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INVERSE PROBLEM FOR HIGHER ORDER ULTRAPARABOLIC EQUATION WITH UNKNOWN MINOR COEFFICIENT AND RIGHT-HAND SIDE FUNCTION

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Abstract. The problem of identifying of the several unknown time dependent parameters in the minor coefficient and in the special type right-hand side function of the semilinear higher order ultraparabolic equation from the additional initial, boundary and integral type overdetermination conditions is considered in this paper. The sufficient conditions of the unique solvability on the interval $[0,T]$, where T is determined by the coefficients of the equation, are obtained.

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1. INTRODUCTION

The initial-boundary value problems for ultraparabolic equations often appear in mathematical modeling of population dynamics, non-isotropic processes in mechanics, Asian options in finance, etc. [3, 8, 11, 14, 15, 18, 20]. If in the equation some coefficients are unknown, then the problems of their identification together with the solution of the equation, are called inverse problems. The problems of recovering of a single time depended function in the right-hand side of the higher order PDEs from the integral type overdetermination conditions were considered in [7, 13], of several parameters in the right-hand side function of higher order ultraparabolic equations – in [16, 19]. In the works [9, 18] the conditions of the unique solvability were obtained for the inverse problems for second order ultraparabolic equations with a single unknown function in its right-hand side, and in [17] – with two unknown parameters in minor coefficient and in the right-hand side function. The problems of identifying of the minor coefficient in the parabolic or hyperbolic equations were studied in [1, 5, 6, 22], of the several coefficients in right-hand side function – in [6, 21]. The authors used the methods: of the integral equations, regularization and the Shauder principle [6, 9], of successive approximations [16–19], of continuation in a parameter [13], the generalized Fourier method [5].

The main aim of the present paper is to find the sufficient conditions of the existence and the uniqueness of the solution for the inverse problem for higher order ultraparabolic equation with a right-hand side function of the special type. The equation has several time dependent unknown parameters in its right-hand side function and in the minor coefficient. The initial, boundary and integral type overdetermination conditions are posed. In order to obtain the results we use the method of successive approximations and the properties of the initial-boundary value problems for the equations. Similar methods were used for the second order PDEs in [2, 17, 18].

2. Formulation of the problem

Let $\Omega \subset \mathbb{R}^n$ and $D \subset \mathbb{R}^l$ be bounded domains with boundaries $\partial\Omega \in C^{m_0}$ and $\partial D \in C^1$ respectively; $n, l, s, m_0 \in \mathbb{N}$; $x \in \Omega$, $y \in D$, $t \in (0,T)$, $T > 0$. Denote $Q_\tau = \Omega \times D \times (0,\tau)$, $\tau \in (0,T]$, $G = \Omega \times D$, $\Sigma_T = \partial\Omega \times D \times (0,T)$, $S_T = \Omega \times \partial D \times (0,T)$, $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1},\dots,\partial x_n^{\alpha_n}}$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, $\gamma, \alpha \in \mathbb{N}^n$.

In the domain Q_T we consider the problem:
to find the sufficient conditions of the existence and the uniqueness of a set of functions $(u(x,y,t), c(t), q_1(t), \dots, q_s(t))$ that satisfy the equation

$$u_t + \sum_{i=1}^{l} \lambda_i(x,y,t) u_{y_i} + \sum_{0<|\alpha|=|\gamma|\le m_0} (-1)^{|\gamma|} D^\gamma (a_{\alpha\gamma}(x,y,t) D^\alpha u)$$

$$+ (c(t) + b(x,y))u + g(x,y,t,u) = \sum_{i=1}^{s} f_i(x,y,t) q_i(t) + f_0(x,y,t) \tag{2.1}$$

and the conditions

$$u(x,y,0) = u_0(x,y), \quad (x,y) \in G; \tag{2.2}$$

$$\left.\frac{\partial^i u}{\partial \nu^i}\right|_{\Sigma_T} = 0 \ \ (i = 0, 1, \dots, m_0 - 1); \qquad u|_{S_T^1} = 0; \tag{2.3}$$

$$\int_G K_i(x,y) u(x,y,t)\, dx dy = E_i(t), \quad t \in [0,T],\ i = 1, \dots, s+1, \tag{2.4}$$

in the sense of Definition 1 (see below). Here ν is the outward unit normal vector to the surface S_T, $S_T^1 = \{(x,y,t) \in S_T : \sum_{i=1}^{l} \lambda_i(x,y,t) \cos(\nu, y_i) < 0\}$. Let us assume that condition

1): there exists $\Gamma_1 \subset \partial D \subset \mathbb{R}^{l-1}$ such that the surface $S_T^1 = \Omega \times \Gamma_1 \times (0,T)$ holds.

Denote $\Gamma_2 = \partial D \backslash \Gamma_1$, $S_T^2 = \{(x,y,t) \in S_T : \sum_{i=1}^{l} \lambda_i(x,y,t) \cos(\nu, y_i) \ge 0\}$.

We shall use the spaces $L^\infty(\cdot), L^2(\cdot), W^{k,2}(\cdot), W_0^{m_0,2}(\cdot)$, $C([0,T]; L^2(G))$, $C^k(\cdot), C^1([0,T]; C^2(\overline{D})), C^1(D; C^1(\overline{\Omega}))$ from [4, pp. 32, 37, 38, 44, 147].

We also introduce the following spaces:

$V_1(Q_T) := \{w : D^\alpha w \in L^2(Q_T)\,(|\alpha| \le m_0),\ \frac{\partial^i w}{\partial \nu^i}\big|_{\Sigma_T} = 0\,(i=0,1,\dots,m_0-1)\};$

$V_2(G) := \{w : w \in L^2(D; W_0^{m_0,2}(\Omega)),\ w_{y_j} \in L^2(G)\ (j=1,\dots,l),\ w|_{\Omega\times\Gamma_1} = 0\};$

$V_3(Q_T) := \{w :\ w \in W^{1,2}(Q_T),\ D^\alpha w \in L^2(Q_T)\ (|\alpha| \le m_0),\ w|_{S_T^1} = 0,$

$\qquad \frac{\partial^i w}{\partial \nu^i}\big|_{\Sigma_T} = 0\ (i = 0, 1, \dots, m_0-1)\};$

$V_4(Q_T) := \{w :\ w \in V_3(Q_T),\ D^\alpha w \in L^2(Q_T)\ (|\alpha| \le 2m_0)\}.$

Definition 1. A set of functions

$$(u(x,y,t),\ c(t),\ q_1(t),\ q_2(t),\ \dots,\ q_s(t)) \tag{2.5}$$

is a solution to the problem (2.1)–(2.4) if $u \in V_4(Q_T) \cap C([0,T]; L^2(G))$, $c \in C([0,T])$, $q_i \in L^2(0,T)$, $i = 1,\dots,s$, and it satisfies (2.1) for almost all $(x,y,t) \in Q_T$ and the conditions (2.2), (2.4) hold.

Assume that following hypotheses hold:

2): $a_{\alpha\gamma} \in L^\infty(Q_T)$, $0 < |\alpha| = |\gamma| \le m_0$,

$$\sum_{0<|\alpha|=|\gamma|\le m_0} \int_\Omega a_{\alpha\gamma}(x,y,t) D^\alpha w D^\gamma w\, dx \ge a_0 \int_\Omega \sum_{|\alpha|=m_0} |D^\alpha w|^2\, dx$$

for almost all $(y,t) \in D \times (0,T)$ and for all $w \in W_0^{m_0,2}(\Omega)$, $a_0 > 0$;

3): $b \in L^\infty(G)$, $b(x,y) \ge b_0$ for almost all $(x,y) \in G$, b_0 is constant;

4): $f_i \in C([0,T]; L^2(G))$, $i = 0,\dots,s$;

5): $g(x,y,t,\xi)$ is measurable with respect to (x,y,t) in the domain Q_T for all $\xi \in \mathbb{R}^1$ and is continuous with respect to ξ for almost all $(x,y,t) \in Q_T$; moreover, there exists a positive constant g_0 such that $|g(x,y,t,\xi) - g(x,y,t,\eta)| \le g_0|\xi - \eta|$ for almost all $(x,y,t) \in Q_T$ and for all $\xi, \eta \in \mathbb{R}^1$;

6): $\lambda_i \in L^\infty(0,T; C(\overline{G}))$, $\lambda_{iy_i} \in L^\infty(Q_T)$ for all $i = 1,\dots,l$;

7): $u_0 \in V_2(G)$;

8): $K_i \in C^1(D; C^1(\overline{\Omega}))$, $K_i\big|_{\partial\Omega\times D} = 0$, $K_i|_{\Omega\times\Gamma_2} = 0$ for all $i = 1,\dots,s+1$;

9): $E_i \in W^{1,2}(0,T)$, $i = 1,\dots,s+1$.

3. Axilary results

First we assume that $c(t) = c^*(t)$, $q_i(t) = q_i^*(t)$, $i = 1,\dots,s$, in (2.1), where $c^*(t) \in C([0,T])$, $q_i^* \in L^2(0,T)$, $i = 1,\dots,s$, are known functions. The results presented in [12], [18], yield the following statements.

Theorem 1. *Suppose that the hypotheses 1) – 7) hold, and, besides:*
i) $a_{\alpha\gamma}$, $D^{\alpha}a_{\alpha\gamma}$, $a_{\alpha\gamma,y_k}$, $b_{y_k} \in L^{\infty}(Q_T)$, $a_{\alpha\gamma,t}$, $f_{iy_k} \in L^2(Q_T)$, $q_j^* \in L^2(0,T)$, $c^* \in C([0,T])$ $(0 < |\alpha| = |\gamma| \le m_0,\ i = 0,\dots,s,\ j = 1,\dots,s,\ k = 1,\dots,l)$;
ii) $|g_{y_i}(x,y,t,\xi)| \le g_1\ \ (i = 1,\dots,l),\ \ g(x,y,t,0)|_{S_T^1} = 0$ *for almost all* $(x,y,t) \in Q_T$ *and all* $\xi \in \mathbb{R}^1$, *where* g_1 *is a positive constant;*
iii) $f_i|_{S_T^1} = 0 \quad (i = 0,\dots,s)$,
then there exists a unique function $u^* \in V_3(Q_T) \cap C([0,T];L^2(G))$, *that satisfies the condition (2.2) and for all functions* $v \in V_1(Q_T)$ *the equality*

$$\int_{Q_T} \Big(u_t^* v + \sum_{i=1}^{l} \lambda_i(x,y,t) u_{y_i}^* v + \sum_{0<|\alpha|=|\gamma|\le m_0} a_{\alpha\gamma}(x,y,t) D^{\alpha}u^* D^{\gamma}v + \\ + (c^*(t) + b(x,y))u^* v + g(x,y,t,u^*)v\Big)\, dx\, dy\, dt \\ = \int_{Q_T} \Big(\sum_{i=1}^{s} f_i(x,y,t) q_i^*(t) + f_0(x,y,t)\Big) v\, dx\, dy\, dt. \qquad (3.1)$$

Moreover, $u^* \in V_4(Q_T) \cap C([0,T];L^2(G))$, u^* *satisfies the condition (2.2) and (2.1) for almost all* $(x,y,t) \in Q_T$ *(so,* u^* *is a solution to the problem (2.1) – (2.3)). The derivatives of* u^* *have the following estimates*

$$\int_{Q_T} \sum_{i=1}^{l} (u_{y_i}^*)^2\, dx\, dy\, dt \le M_0, \qquad \int_{Q_T} (u_t^*)^2\, dx\, dy\, dt \le M, \qquad (3.2)$$

where the constants M_0, M *depend on* u_0, *and on the coefficients and the right-hand side function of (2.1).*

The proof is carried out according to the scheme of proving of Theorem 1, 2 and Lemma 1 [12] and Theorem 3.4.3 [18, p. 97].

Further for functions $w \in W_0^{k,2}(\Omega)$ we shall use Friedrichs' inequalities:

$$\int_{\Omega} \sum_{|\alpha|=j} |D^{\alpha}w|^2\, dx \le \gamma_{k,j} \int_{\Omega} \sum_{|\alpha|=k} |D^{\alpha}w|^2\, dx, \quad j = 0,1,\dots,k, \qquad (3.3)$$

where constant $\gamma_{k,j}$ depends on Ω, k, j. Denote $\Gamma_k = \sum_{j=0}^{k} \gamma_{k,j}$.

4. Reduction of the problem to the equivalent problem

Equation (2.1) and conditions (2.4) for $j = 1,\dots,s+1$ imply the equalities

$$\sum_{i=1}^{s} q_i(t) \int_G K_j(x,y) f_i(x,y,t)\, dx\, dy - c(t)E_j(t) = F_j(t), \quad t \in [0,T], \qquad (4.1)$$

where $F_j(t) := E'_j(t) - \int_G \big(K_j(x,y)(f_0(x,y,t) - b(x,y)u - g(x,y,t,u)) + \sum_{i=1}^{l}(\lambda_i(x,y,t)K_j(x,y))_{y_i}u - \sum_{0<|\alpha|=|\gamma|\le m_0}(-1)^{|\alpha|}D^\alpha(D^\gamma K_j(x,y)a_{\alpha\gamma}(x,y,t))u\big)dxdy.$

Denote $B(t) := [b_{ij}(t)]_{(s+1)\times(s+1)}$, where $b_{ij}(t) = \int_G K_i(x,y)f_j(x,y,t)\,dxdy$, $b_{i,s+1}(t) = -E_i(t)$, $i = 1,\dots,s+1$, $j = 1,\dots,s$, $\Delta(t) := \det B(t)$, $A_{ij}(t)$ – the algebraical complements of the elements of $B(t)$. Let

$$\Delta(t) \neq 0 \text{ for all } t \in [0,T] \tag{4.2}$$

and for $i = 1,\dots,s+1$, $j = 1,\dots,s+1$

$$\alpha_{ij}(x,y,t) := A_{ji}(t)(\Delta(t))^{-1}\big(-K_j(x,y)b(x,y) + \sum_{i=1}^{l}(\lambda_i(x,y,t)K_j(x,y))_{y_i} - \\ - \sum_{0<|\alpha|=|\gamma|\le m_0}(-1)^{|\alpha|}D^\alpha(D^\gamma K_j(x,y)a_{\alpha\gamma}(x,y,t))\big),$$

$$\beta_{ij}(t) := A_{ji}(t)(\Delta(t))^{-1}\big(E'_j(t) - \int_G K_j(x,y)f_0(x,y,t)\,dxdy\big).$$

Then from (4.1) for $t \in [0,T]$ and $i = 1,\dots,s$, we obtain

$$q_i(t) = \sum_{j=1}^{s+1}\left(\beta_{ij}(t) - \int_G \left(\alpha_{ij}(x,y,t)u - \frac{A_{ji}(t)}{\Delta(t)}K_j(x,y)g(x,y,t,u)\right)dxdy\right), \tag{4.3}$$

$$c(t) = \sum_{j=1}^{s+1}\left(\beta_{s+1,j}(t) - \int_G \left(\alpha_{s+1,j}(x,y,t)u - \\ - \frac{A_{j,s+1}(t)}{\Delta(t)}K_j(x,y)g(x,y,t,u)\right)dxdy\right). \tag{4.4}$$

From here it follows that the solution of the problem (2.1)–(2.4) satisfies (2.1) for almost all $(x,y,t) \in Q_T$ and (2.2), (4.3), (4.4). By analogous considerations as in Lemma 1 [17] it could be proved that the reverse statement is also correct. Therefore the following lemma holds

Lemma 1. *Let the assumptions of Theorem 1, hypotheses 8), 9) and (4.2) hold. The set of functions (2.5), where $u \in V_4(Q_T) \cap C([0,T]; L^2(G))$, $c \in C([0,T])$, $q_i \in L^2(0,T)$, $i = 1,\dots,s$, is a solution to the problem (2.1)–(2.4) if and only if this set satisfies (2.1) for almost all $(x,y,t) \in Q_T$ and (2.2), (4.3), (4.4) hold.*

5. Main results

Further we shall use the following denotations:

$$C_{1,i} := 2(s+1)\sum_{j=1}^{s+1}\int_0^T\left(|\beta_{ij}(t)| + \int_G\left|\frac{A_{ji}(t)}{\Delta(t)}K_j(x,y)g(x,y,t,0)\right|dxdy\right)^2 dt,$$

$$C_{2,i} := 2(s+1)\sum_{j=1}^{s+1}\sup_{[0,T]}\int_G\left(|\alpha_{ij}(x,y,t))| + g_0\left|\frac{A_{ji}(t)}{\Delta(t)}K_j(x,y)\right|\right)^2 dxdy,$$

$$C_1 := \sum_{i=1}^{s} C_{1,i}, \qquad C_2 := \sum_{i=1}^{s} C_{2,i},$$

$$C_3 := 2(s+1)\sup_{[0,T]}\sum_{j=1}^{s+1}\left(|\beta_{s+1,j}(t)| + \int_G\left|\frac{A_{j,s+1}(t)}{\Delta(t)}K_j(x,y)g(x,y,t,0)\right|dxdy\right)^2,$$

$$C_4 := 2(s+1)\sum_{j=1}^{s+1}\sup_{[0,T]}\int_G\left(|\alpha_{s+1,j}(x,y,t))| + g_0\left|\frac{A_{j,s+1}(t)}{\Delta(t)}K_j(x,y)\right|\right)^2 dxdy.$$

Assume that there exist such numbers $T > 0$ and $\delta > 0$ that the following inequalities are fulfilled:

$$\tilde{f}C_2T < \delta, \quad \varkappa > 0, \quad M_7 < 1, \tag{5.1}$$

where $\tilde{f} = \max_i \sup_{[0,T]}\int_G |f_i(x,y,t)|^2 dxdy$,

$$M_1 := \frac{1}{\delta}\int_{Q_T}\left(f_0(x,y,t))^2 + (g(x,y,t,0))^2\right)dxdydt + \int_G (u_0(x,y))^2 dxdy,$$

$$M_2 := \left(C_3 + C_4M_1 + \frac{C_4\tilde{f}(C_1+C_2M_1T)}{\delta - C_2\tilde{f}T}\right)^{1/2},$$

$$\lambda^1 := \max_i \operatorname*{ess\,sup}_{Q_T}|\lambda_{iy_i}(x,y,t)|, \quad \varkappa := \frac{2a_0}{\gamma_{m_0,0}} + 2b_0 - \lambda^1 l - 2M_2 - 2g_0 - (s+2)\delta,$$

$$M_3 := \frac{\delta(C_1+C_2M_1T)}{\delta - C_2\tilde{f}T}, \qquad M_4 := M_1 + \frac{1}{\delta}\tilde{f}M_3, \qquad M_5 := \max\{M_4;\tilde{f}\},$$

$$M_6 := \frac{M_5}{\varkappa\delta}, \qquad M_7 := \frac{1}{2}(C_2+C_4)M_6, \qquad M_8 := \sqrt{\frac{C_4M_5}{2\delta}}.$$

Theorem 2. *Let the hypotheses (4.2), (5.1) and 1) − 9) hold. Then the problem (2.1)–(2.4) has at most one solution.*

If, besides, $a_{\alpha\gamma}, D^\alpha a_{\alpha\gamma}, a_{\alpha\gamma,y_k}, b_{y_k} \in L^\infty(Q_T)$, $a_{\alpha\gamma,t}, f_{iy_k} \in L^2(Q_T)$ $(0<|\alpha|=|\gamma|\le m_0,\ i=0,\dots,s,\ k=1,\dots,l)$ and the assumptions ii), iii) of Theorem 1 are true, then a solution to the problem (2.1)–(2.4) exists in Q_T.

Proof. Existence. In order to prove the existence of the solution we use the method of successive approximations. We construct the approximation of the solution to the problem (2.1)–(2.4) as follows:

$$c^1(t) := 0,\ q_i^1(t) := 0, \qquad t \in [0,T],\ i = 1,\dots,s,$$

$$q_i^m(t) = \sum_{j=1}^{s+1} \Bigg(\beta_{ij}(t) - \int\limits_G \big(\alpha_{ij}(x,y,t)u^{m-1} - \frac{A_{ji}(t)}{\Delta(t)} K_j(x,y) \times$$

$$\times g(x,y,t,u^{m-1}) \big)\, dx\, dy \Bigg), t \in [0,T],\ i = 1,\dots,s,\ m \geq 2, \quad (5.2)$$

$$c^m(t) = \sum_{j=1}^{s+1} \Bigg(\beta_{s+1,j}(t) - \int\limits_G \alpha_{s+1,j}(x,y,t)u^{m-1}\, dx\, dy$$

$$+ \int\limits_G \frac{A_{j,s+1}(t)}{\Delta(t)} K_j(x,y) g(x,y,t,u^{m-1})\, dx\, dy \Bigg),\ t \in [0,T],\ \ m \geq 2, \quad (5.3)$$

u^m satisfies the equality

$$\int\limits_{Q_\tau} \big(u_t^m v + \sum_{i=1}^{l} \lambda_i(x,y,t) u_{y_i}^m v + \sum_{0<|\alpha|=|\gamma|\leq m_0} a_{\alpha\gamma}(x,y,t) D^\alpha u^m D^\gamma v +$$

$$+ (c^m(t) + b(x,y)) u^m v + g(x,y,t,u^m) v \big)\, dx\, dy\, dt$$

$$= \int\limits_{Q_\tau} (\sum_{i=1}^{s} f_i(x,y,t) q_i^m(t) + f_0(x,y,t)) v\, dx\, dy\, dt,\ \tau \in (0,T],\ m \geq 1 \quad (5.4)$$

for all $v \in V_1(Q_T)$ and the condition

$$u^m(x,y,0) = u_0(x,y),\ \ (x,y) \in G. \quad (5.5)$$

It follows from (5.2), (5.3), that

$$q_i^m \in L^2(0,T),\ i = 1,\dots,s,\ m \geq 2,\ \ c^m \in C([0,T]),\ m \geq 2.$$

According to Theorem 1 for each $m \in \mathbb{N}$ there exists a unique function $u^m \in V_4(Q_T) \cap C([0,T];L^2(G))$, which satisfies (5.4), (5.5).

At first we shall find some estimates for the approximations $(u^m(x,y,t), c^m(t), q_1^m(t), q_2^m(t), \dots, q_s^m(t))$, and show that $c^m(t)$ for all $m \in \mathbb{N}$ and $t \in [0,T]$ could be bounded from below by the same constant. Let $c^m(t) \geq c_{0m}$ for all $t \in [0,T]$, where $c_{0m} \in \mathbb{R}$. If we choose $v = u^m$ in (5.4)

$$\int\limits_{Q_\tau} \big(u_t^m u^m + \sum_{i=1}^{l} \lambda_i(x,y,t) u_{y_i}^m u^m + \sum_{0<|\alpha|=|\gamma|\leq m_0} a_{\alpha\gamma}(x,y,t) D^\alpha u^m D^\gamma u^m +$$

$$+ (c^m(t) + b(x,y))(u^m)^2 + g(x,y,t,u^m) u^m \big)\, dx\, dy\, dt$$

$$= \int\limits_{Q_\tau} (\sum_{i=1}^{s} f_i(x,y,t) q_i^m(t) + f_0(x,y,t)) u^m\, dx\, dy\, dt,\ \tau \in (0;T],\ \ m \geq 1, \quad (5.6)$$

and take into account the hypotheses 2) – 7), then from (5.6) we get the inequalities

$$\int_G (u^m(x,y,\tau))^2\,dx\,dy+\int_{S^2_\tau}\sum_{i=1}^{l}\lambda_i(x,y,t)(u^m)^2\cos(\nu,y_i)\,d\sigma$$
$$+\int_{Q_\tau}\Bigg(2a_0\sum_{|\alpha|=m_0}|D^\alpha u^m|^2+(2b_0-\lambda^1 l+2c_{0m}-2g_0-(s+2)\delta)(u^m)^2\Bigg)dxdydt$$
$$\le\frac{1}{\delta}\int_{Q_\tau}\Bigg(\sum_{i=1}^{s}(f_i(x,y,t))^2(q_i^m(t))^2+(f_0(x,y,t))^2+(g(x,y,t,0))^2\Bigg)\,dx\,dy\,dt$$
$$+\int_G (u_0(x,y))^2\,dx\,dy,\ \tau\in(0;T],\ \ m\ge 1. \tag{5.7}$$

After using the inequality (3.3) in the third term of (5.7), we obtain

$$\int_G (u^m(x,y,\tau))^2\,dx\,dy+\int_{S^2_\tau}\sum_{i=1}^{l}\lambda_i(x,y,t)(u^m)^2\cos(\nu,y_i)\,d\sigma$$
$$+\int_{Q_\tau}\left(\frac{2a_0}{\gamma_{m_0,0}}-\lambda^1 l+2c_{0m}+2b_0-2g_0-(s+2)\delta\right)(u^m)^2\,dx\,dy\,dt$$
$$\le\frac{1}{\delta}\int_{Q_\tau}\Bigg(\sum_{i=1}^{s}(f_i(x,y,t))^2(q_i^m(t))^2+(f_0(x,y,t))^2+$$
$$+(g(x,y,t,0))^2\Bigg)\,dx\,dy\,dt+\int_G (u_0(x,y))^2\,dx\,dy,\ \tau\in(0;T],\ \ m\ge 1. \tag{5.8}$$

Using the assumption $\frac{2a_0}{\gamma_{m_0,0}}-\lambda^1 l+2c_{0m}+2b_0-2g_0-(s+2)\delta\ge 0$, from (5.8) for $\tau\in(0;T]$, $m\ge 1$, we get the estimates

$$\int_G (u^m(x,y,\tau))^2\,dx\,dy\le M_1+\frac{1}{\delta}\int_{Q_\tau}\sum_{i=1}^{s}(f_i(x,y,t))^2(q_i^m(t))^2\,dx\,dy\,dt. \tag{5.9}$$

Now square both sides of (5.2) and after using Hölder inequality, and integrating the result with respect to t, we get the estimates

$$\int_0^T (q_i^m(t))^2\,dt\le C_{1,i}+C_{2,i}\int_{Q_T}(u^{m-1})^2\,dx\,dy\,dt,\ m\ge 2,\ i=1,\dots,s, \tag{5.10}$$

rising up the both sides of (5.3) to the square and using Hölder inequality, we get the estimate

$$(c^m(t))^2\le C_3+C_4\int_G (u^{m-1})^2\,dx\,dy,\qquad t\in[0;T],\,m\ge 2. \tag{5.11}$$

From the system of inequalities (5.9)–(5.11) under the condition (5.1) it is easy to proof the estimates

$$|c^m(t)| \le M_2, \qquad t \in [0,T],\ m \ge 1, \tag{5.12}$$

$$\sum_{i=1}^{s} \int_0^T |q_i^m(t)|^2\,dt \le M_3, \qquad m \ge 1, \tag{5.13}$$

$$\int_G |u^m(x,y,t)|^2\,dx\,dy \le M_4, \qquad t \in [0,T],\ m \ge 1. \tag{5.14}$$

Remark, that M_2 is independent on m and if we take $-M_2$ instead of c_{0m} for each m and take into account the condition (5.1), we get

$$\frac{2a_0}{\gamma_{m_0,0}} - \lambda^1 l + 2c_{0m} + 2b_0 - 2g_0 - (s+2)\delta = \varkappa > 0.$$

Thus, $c^m(t) \ge -M_2$ for all $m \in \mathbb{N}$.

Further we show that $\{(u^m(x,y,t), c^m(t), q_1^m(t), q_2^m(t), \dots, q_s^m(t))\}_{m=1}^{\infty}$ converges to the solution of the problem (2.1)–(2.4). Denote for $m \ge 2$

$$s^m(t) := c^m(t) - c^{m-1}(t), \qquad r_i^m(t) := q_i^m(t) - q_i^{m-1}(t),\ i = 1,\dots,s,$$
$$z^m := z^m(x,y,t) = u^m(x,y,t) - u^{m-1}(x,y,t).$$

From (5.5) we get $z^m(x,y,0) = 0,\ (x,y) \in G,\ m \ge 2$. Moreover, using (5.4) with $v = z^m$ we obtain the equality

$$\begin{aligned}
\frac{1}{2}\int_G |z^m(x,y,\tau)|^2\,dx\,dy + \int_{Q_\tau} \Big(\sum_{i=1}^{l} \lambda_i(x,y,t) z^m_{y_i} z^m + b(x,y)(z^m)^2 \\
+ \sum_{0<|\alpha|=|\gamma|\le m_0} a_{\alpha\gamma}(x,y,t) D^\alpha z^m D^\gamma z^m + (c^m(t)u^m - c^{m-1}(t)u^{m-1})z^m \\
+ (g(x,y,t,u^m) - g(x,y,t,u^{m-1}))z^m \Big)\,dx\,dy\,dt = \\
= \int_{Q_\tau} \sum_{i=1}^{s} f_i(x,y,t) r_i^m(t) z^m\,dx\,dy\,dt,\ \tau \in (0,T],\ m \ge 2.
\end{aligned} \tag{5.15}$$

Notice that

$$\int_{Q_\tau} \sum_{i=1}^{s} f_i(x,y,t) r_i^m(t) z^m\,dx\,dy\,dt \le \frac{s\delta}{2}\int_{Q_\tau} (z^m)^2\,dx\,dy\,dt + \frac{\tilde f}{2\delta}\int_0^T \sum_{i=1}^{s} (r_i^m(t))^2\,dt$$

and

$$(c^m(t)u^m - c^{m-1}(t)u^{m-1})z^m = c^m(t)(z^m)^2 + s^m(t)u^{m-1}z^m,$$

therefore

$$\int_{Q_\tau} (c^m(t)u^m - c^{m-1}(t)u^{m-1})z^m\,dx\,dy\,dt \ge \left(-M_2 - \frac{\delta}{2}\right)\int_{Q_\tau} (z^m)^2\,dx\,dy\,dt$$

$$-\frac{1}{2\delta}\int_0^\tau (s^m(t))^2\left(\int_{G_t}(u^{m-1})^2\,dx\,dy\right)dt \geq \left(-M_2-\frac{\delta}{2}\right)\int_{Q_\tau}(z^m)^2\,dx\,dy\,dt$$

$$-\frac{M_4}{2\delta}\int_0^\tau (s^m(t))^2\,dt, \qquad \tau\in(0,T],\ m\geq 2. \quad (5.16)$$

Then, taking into account 2) – 7) and (5.16), from (5.15) we get inequalities

$$\int_G (z^m(x,y,\tau))^2\,dx\,dy+\int_{S^2_\tau}\sum_{i=1}^{l}\lambda_i(x,y,t)(z^m)^2\cos(\nu,y_i)\,d\sigma$$

$$+\int_{Q_\tau}\left(2a_0\sum_{|\alpha|=m_0}|D^\alpha z^m|^2+\big(2b_0-\lambda^1 l-2g_0-2M_2-(s+2)\delta\big)(z^m)^2\right)dx\,dy\,dt$$

$$\leq \frac{M_4}{\delta}\int_0^T (s^m(t))^2\,dt+\frac{\tilde f}{\delta}\int_0^T\sum_{i=1}^{s}(r_i^m(t))^2\,dt,\ \tau\in(0;T],\ m\geq 2. \quad (5.17)$$

After applying (3.3) to the third term of (5.17), we get the estimates

$$\int_G (z^m(x,y,\tau))^2\,dxdy+\int_{S^2_\tau}\sum_{i=1}^{l}\lambda_i(x,y,t)(z^m)^2\cos(\nu,y_i)d\sigma+\varkappa\int_{Q_\tau}(z^m)^2\,dxdydt$$

$$\leq \frac{M_4}{\delta}\int_0^T (s^m(t))^2\,dt+\frac{\tilde f}{\delta}\int_0^T\sum_{i=1}^{s}(r_i^m(t))^2\,dt,\ \tau\in(0;T],\ m\geq 2. \quad (5.18)$$

Taking into account (5.1), from (5.18) for $m\geq 2$ we find the estimates

$$\int_G (z^m(x,y,\tau))^2\,dx\,dy\leq\frac{M_5}{\delta}\int_0^T\left((s^m(t))^2+\sum_{i=1}^{s}(r_i^m(t))^2\right)dt,\tau\in(0;T], \quad (5.19)$$

and

$$\int_{Q_T}(z^m)^2\,dx\,dy\,dt\leq M_6\int_0^T\left((s^m(t))^2+\sum_{i=1}^{s}(r_i^m(t))^2\right)dt, \qquad m\geq 2. \quad (5.20)$$

Besides, from (5.2) and (5.3) it follows that

$$\int_0^T\sum_{i=1}^{s}(r_i^m(t))^2\,dt\leq\frac{C_2}{2}\int_{Q_T}(z^{m-1})^2\,dx\,dy\,dt, \qquad m\geq 2, \quad (5.21)$$

$$\int_0^T (s^m(t))^2\,dt \le \frac{C_4}{2}\int_{Q_T}(z^{m-1})^2\,dx\,dy\,dt, \qquad m\ge 2, \tag{5.22}$$

$$(s^m(t))^2 \le \frac{C_4}{2}\int_G (z^{m-1}(x,y,t))^2\,dx\,dy, \qquad t\in[0,T],\, m\ge 2. \tag{5.23}$$

Then (5.21), (5.22) and (5.20) imply

$$\int_0^T \Big((s^m(t))^2+\sum_{i=1}^{s}(r_i^m(t))^2\Big)\,dt \le M_7\int_0^T \Big((s^{m-1}(t))^2+\sum_{i=1}^{s}(r_i^{m-1}(t))^2\Big)\,dt$$
$$\le (M_7)^{m-2}\int_0^T \Big((s^2(t))^2+\sum_{i=1}^{s}(r_i^2(t))^2\Big)\,dt, \qquad m\ge 2, \tag{5.24}$$

and with the use of (5.19), from (5.23) for $m\ge 2$ we get

$$|s^m(t)| \le M_8\left(\int_0^T \Big((s^{m-1}(t))^2+\sum_{i=1}^{s}(r_i^{m-1}(t))^2\,dt\Big)\right)^{\frac12},\ t\in[0,T]. \tag{5.25}$$

Afterwards, using (5.24), (5.25) and the assumption $M_7<1$, we show that the estimates

$$|c^{m+k}(t)-c^m(t)| \le \sum_{i=m+1}^{m+k}|s^i(t)| \le M_8\sum_{i=m+1}^{m+k}\left(\int_0^T \Big(\sum_{l=1}^{s}({r_l}^{i-1}(t))^2+\right.$$
$$\left.+(s^{i-1}(t))^2\Big)\,dt\right)^{\frac12} \le \sum_{i=m+1}^{m+k} M_8(M_7)^{\frac{i-3}{2}}\left(\int_0^T \Big((s^2(t))^2+\sum_{i=1}^{s}(r_i^2(t))^2\Big)\,dt\right)^{\frac12}$$
$$\le M_8\frac{(M_7)^{\frac{m-2}{2}}}{1-(M_7)^{\frac12}}\left(\int_0^T \Big((s^2(t))^2+\sum_{i=1}^{s}(r_i^2(t))^2\Big)\,dt\right)^{\frac12} \tag{5.26}$$

and

$$\left(\int_0^T ({q_l}^{m+k}(t)-{q_l}^m(t))^2\,dt\right)^{\frac12} \le \sum_{i=m+1}^{m+k}\left(\int_0^T \Big(({r_l}^i(t))^2+(s^i(t))^2\Big)\,dt\right)^{\frac12}$$
$$\le \frac{(M_7)^{\frac{m-1}{2}}}{1-(M_7)^{\frac12}}\left(\int_0^T \Big((s^2(t))^2+\sum_{l=1}^{s}(r_l^2(t))^2\Big)\,dt\right)^{\frac12},\ l=1,\dots,s, \tag{5.27}$$

hold for all $k,m \in \mathbb{N},\ m \geq 3$. From (5.26), (5.27) we conclude that for any $\varepsilon > 0$, there exists m_0 such that for all $k,m \in \mathbb{N},\ m > m_0$, the inequalities $\|c^{m+k}(t) - c^m(t);C([0,T])\| \leq \varepsilon$ and $\sum_{i=1}^{s} \|q_i^{m+k}(t) - q_i^m(t);L^2(0,T)\| \leq \varepsilon$ are true and, hence, the sequence $\{c^m\}_{m=1}^{\infty}$ is fundamental in $C([0,T])$, and $\{q_i^m\}_{m=1}^{\infty}$, $i = 1,\ldots,s$, are fundamental in $L^2(0,T)$. Therefore, (5.19) and (5.17) imply that $\{u^m\}_{m=1}^{\infty}$ is fundamental in $L^2(Q_T) \cap C([0,T];L^2(G))$ and $\{D^\alpha u^m\}_{m=1}^{\infty}$, $|\alpha| \leq m$, is fundamental in $L^2(Q_T)$ and, hence, as $m \to \infty$

$$u^m \to u \text{ in } L^2(Q_T) \cap C([0,T];L^2(G)),\ q_i^m \to q_i \text{ in } L^2(0,T),\ i=1,\ldots,s.$$
$$D^\alpha u^m \to D^\alpha u \text{ in } L^2(Q_T),\ |\alpha| \leq m, \quad c^m \to c \text{ in } C([0,T]). \tag{5.28}$$

Formula (3.2) implies the following estimates

$$\int_{Q_T} \sum_{i=1}^{l} (u_{y_i}^m)^2\, dx\,dy\,dt \leq M_0, \qquad \int_{Q_T} (u_t^m)^2\, dx\,dy\,dt \leq M, \tag{5.29}$$

and, by virtue of the inequalities (5.12), (5.13), the constants $M_0,\ M$ are independent of m and (5.29) is true for all $m \in \mathbb{N}$. In view of (5.29), we can select a subsequence of sequence $\{u^m\}_{m=1}^{\infty}$ (we preserve the same notation for this subsequence), such that

$$u_t^m \to u_t \text{ and } u_{y_i}^m \to u_{y_i} \text{ weakly in } L^2(Q_T),\ i = 1,\ldots,l, \text{ as } m \to \infty. \tag{5.30}$$

Passing to the limit in (5.2) – (5.4) as $m \to \infty$ and taking into account (5.28), (5.30), we get that the set of functions (2.5) satisfies the system of equations (4.3), (4.4) and

$$\int_{Q_\tau} \Big(u_t v + \sum_{i=1}^{l} \lambda_i(x,y,t) u_{y_i} v + \sum_{0<|\alpha|=|\gamma|\leq m_0} a_{\alpha\gamma}(x,y,t) D^\alpha u D^\gamma v$$
$$+ (c(t) + b(x,y))uv + g(x,y,t,u)v\Big)\, dx\,dy\,dt$$
$$= \int_{Q_\tau} \Big(\sum_{i=1}^{s} f_i(x,y,t) q_i(t) + f_0(x,y,t)\Big) v\, dx\,dy\,dt \tag{5.31}$$

for all $\tau \in (0;T]$ and $v \in V_1(Q_T)$. From (5.31) we obtain that

$$\int_{\Omega} \Big(u_t w + \sum_{i=1}^{l} \lambda_i(x,y,t) u_{y_i} w + \sum_{0<|\alpha|=|\gamma|\leq m_0} a_{\alpha\gamma}(x,y,t) D^\alpha u D^\gamma w + (c(t)+$$
$$+ b(x,y))uw + g(x,y,t,u)w\Big)\, dx = \int_{\Omega} \Big(\sum_{i=1}^{s} f_i(x,y,t) q_i(t) + f_0(x,y,t)\Big) w\, dx \tag{5.32}$$

for almost all $(y,t) \in D \times (0;T)$ and for all $w \in W_0^{m_0,2}(\Omega)$. From (5.32) we derive that u for almost all $(y,t) \in D \times (0;T)$ is a weak solution to the Dirichlet problem for the elliptic equation

$$\sum_{0<|\alpha|=|\gamma|\le m_0} D^{\gamma}(a_{\alpha\gamma}(x,y,t)D^{\alpha}u) = F(x,y,t),\ x\in\Omega; \tag{5.33}$$
$$\left.\frac{\partial^i u}{\partial \nu^i}\right|_{\Omega} = 0\ (i=0,\,1,\,\dots,m_0-1),$$

where $F(x,y,t)=\sum\limits_{i=1}^{s} f_i(x,y,t)q_i(t)+f_0(x,y,t)-u_t-\sum\limits_{i=1}^{l}\lambda_i(x,y,t)u_{y_i}-(c(t)+b(x,y))u-g(x,y,t,u)$. Since function $F(x,y,t)\in L^2(\Omega)$ for almost all $(y,t)\in D\times(0,T)$, there exists a unique weak solution u of the problem (5.33). Then, from [23] and the condition $\partial\Omega\in C^{m_0}$ we conclude that $D^{\alpha}u^*(\cdot,y,t)\in L^2(\Omega)$, $|\alpha|\le 2m_0$. Thus, $u^*(\cdot,y,t)\in W_0^{m_0,2}(\Omega)\cap W^{2m_0,2}(\Omega)$. Using the scheme [10, p. 219] we prove that the function $u^*(x,y,t)$ satisfies the equation (2.1) for almost all $(x,y,t)\in Q_T$.

Hence, $u\in V_4(Q_T)\cap C([0,T];L^2(G))$, the set $(u(x,y,t),c(t),q_1(t),\dots,q_s(t))$ satisfies (2.1) for almost all $(x,y,t)\in Q_T$, and by virtue of Lemma 1 $(u(x,y,t),c(t),q_1(t),\dots,q_s(t))$ is a solution to the problem (2.1)–(2.4) in Q_T.

Uniqueness. Assume that $(u^{(l)}(x,y,t),c^{(l)}(t),q_1^{(l)}(t),\dots,q_s^{(l)}(t))$, $l=1,2$, are two solutions to problem (2.1)–(2.4). Denote $\tilde u(x,y,t):=u^{(1)}(x,y,t)-u^{(2)}(x,y,t)$, $\tilde c(t):=c^{(1)}(t)-c^{(2)}(t)$, $\tilde q_i(t):=q_i^{(1)}(t)-q_i^{(2)}(t)$. Then $\tilde u(x,y,0)\equiv 0$, the set of functions $(\tilde u(x,y,t),\tilde c(t),\tilde q_1(t),\dots,\tilde q_s(t))$ satisfies the equality

$$\int\limits_{Q_T}\Big(\tilde u_t v+\sum_{i=1}^{l}\lambda_i(x,y,t)\tilde u_{y_i}v+\sum_{0<|\alpha|=|\gamma|\le m_0} a_{\alpha\gamma}(x,y,t)D^{\alpha}\tilde u D^{\gamma}v+b(x,y)\tilde u v+$$
$$+(c^{(1)}(t)u^{(1)}-c^{(2)}(t)u^{(2)})v+(g(x,y,t,u^{(1)})-$$
$$-g(x,y,t,u^{(2)}))v\Big)\,dx\,dy\,dt=\int\limits_{Q_T}\sum_{i=1}^{s} f_i(x,y,t)\tilde q_i(t)v\,dx\,dy\,dt, \tag{5.34}$$

for all $v\in V_1(Q_T)$, and the system of equalities

$$\tilde c(t)=\sum_{j=1}^{s+1}\int\limits_{G}\Big(-\alpha_{s+1,j}(x,y,t)\tilde u+\frac{A_{j,s+1}(t)}{\Delta(t)}K_j(x,y)\big(g(x,y,t,u^{(1)})-$$
$$-g(x,y,t,u^{(2)})\big)\Big)\,dx\,dy,\ t\in[0,T], \tag{5.35}$$

$$\tilde q_i(t)=\sum_{j=1}^{s+1}\int\limits_{G}\Big(-\alpha_{ij}(x,y,t)\tilde u+\frac{A_{ji}(t)}{\Delta(t)}K_j(x,y)\big(g(x,y,t,u^{(1)})-$$
$$-g(x,y,t,u^{(2)})\big)\Big)\,dx\,dy,\ t\in[0,T],\ i=1,\dots,s. \tag{5.36}$$

From (5.35), (5.36) and 5) we derive the inequalities

$$\int_0^T \sum_{i=1}^s (\tilde{q}_i(t))^2 dt \le \frac{C_2}{2} \int_{Q_T} (\tilde{u})^2 dxdydt, \quad \int_0^T (\tilde{c}(t))^2 dt \le \frac{C_4}{2} \int_{Q_T} (\tilde{u})^2 dxdydt, \tag{5.37}$$

and after choosing $v = \tilde{u}$, in (5.34) we get

$$\int_{Q_T} \Big(\tilde{u}_t \tilde{u} + \sum_{i=1}^l \lambda_i(x,y,t) \tilde{u}_{y_i} \tilde{u} + \sum_{0<|\alpha|=|\gamma|\le m_0} a_{\alpha\gamma}(x,y,t) D^\alpha \tilde{u} D^\gamma \tilde{u} +$$
$$+ (c^{(1)}(t)u^{(1)} - c^{(2)}(t)u^{(2)})\tilde{u} + b(x,y)(\tilde{u})^2 + (g(x,y,t,u^{(1)}) -$$
$$- g(x,y,t,u^{(2)}))\tilde{u}\Big) \, dxdydt = \int_{Q_T} \sum_{i=1}^s f_i(x,y,t) \tilde{q}_i(t) \tilde{u} \, dxdydt.$$

From here by the same way as from (5.15) we got (5.20), we find the following estimate:

$$\int_{Q_T} (\tilde{u})^2 dxdydt \le M_6 \int_0^T \Big((\tilde{c}(t))^2 + \sum_{i=1}^s (\tilde{q}_i(t))^2\Big) dt \tag{5.38}$$

and, taking into account (5.37), from (5.38) we obtain

$$(1 - M_7) \int_{Q_T} (\tilde{u})^2 dxdydt \le 0.$$

Since $M_7 < 1$, we conclude that $u^{(1)} = u^{(2)}$ in Q_T. Then (5.37) imply $c^{(1)}(t) \equiv c^{(2)}(t)$, $q_i^{(1)}(t) \equiv q_i^{(2)}(t)$, $i = 1, \dots, s$. □

6. Conclusion

We obtained the sufficient conditions of the unique solvability for the inverse problem for higher order semilinear ultraparabolic equation with the unknown time dependent functions in the minor coefficient and in the right-hand side function of the equation.

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