamma(x_0)}F_t(\prod_{j=1}^m(b_j-(b_j)_Q)f_2)(y)||\\
&=H_1(x)+H_2(x)+H_3(x)+H_4(x).
\end{aligned}
$$

For $H_1(x)$, we take $1 < p < n/\delta$, $1/s = 1/r - \delta/n$. By Hölder's inequality and Lemma 1,2, we have

$$\frac{1}{|Q|}\int_Q H_1(x)dx$$
$$\leq \left(\frac{1}{|Q|}\int_Q |\prod_{j=1}^m (b_j(x)-(b_j)_Q)|^{q'}dx\right)^{1/q'} \left(\frac{1}{|Q|}\int_Q |S_\delta(f)(x)|^q dx\right)^{1/q}$$
$$\leq C||\vec{b}||_{BMO}|Q|^{-1/q}\left(\int_Q |f(x)|^p dx\right)^{1/p}$$
$$\leq C||\vec{b}||_{BMO} d^{-n(1/p-\delta/n)}||f\chi_Q||_{L^p}$$
$$\leq C||\vec{b}||_{BMO}||f||_{B_p^\delta}.$$

For $H_2(x)$, taking $1 < p < n/\delta$, $1/s = 1/r - \delta/n$, and $1/s' + 1/s = 1$, we obtain that

$$\frac{1}{|Q|}\int_Q H_2(x)dx$$
$$\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\left(\frac{1}{|Q|}\int_Q |(\vec{b}(x)-\vec{b}_Q)_\sigma|^{s'}dx\right)^{1/s'}$$
$$\left(\frac{1}{|Q|}\int_Q |S_\delta((\vec{b}-\vec{b}_Q)_{\sigma^c})f)(x)|^s dx\right)^{1/s}$$
$$\leq C\sum_{j=1}^{m-1}||\vec{b}_\sigma||_{BMO}|Q|^{-1/s}\left(\int_{R^n}|(b(\vec{x})-\vec{b}_Q)_{\sigma^c})f(x)|^r\chi_Q dx\right)^{1/r}$$
$$\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\vec{b}_\sigma||_{BMO}\left(\frac{1}{|Q|}\int_Q |(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}|^{pr/(p-r)}dx\right)^{(p-r)/pr}$$
$$|Q|^{(\delta/n-1/p)}||f\chi_Q||_{L^p}$$
$$\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\vec{b}_\sigma||_{BMO}||\vec{b}_{\sigma^c}||_{BMO} d^{-n(1/p-\delta/n)}||f\chi_Q||_{L^p}$$
$$\leq C||\vec{b}||_{BMO}||f||_{B_p^\delta}.$$

For $H_3(x)$, taking $1 < p < n/\delta$, $1/s = 1/r - \delta/n$ and $1/s' + 1/s = 1$, we get

$$\frac{1}{|Q|}\int_Q H_3(x)dx$$

$$
\begin{aligned}
&\le \left(\frac{1}{|Q|}\int_Q |\mu_\delta((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_1)(x)|^s dx\right)^{1/s}\\
&\le C|Q|^{-1/s}||((b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)f\chi_Q||_{L^r}\\
&\le C\left(\frac{1}{|Q|}\int_Q |(b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)|^{pr/(p-r)}dx\right)^{(p-r)/pr}\\
&\quad d^{-n(1/p-\delta/n)}||f\chi_Q||_{L^p}\\
&\le C||\vec{b}||_{BMO}||f||_{B_p^\delta}.
\end{aligned}
$$

For $H_4(x)$, we have

$$
\begin{aligned}
&H_4(x)\\
&\le \left[\int\int_{R_+^{n+1}}\left(\int_{Q^c}|\chi_{\Gamma(x)}-\chi_{\Gamma(x_0)}|\right.\right.\\
&\left.\left.\prod_{j=1}^m |b_j(z)-(b_j)_Q||\psi_t(y-z)||f(z)|dz\right)^2\frac{dydt}{t^{n+1}}\right]^{1/2}\\
&\le C\sum_{k=0}^\infty \int_{2^{k+1}Q\setminus 2^kQ}|x_0-x|^{1/2}|x_0-z|^{-(n+1/2-\delta)}|\prod_{j=1}^m(b_j(z)-(b_j)_Q)||f(z)|dz\\
&\le C\sum_{k=1}^\infty 2^{-k/2}\frac{1}{|2^kQ|^{1-\delta/n}}\int_{2^kQ}\left|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right||f(y)|dy\\
&\le C\sum_{k=1}^\infty 2^{-k/2}\frac{1}{|2^kQ|^{1-\delta/n}}\left(\int_{2^kQ}\left|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right|^{p/(p-1)}dy\right)^{(p-1)/p}\\
&\quad\times\left(\int_{2^kQ}|f(y)|^p dy\right)^{1/p}\\
&\le C\sum_{k=1}^\infty 2^{-k/2}\left(\frac{1}{|2^kQ|}\int_{2^kQ}\left|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right|^{p/(p-1)}dy\right)^{(p-1)/p}\\
&\quad\times|2^kQ|^{-(1/p-\delta/n)}||f\chi_{2^kQ}||_{L^p}\\
&\le C||\vec{b}||_{BMO}||f||_{B_p^\delta}.
\end{aligned}
$$

This completes the proof of Theorem 2. □

Theorem 3. *Let* $0 < \delta < n$ *and* $\vec{b} = (b_1, \cdots, b_m)$ *with* $b_j \in BMO(R^n)$ *for* $1 \le j \le m$. *Assume that the following inequality holds for any* $H^1(R^n)-$*atom* a *supported on a certain cube* Q *and for* $u \in Q$.

$$\sum_{j=1}^{m} \sum_{\sigma \in C_j^m} \int_{(2Q)^c} \left[|(b(x) - b_Q)_{\sigma^c}| \right.$$

$$\left. \left(\int\int_{\Gamma(x)} \left(\int_Q |(b(z) - b_Q)_\sigma \psi_t(y-u) a(z)| dz \right)^2 \frac{dy dt}{t^{n+1}} \right)^{1/2} \right]^{n/(n-\delta)} dx$$

$$\le C$$

Then $S_\delta^{\vec{b}}$ *is bounded from* $H^1(R^n)$ *to* $L^{n/(n-\delta)}(R^n)$.

Proof of Theorem 3. Let a be an atom supported in some cube Q and $u \in Q$. We write

$$\int_{R^n} |S_\delta^{\vec{b}}(a)(x)|^{n/(n-\delta)} dx$$

$$= \int_{2Q} |S_\delta^{\vec{b}}(a)(x)|^{n/(n-\delta)} dx + \int_{(2Q)^c} |S_\delta^{\vec{b}}(a)(x)|^{n/(n-\delta)} dx = I + II.$$

For I, take $1 < p < n/\delta$ and $1/q = 1/p - \delta/n$. Then we have

$$I \le ||\mu_\delta^{\vec{b}}(a)||_{L^q}^{n/(n-\delta)} |2Q|^{1-n/((n-\delta)q)} \le C||a||_{L^p}^{n/(n-\delta)} |Q|^{1-n/((n-\delta)q)} \le C.$$

For II, we first calculate $F_t^{\vec{b}}(a)(x)$. When $m = 1$, we have

$$\begin{aligned} |F_t^{b_1}(a)(x,y)| &\le |\int_Q \psi_t(y-z) a(z)(b_1(x) - (b_1)_Q) dz| \\ &+ |\int_Q (\psi_t(y-z) - \psi_t(y-u)) a(z)(b_1(z) - (b_1)_Q) dz| \\ &+ |\int_Q \psi_t(y-u)(b_1(z) - (b_1)_Q) a(z) dz| \\ &= \nu_1' + \nu_2' + \nu_3', \end{aligned}$$

so

$$S_\delta^{b_1}(a)(x) = ||\chi_{\Gamma(x)} F_t^{b_1}(a)(x,y)||$$

$$\le \left(\int\int_{\Gamma(x)} |\nu_1'|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2} + \left(\int\int_{\Gamma(x)} |\nu_2'|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2}$$

$$+ \left(\int\int_{\Gamma(x)} |\nu_3'|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2}$$

$$= A'(x) + B'(x) + C'(x).$$

For $A'(x)$, we have

$$A'(x) \leq S_\delta(a)(x)|b_1(x) - (b_1)_Q|,$$

thus

$$\left(\int_{(2Q)^c} (A'(x))^{n/(n-\delta)} dx\right)^{(n-\delta)/n}$$

$$= (\int_{(2Q)^c} [|b_1(x) - (b_1)_Q|$$

$$\times \left(\int\int_{\Gamma(x)} |\int_Q (\psi_t(y-z) - \psi_t(y-u))a(z)dz|^2 \frac{dydt}{t^{n+1}}\right)^{1/2}]^{n/(n-\delta)} dx)^{(n-\delta)/n}$$

$$\leq C|Q|^{1+\varepsilon/n}||a||_{L^\infty}$$

$$\sum_{k=1}^{\infty} \left(\int_{2^{k+1}Q} \left(\frac{|2^kQ|^{\delta/n}}{|2^kQ|^{(n+\varepsilon)/n}} |b_1(x) - (b_1)_Q|\right)^{n/(n-\delta)} dx\right)^{(n-\delta)/n}$$

$$\leq C \sum_{k=1}^{\infty} 2^{-k\varepsilon} \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |b_1(x) - (b_1)_Q|^{n/(n-\delta)} dx\right)^{(n-\delta)/n}$$

$$\leq C||b_1||_{BMO}.$$

For $B'(x)$, we have

$$B'(x)$$

$$\leq C \left(\int\int_{\Gamma(x)} \left(\int_Q \frac{t|u-z|^\varepsilon}{(t+|y-u|)^{n+1+\varepsilon-\delta}} |a(z)||b(z) - (b_1)_Q|dz\right)^2 \frac{dydt}{t^{n+1}}\right)^{1/2}$$

$$\leq C \left(\int_0^\infty \frac{tdt}{(t+|x-u|)^{2(n+1+\varepsilon-\delta)}}\right)^{1/2} \int_Q |u-z|^\varepsilon |a(z)||b_1(z) - (b_1)_Q|dz$$

$$\leq C||b_1||_{BMO}|x-u|^{-(n+\varepsilon-\delta)}|Q|^{1+\varepsilon/n}||a||_{L^\infty},$$

thus

$$\left(\int_{(2Q)^c} (B'(x))^{n/(n-\delta)} dx\right)^{(n-\delta)/n}$$

$$\leq C||b_1||_{BMO}||a||_{L^\infty} \left[\sum_{k=1}^{\infty} \int_{2^{k+1}Q\setminus 2^kQ} \left(\frac{|Q|^{1+\varepsilon/n}}{|x-u|^{n+\varepsilon-\delta}}\right)^{n/(n-\delta)} dx\right]^{(n-\delta)/n}$$

$$\leq C|||b_1||_{BMO}\sum_{k=1}^{\infty}2^{-k\varepsilon}$$
$$\leq C|||b_1||_{BMO}.$$

From that we know, if

$$\begin{aligned}&\int_{(2Q)^c}(C'(x))^{n/(n-\delta)}dx\\&=\int_{(2Q)^c}\left[\left(\int\int_{\Gamma(x)}\left(\int_Q|(b_1(z)-(b_1)_Q)\right.\right.\right.\\&\qquad\left.\left.\left.\psi_t(y-u)a(z)|dz\right)^2\frac{dydt}{t^{n+1}}\right)^{1/2}\right]^{n/(n-\delta)}dx\leq C,\end{aligned}$$

then

$$\int_{R^n}|S_\delta^{b_1}(a)(x)|^{n/(n-\delta)}dx\leq C.$$

When $m>1$, we have

$$\begin{aligned}|F_t^{\vec{b}}(a)(x,y)|&\leq\left|\prod_{j=1}^m(b_j(x)-(b_j)_Q)\int_Q\psi_t(y-z)a(z)dz\right|\\&+\sum_{j=1}^m\sum_{\sigma\in C_j^m}\left|(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}\int_Q(\psi_t(y-z)-\psi_t(y-u))(\vec{b}(z)-\vec{b}_Q)_\sigma a(z)dz\right|\\&+\sum_{j=1}^m\sum_{\sigma\in C_j^m}\left|(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}\int_Q\psi_t(y-u)(\vec{b}(z)-\vec{b}_Q)_\sigma a(z)dz\right|\\&=\nu_1+\nu_2+\nu_3,\end{aligned}$$

so

$$\begin{aligned}S_\delta^{\vec{b}}(a)(x)&=||\chi_{\Gamma(x)}F_t^{\vec{b}}(a)(x,y)||\\&\leq\left(\int\int_{\Gamma(x)}|\nu_1|^2\frac{dydt}{t^{n+1}}\right)^{1/2}+\left(\int\int_{\Gamma(x)}|\nu_2|^2\frac{dydt}{t^{n+1}}\right)^{1/2}\\&\quad+\left(\int\int_{\Gamma(x)}|\nu_3|^2\frac{dydt}{t^{n+1}}\right)^{1/2}\\&=A(x)+B(x)+C(x).\end{aligned}$$

For $A(x)$, we have

$$A(x) = \left(\int\int_{\Gamma(x)} \prod_{j=1}^{m} |b_j(x) - (b_j)_Q|^2 | \int_Q \psi_t(y-z)a(z)dz|^2 \frac{dydt}{t^{n+1}} \right)^{1/2}$$

$$= \prod_{j=1}^{m} |b_j(x) - (b_j)_Q| S_\delta(a)(x),$$

thus

$$\left(\int_{(2Q)^c} (A(x))^{n/(n-\delta)} dx \right)^{(n-\delta)/n}$$

$$\leq C||a||_{L^\infty}$$

$$\left[\sum_{k=1}^{\infty} \int_{2^{k+1}Q \setminus 2^k Q} \left(\frac{|Q|^{1+\varepsilon/n}}{|x-u|^{n+\varepsilon-\delta}} \prod_{j=1}^{m} |b_j(x) - (b_j)_Q| \right)^{n/(n-\delta)} dx \right]^{(n-\delta)/n}$$

$$\leq C|Q|^{1+\varepsilon/n}||a||_{L^\infty}$$

$$\sum_{k=1}^{\infty} \left(\int_{2^{k+1}Q} \left(\frac{|2^k Q|^{\delta/n}}{|2^k Q|^{(n+\varepsilon)/n}} \prod_{j=1}^{m} |b_j(x) - (b_j)_Q| \right)^{n/(n-\delta)} dx \right)^{(n-\delta)/n}$$

$$\leq C \sum_{k=1}^{\infty} 2^{-k\varepsilon} \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} \left(\prod_{j=1}^{m} |b_j(x) - (b_j)_Q| \right)^{n/(n-\delta)} dx \right)^{(n-\delta)/n}$$

$$\leq C||\vec{b}||_{BMO}.$$

For $B(x)$, we have

$$B(x) = (\int\int_{\Gamma(x)} |\sum_{j=1}^{m} \sum_{\sigma \in C_j^m} (\vec{b}(x) - \vec{b}_Q)_{\sigma^c}$$

$$\times \int_Q (\psi_t(y-z) - \psi_t(y-u))(\vec{b}(z) - \vec{b}_Q)_\sigma a(z)dz|^2 \frac{dydt}{t^{n+1}})^{1/2}$$

$$\leq C \sum_{j=1}^{m} \sum_{\sigma \in C_j^m} |(\vec{b}(x) - \vec{b}_Q)_{\sigma^c}|$$

$$\times \left(\int\int_{\Gamma(x)} \left(\int_Q \frac{t|u-z|^\varepsilon}{(t+|y-u|)^{n+1+\varepsilon-\delta}} |(\vec{b}(z) - \vec{b}_Q)_\sigma||a(z)|dz \right)^2 \frac{dydt}{t^{n+1}} \right)^{1/2}$$

$$\leq C\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}|(\vec{b}(x)-b_Q)_{\sigma^c}|\left(\int\int_{\Gamma(x)}\frac{t^{1-n}dydt}{(t+|y-u|)^{2(n+1+\varepsilon-\delta)}}\right)^{1/2}$$

$$\times\int_Q|u-z|^{\varepsilon}|a(z)||(\vec{b}(z)-\vec{b}_Q)_\sigma|dz$$

$$\leq C\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}|(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}||x-u|^{-(n+\varepsilon)}|Q|^{1+\varepsilon/n-\delta}||a||_{L^\infty}||\vec{b}_\sigma||_{BMO},$$

thus

$$\left(\int_{(2Q)^c}(B(x))^{n/(n-\delta)}dx\right)^{(n-\delta)/n}$$

$$\leq C\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}\sum_{k=1}^{\infty}2^{-k\varepsilon}$$

$$\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}\left|(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}\right|^{n/(n-\delta)}dx\right)^{(n-\delta)/n}||\vec{b}_\sigma||_{BMO}$$

$$\leq C||\vec{b}||_{BMO}.$$

So, if

$$\int_{(2Q)^c}(C(x))^{n/(n-\delta)}dx$$

$$=\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}\int_{(2Q)^c}\left[|(b(x)-b_Q)_{\sigma^c}|\left(\int\int_{\Gamma(x)}\left(\int_Q|(b(z)-b_Q)_\sigma\right.\right.\right.$$

$$\left.\left.\left.\psi_t(y-u)a(z)|dz\right)^2\frac{dydt}{t^{n+1}}\right)^{1/2}\right]^{n/(n-\delta)}dx$$

$$\leq C,$$

then

$$\int_{R^n}|S_\delta^{\vec{b}}(a)(x)|^{n/(n-\delta)}dx\leq C.$$

This completes the proof of Theorem 3. □

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