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EFFICIENT ESTIMATE OF THE REMAINDER FOR THE DIRICHLET FUNCTION $\eta(p)$ FOR $p \in \mathbb{R}^+$

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Abstract. For any $k,n,q \in \mathbb{N}$, where $n \geq 2k+1 \geq 3$, and $p \in \mathbb{R}^+$ the nth remainder $\rho(n,p)$ of the Dirichlet eta function $\eta(p)$, known also as the alternating Riemann zeta function,

$$\eta(p) := \sum_{i=1}^{\infty} (-1)^{i+1} \frac{1}{i^p},$$

is given in the form

$$\rho(n,p) = \Delta_q(n,p) + \delta_q(k,p),$$

where k and q are parameters controlling the magnitude of the error term $\delta_q(k,p)$. The function $\Delta_q(n,p)$ consists of $\lfloor q/2 \rfloor + 1$ simple summands and $\delta_q(k,n,p)$ is estimated as

$$\left|\delta_q(k,p)\right| < \frac{5}{3} \cdot \frac{p^{(q-1)}}{\pi^{q-1}} \cdot \frac{1}{(2k)^{p+q-1}},$$

where $p^{(q-1)}$ is the rising Pochhammer product.

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1. Introduction

To obtain the numerical values of the sequence $n \mapsto H^*\left(n, \frac{1}{1000}\right)$,

$$\begin{aligned} H^*\left(n, \tfrac{1}{1000}\right) := \sum_{j=1}^{n} (-1)^{j+1} \frac{1}{j^{1/1000}} \\ = 1.0000\ldots - 0.9993\ldots + 0.9989\ldots - 0.9986\ldots \\ + 0.9983 - 0.9982\ldots + \cdots + \frac{(-1)^{n+1}}{n^{1/1000}} \quad (n \in \mathbb{N}), \end{aligned}$$

the Dirichlet eta function,[1]

$$\eta(p) := \sum_{j=1}^{\infty} (-1)^{j+1} \frac{1}{j^p} \quad (p > 0),$$

is not a very useful tool although $H^*\big(n, \frac{1}{1000}\big) \approx \lim\limits_{n\to\infty} H^*\big(n, \frac{1}{1000}\big) = \eta\big(\frac{1}{1000}\big) \approx \lim\limits_{p\downarrow 0} \eta(p) = \frac{1}{2}$ [1, Proposition 5.3]. The approximation $H^*\big(n, \frac{1}{1000}\big) \approx \eta\big(\frac{1}{1000}\big)$ is good only for "very large" n due to slow convergence of $H^*\big(n, \frac{1}{1000}\big)$ illustrated[2] in Figure 1.

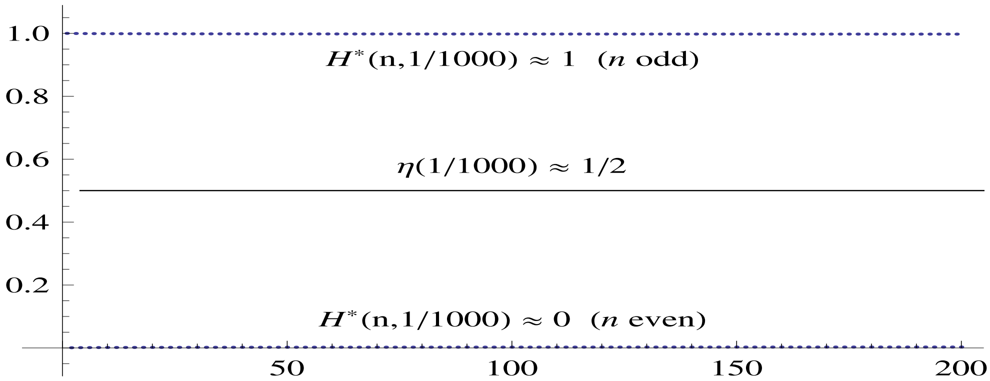

FIGURE 1. The graph of the sequence $n \mapsto H^*\big(n, \frac{1}{1000}\big)$.

We also remark that for any (fixed) $n \in \mathbb{N}$ we have

$$\lim_{p\downarrow 0} \sum_{j=1}^{n} \frac{(-1)^{j+1}}{j^p} = \lim_{p\downarrow 0} \begin{cases} 1 - \sum_{i=1}^{(n-1)/2} \left(\frac{1}{(2i)^p} - \frac{1}{(2i+1)^p} \right), & n \text{ odd} \\ \sum_{i=1}^{n/2} \left(\frac{1}{(2i-1)^p} - \frac{1}{(2i)^p} \right), & n \text{ even} \end{cases} = \begin{cases} 1, & n \text{ odd} \\ 0, & n \text{ even.} \end{cases}$$

Motivated by the results obtained by L. Tóth – J. Bukor [5], V. Timofte [4], and the Sîntămărian's papers [2, 3], we shall accurately estimate the remainder

$$\rho(n,p) := \eta(p) - H^*(n,p) \quad (n \in \mathbb{N},\ p \in \mathbb{R}^+), \tag{1.1}$$

where

$$H^*(n,p) := \sum_{j=1}^{n} (-1)^{j+1} \frac{1}{j^p} \tag{1.2}$$

is called the nth alternating generalized harmonic number of order p.

[1] known also as the alternating Riemann zeta function

[2] In this article all figures are plotted by Stephen Wolfram's Mathematica Software, Version 7.0.

2. Auxiliary results

In the sequel we shall use special sums[3]

$$\sigma_q^*(x,p) := \sum_{i=1}^{\lfloor q/2 \rfloor} \left(4^i - 1\right) \frac{B_{2i} \cdot p^{(2i-1)}}{x^{p+2i-1} \cdot (2i)!} \quad (q \in \mathbb{N},\, p, x \in \mathbb{R}), \tag{2.1}$$

where B_k is the kth Bernoulli coefficient (or Bernoulli number)[4], defined by the identity $\frac{t}{e^t - 1} \equiv \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}$ $(|t| < 2\pi)$. Here, for $x \in \mathbb{R}$ and $k \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$ the symbol $x^{(k)}$ means the upper (rising) Pochhammer product defined as

$$x^{(0)} := 1 \quad \text{and} \quad x^{(k)} := \prod_{i=0}^{k-1} (x+i) = x(x+1) \cdot \ldots \cdot (x+k-1) \quad (k \in \mathbb{N}). \tag{2.2}$$

According to [1, Theorem 3.2] there holds the following lemma.

Lemma 1. *For $p \in \mathbb{R}^+$ and for any $k, n, q \in \mathbb{N}$, where $n \geq 2k+1 \geq 3$, we have*

$$H^*(n,p) = H_q^*(k,n,p) + r_q^*(k,p), \tag{2.3}$$

where

$$H_q^*(k,n,p) := A_q^*(k,p) + B_q^*(n,p), \tag{2.4}$$

$$A_q^*(k,p) := H^*(2k-1,p) - \frac{1}{2\left(2k\right)^p} - \sigma_q^*\left(2k,p\right), \tag{2.5}$$

$$B_q^*(n,p) := \frac{(-1)^{n+1}}{2\left(2\lfloor \frac{n+1}{2} \rfloor\right)^p} + \sigma_q^*\left(2\lfloor \tfrac{n+1}{2} \rfloor, p\right), \tag{2.6}$$

$$\left|r_q^*(k,p)\right| \leq + \frac{p^{(q-1)} \cdot \pi^{p+1}}{3(2k\pi)^{p+q-1}} \tag{2.7}$$

$$. \tag{2.8}$$

Referring to [1, Theorem 5.2] we have also the next lemma.

Lemma 2. *For $p \in \mathbb{R}^+$ and for any $k, q \in \mathbb{N}$, there holds the following relation (see (2.1)),*

$$\eta(p) = H^*(2k-1,p) - \frac{1}{2\left(2k\right)^p} - \sigma_q^*\left(2k,p\right) + \varepsilon_q(k,p), \tag{2.9}$$

where

$$\left|\varepsilon_q(k,p)\right| < \frac{p^{(q-1)}}{3\pi^{q-2}(2k)^{p+q-1}}. \tag{2.10}$$

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[3] By definition, $\sum_{i=m}^{n} x_i = 0$ if $m > n$.

[4] $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = B_5 = B_7 = \ldots = 0$, $B_4 = B_8 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$, ...

3. The main result

From Lemmas 2 and 1 immediately follows the next result.

Theorem 1. *For $p \in \mathbb{R}^+$ and every $k,n,q \in \mathbb{N}$, with $n \geq 2k+1 \geq 3$, the nth remainder $\rho(n,p) := \eta(p) - \sum_{j=1}^{n}(-1)^{j+1}\frac{1}{j^p}$ is given in the form*

$$\rho(n,p) = \Delta_q(n,p) + \delta_q(k,p), \tag{3.1}$$

where

$$\Delta_q(n,p) = \frac{(-1)^n}{2\left(2\lfloor\frac{n+1}{2}\rfloor\right)^p} - \sigma_q^*\left(2\lfloor\tfrac{n+1}{2}\rfloor, p\right), \tag{3.2}$$

$$\left|\delta_q(k,p)\right| < \frac{5p^{(q-1)}}{3\pi^{q-1}} \cdot \frac{1}{(2k)^{p+q-1}}. \tag{3.3}$$

Proof. We have

$$\begin{aligned}
\rho(n,p) &= \eta(p) - H^*(n,p) \\
&= \left(\cancel{H^*(2k-1,p)} - \frac{1}{2(2k)^p} - \cancel{\sigma_q^*(2k,p)} + \varepsilon_q(k,p)\right) \\
&\quad - \left(\cancel{H^*(2k-1,p)} - \frac{1}{2(2k)^p} - \cancel{\sigma_q^*(2k,p)} + \frac{(-1)^{n+1}}{2\left(2\lfloor\frac{n+1}{2}\rfloor\right)^p}\right. \\
&\qquad \left. + \sigma_q^*\left(2\lfloor\tfrac{n+1}{2}\rfloor, p\right) + r_q^*(k,p)\right) \\
&= \varepsilon_q(k,p) - \frac{(-1)^{n+1}}{2\left(2\lfloor\frac{n+1}{2}\rfloor\right)^p} - \sigma_q^*\left(2\lfloor\tfrac{n+1}{2}\rfloor, p\right) - r_q^*(k,p),
\end{aligned}$$

where

$$\begin{aligned}
\left|\delta_q(k,p)\right| &= \left|\varepsilon_q(k,p) - r_q^*(k,p)\right| \\
&< \frac{p^{(q-1)}}{3\pi^{q-2}(2k)^{p+q-1}} + \frac{p^{(q-1)}}{3(2k\pi)^{p+q-1}} \\
&= \frac{p^{(q-1)}}{3\pi^{q-2}(2k)^{p+q-1}}\left(1 + \pi^{-(p+1)}\right) \\
&< \frac{p^{(q-1)}}{3\pi^{q-2}(2k)^{p+q-1}} \cdot \frac{5}{\pi}.
\end{aligned}$$

□

In our Theorem k and q are parameters controlling the magnitude of the error term $\delta_q(k,p) = \varepsilon_q(k,p) - r_q(k,p)$. We add that, for $0 < p \leq 1$ and $q \in \mathbb{N}$, using the Stirling factorial formula, we estimate

$$p^{(q-1)} = p(p+1)\cdots(p+q-2) \leq (q-1)! = \frac{q!}{q} < \frac{1}{q}\left(\frac{q}{e}\right)^q \sqrt{2\pi q} \cdot e^{\frac{1}{12q}}.$$

Thus, for $n \ge 2k+1 \ge 3$ and $0 < p \le 1$, we obtain the estimates

$$\left|\delta_q(k,p)\right| < \frac{5p^{(q-1)}}{3\pi^{q-1}} \cdot \frac{1}{(2k)^{p+q-1}} < \frac{5}{3}\sqrt{\frac{2\pi}{q}}\left(\frac{q}{e}\right)^q e^{\frac{1}{12q}} \cdot \frac{1}{(2\pi k)^{q-1}}$$
$$= \frac{5}{3e}\sqrt{2\pi q}\left(\frac{q}{2e\pi k}\right)^{q-1} e^{\frac{1}{12q}} < 2\sqrt{q}\left(\frac{q}{2e\pi k}\right)^{q-1}. \tag{3.4}$$

Figure 2 shows the graph of the sequence $n \mapsto \Delta_5(n, 1/1000)$.

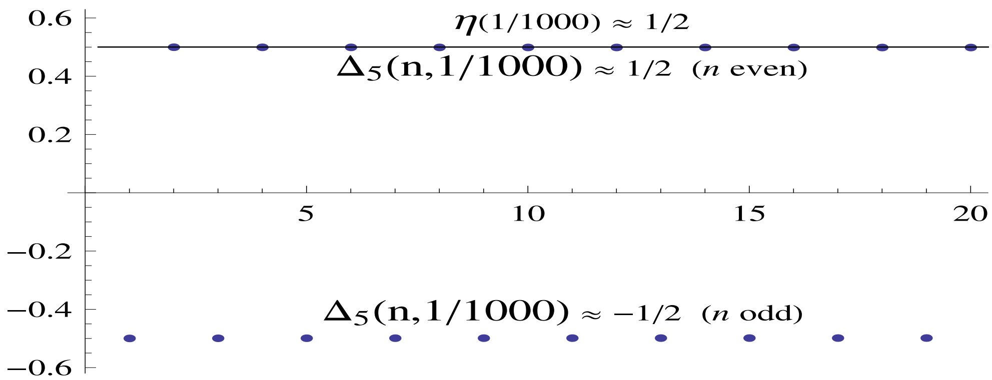

FIGURE 2. The graph of the sequence $n \mapsto \Delta_5\left(n, \frac{1}{1000}\right)$.

Since $\sigma_1(x,p) \equiv 0$, we obtain from Theorem the first corollary.

Corollary 1. *For $p \in \mathbb{R}^+$ and any $k,n \in \mathbb{N}$, with $n \ge 2k+1 \ge 3$, we have*

$$\sum_{j=1}^{n}(-1)^{j+1}\frac{1}{j^p} + \frac{(-1)^n}{2\left(2\lfloor\frac{n+1}{2}\rfloor\right)^p} - \frac{5}{3(2k)^p}$$
$$< \eta(p) < \sum_{j=1}^{n}(-1)^{j+1}\frac{1}{j^p} + \frac{(-1)^n}{2\left(2\lfloor\frac{n+1}{2}\rfloor\right)^p} + \frac{5}{3(2k)^p}.$$

Setting $q = 5$ in Theorem we obtain the following result.

Corollary 2. *For $p \in \mathbb{R}^+$ and $k,n \in \mathbb{N}$, such that $n \ge 2k+1 \ge 3$, the following equation follows*

$$\eta(p) = \sum_{j=1}^{n}(-1)^{j+1}\frac{1}{j^p} + \frac{(-1)^n}{2\left(2\lfloor\frac{n+1}{2}\rfloor\right)^p} - \frac{p}{4\left(2\lfloor\frac{n+1}{2}\rfloor\right)^{p+1}}$$
$$+ \frac{p(p+1)(p+2)}{48\left(2\lfloor\frac{n+1}{2}\rfloor\right)^{p+3}} + \delta_5(k,p),$$

where

$$\left|\delta_5(k,p)\right| < \frac{5p(p+1)(p+2)(p+3)}{3\pi^4\left(2k\right)^{p+4}}\,.$$

Example 1. Since $\left|\delta_5(10,0.001)\right| < 10^{-9}$ the approximation

$$\eta(0.001)\approx S_n^* := \sum_{j=1}^{n}(-1)^{j+1}\frac{1}{j^{0.001}} + \frac{(-1)^n}{2\left(2\lfloor\frac{n+1}{2}\rfloor\right)^{0.001}} - \frac{0.001}{4\left(2\lfloor\frac{n+1}{2}\rfloor\right)^{1.001}} + \frac{0.001\times 1.001\times 2.001}{48\left(2\lfloor\frac{n+1}{2}\rfloor\right)^{3.001}}$$

is very accurate for every $n \geq 21$ as is seen from Figure 3 where are plotted the graphs of the sequence $n \mapsto S_n^*$ and the constant function $n \mapsto \eta(0.001) = H^*(\infty,0.001) = 0.5002257\ldots$. The corrected term $\Delta_5(n,0.001)$ is crucial since, for example, we have

$H^*(10000\mathbf{0},0.001) = 0.005949215\ldots$, $H^*(10000\mathbf{1},0.001) = 0.994502300\ldots$, $\Delta_5(10000\mathbf{0},0.001) = 0.494276\mathbf{544}\ldots$ and $\Delta_5(10000\mathbf{1},0.001) = -0.494276\mathbf{539}\ldots$.

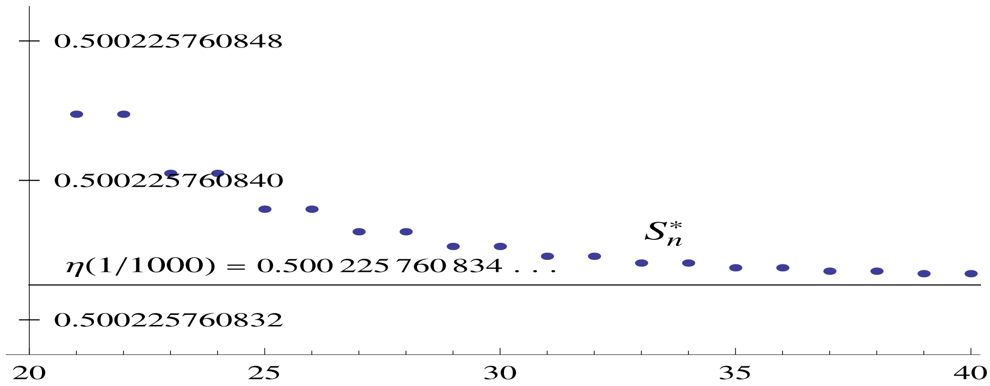

FIGURE 3. The graph of the sequence $n \mapsto S_n^*$.

Obviously, using the approximation $\eta(0.001) \approx S_n^*$ we directly obtain accurate approximation of $H^*\left(n,\frac{1}{1000}\right)$ from Introduction:

$$H^*\left(n,\frac{1}{1000}\right) \approx \eta(0.001) + \frac{(-1)^{n+1}}{2\left(2\lfloor\frac{n+1}{2}\rfloor\right)^{0.001}} + \frac{0.001}{4\left(2\lfloor\frac{n+1}{2}\rfloor\right)^{1.001}} - \frac{0.001\times 1.001\times 2.001}{48\left(2\lfloor\frac{n+1}{2}\rfloor\right)^{3.001}} \qquad (n\in\mathbb{N}).$$

CONCLUSION

The paper provides an estimate of the remainder of the series representing Dirichlet's function $\eta(p)$ for $p \in \mathbb{R}^+$. The presented estimate is effective also in the case of slowly convergent series occurring for $p \approx 0$.

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