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SOLUTIONS OF HOMOGENEOUS FRACTIONAL p-KIRCHHOFF EQUATIONS IN $\mathbb{R}^N$

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Abstract. In this note, we furnish a transformation such that solutions of the fractional p-Kirchhoff equation in $\mathbb{R}^N$ are easily obtained from known solutions of the corresponding fractional p-Laplace equation. As an application, we classify all positive solutions of some (fractional) p-Kirchhoff equations with sub-critical or critical nonlinearities and Hénon-Hardy potentials. Similar results for Kirchhoff type systems are also discussed.

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1. INTRODUCTION AND MAIN RESULTS

Throughout this note, we always assume that $N \geq 1$ and $M : [0,+\infty) \to [0,+\infty)$. We also suppose that $0 < s \leq 1 < p$ and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is measurable and homogeneous in the first variable. More precisely,

(F) $f(x,u) = |x|^\alpha f\left(\frac{x}{|x|}, u\right)$ for some $\alpha \neq -ps$ and all $(x,u) \in \mathbb{R}^N \setminus \{0\} \times \mathbb{R}$.

We study solutions of the following nonlocal equation

$$M\left([u]_{s,p}^p\right)(-\Delta)_p^s u = f(x,u) \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where

$$[u]_{1,p} = \left(\int_{\mathbb{R}^N} |\nabla u|^p \, dx\right)^{1/p},$$

$$(-\Delta)_p^1 u(x) = -\operatorname{div}(|\nabla u(x)|^{p-2} \nabla u(x))$$

and for $s \in (0,1)$,

$$[u]_{s,p} = \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} \, dx dy\right)^{1/p},$$

$$(-\Delta)_p^s u(x) = \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}[u(x)-u(y)]}{|x-y|^{N+ps}} \, dy,$$

up to normalization factors, here $B_\varepsilon(x)$ is the ball centered at $x \in \mathbb{R}^N$ with radius $\varepsilon > 0$. We also denote

$$\langle u, \varphi \rangle_{1,p} = \int_{\mathbb{R}^N} |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla \varphi(x)\, dx$$

and

$$\langle u, \varphi \rangle_{s,p} = \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}[u(x)-u(y)][\varphi(x)-\varphi(y)]}{|x-y|^{N+ps}}\, dx dy$$

for $s \in (0,1)$. Note that $[u]_{s,p}^p = \langle u, u \rangle_{s,p}$. For the sake of simplicity and usual convention, we also denote $-\Delta_p u = (-\Delta)_p^1 u$ and $(-\Delta)^s u = (-\Delta)_2^s u$.

A general functional framework for (1.1) is the local (fractional) Sobolev space $W^{s,p}_{\mathrm{loc}}(\mathbb{R}^N)$ consisting of all measurable function u on $\mathbb{R}^N$ such that $u\varphi \in W^{s,p}(\mathbb{R}^N)$ for all $\varphi \in C_c^\infty(\mathbb{R}^N)$, where $W^{s,p}(\mathbb{R}^N) = \{u \in L^p(\mathbb{R}^N) : [u]_{s,p} < \infty\}$ and $C_c^\infty(\mathbb{R}^N)$ is the space of smooth functions with compact support in $\mathbb{R}^N$.

A function $u \in W^{s,p}_{\mathrm{loc}}(\mathbb{R}^N)$ is called a solution of (1.1) if $[u]_{s,p} < \infty$, $f(x,u) \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and

$$M\left([u]_{s,p}^p\right) \langle u, \varphi \rangle_{s,p} = \int_{\mathbb{R}^N} f(x, u(x)) \varphi(x)\, dx \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^N).$$

Equation (1.1) is analogous to the stationary problem of a physical model which were first introduced by Kirchhoff [10] to describe the transversal oscillations of elastic strings. This type of problems received much attention of several researchers after the work of Lions [13], where a functional analysis framework was proposed to attack it.

Due to the presence of the nonlocal term $M([u]_{s,p}^p)$, problem (1.1) is no longer a pointwise identity even if $s = 1$. This phenomenon causes some mathematical difficulties which make the study of such problems particularly interesting. In the last decade, working directly with nonlocal term via variational methods, several authors have established many interesting results about the existence and nonexistence of positive solutions, sign-changing solutions, ground state solutions, least energy nodal solutions, multiplicity of solutions, semi-classical limit and concentrations of solutions to Kirchhoff and p-Kirchhoff problems, see e.g. [5, 8, 11] and the references therein. Recently, many authors also study fractional p-Kirchhoff problems (1.1) when $s \in (0,1)$. Variational results for these and related problems are established in [6, 18, 19, 27] and references therein.

There is, however, another simple method that helps eliminate nonlocal term of some Kirchhoff equations in $\mathbb{R}^N$. In [8], the authors used a transformation that allows them to obtain solutions of the autonomous Kirchhoff equation

$$-M\left(\int_{\mathbb{R}^N} |\nabla u|^2\, dx\right) \Delta u = h(u) \quad \text{in } \mathbb{R}^N$$

from the corresponding local equation. In this note we will extends this transformation to fractional p-Kirchhoff equation (1.1) with homogeneous nonlinearity $f(x,u)$. This method allows us to reduce (1.1) to its corresponding fractional p-Laplace equation

$$(-\Delta)_p^s u = f(x,u) \quad \text{in } \mathbb{R}^N. \tag{1.2}$$

For any function u on $\mathbb{R}^N$ and $\lambda > 0$, we denote $u^\lambda(x) \equiv u(\frac{x}{\lambda})$. Our first result is the following theorem, which extends [8, Theorem 1-3].

Theorem 1. *Assume that $0 < s \leq 1 < p$ and f satisfies (F). Then u is a solution of (1.1) if and only if $M\left([u]_{s,p}^p\right) = 0$ and $f(x,u) = 0$ a.e. in $\mathbb{R}^N$, or $u = u_0^\lambda$, where u_0 is a solution of (1.2) and $\lambda > 0$ is a solution of the algebraic equation $M(\lambda^{N-ps}[u_0]_{s,p}^p) = \lambda^{\alpha+ps}$.*

Our second theorem is an analogous result for the Kirchhoff type system

$$\begin{cases} M_1\left([u]_{s_1,p_1}^{p_1}\right)(-\Delta)_{p_1}^{s_1} u = f_1(x,u,v) & \text{in } \mathbb{R}^N, \\ M_2\left([v]_{s_2,p_2}^{p_2}\right)(-\Delta)_{p_2}^{s_2} v = f_2(x,u,v) & \text{in } \mathbb{R}^N. \end{cases} \tag{1.3}$$

We say that $(u,v) \in W_{\text{loc}}^{s_1,p_1}(\mathbb{R}^N) \times W_{\text{loc}}^{s_2,p_2}(\mathbb{R}^N)$ is a solution of (1.3) if $[u]_{s_1,p_1} < \infty$, $[v]_{s_2,p_2} < \infty$, $f_1(x,u,v) \in L_{\text{loc}}^1(\mathbb{R}^N)$, $f_2(x,u,v) \in L_{\text{loc}}^1(\mathbb{R}^N)$ and

$$\begin{cases} M_1\left([u]_{s_1,p_1}^{p_1}\right)\langle u,\varphi\rangle_{s_1,p_1} = \int_{\mathbb{R}^N} f_1(x,u(x),v(x))\varphi(x)\,dx, \\ M_2\left([v]_{s_2,p_2}^{p_2}\right)\langle v,\varphi\rangle_{s_2,p_2} = \int_{\mathbb{R}^N} f_2(x,u(x),v(x))\varphi(x)\,dx, \end{cases}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^N)$.

The corresponding system of (1.3) when $M_1 \equiv M_2 \equiv 1$ is

$$\begin{cases} (-\Delta)_{p_1}^{s_1} u = f_1(x,u,v) & \text{in } \mathbb{R}^N, \\ (-\Delta)_{p_2}^{s_2} v = f_2(x,u,v) & \text{in } \mathbb{R}^N. \end{cases} \tag{1.4}$$

Theorem 2. *Assume that $0 < s_1, s_2 \leq 1 < p_1, p_2$, $M_1, M_2 : [0,+\infty) \to [0,+\infty)$, f_1, f_2 are measurable and $f_1(x,u,v) = |x|^{\alpha_1} f_1\left(\frac{x}{|x|},u,v\right)$, $f_2(x,u,v) = |x|^{\alpha_2} f_2\left(\frac{x}{|x|},u,v\right)$ for some $\alpha_1,\alpha_2 \in \mathbb{R}$ and all $(x,u,v) \in \mathbb{R}^N \setminus \{0\} \times \mathbb{R} \times \mathbb{R}$. Then $(u_0^\lambda, v_0^\lambda)$ is a solution of (1.3) if (u_0,v_0) is a solution of (1.4) and $\lambda > 0$ is a solution of the algebraic system*

$$\begin{cases} M_1(\lambda^{N-p_1 s_1}[u_0]_{s_1,p_1}^{p_1}) = \lambda^{\alpha_1+p_1 s_1}, \\ M_2(\lambda^{N-p_2 s_2}[v_0]_{s_2,p_2}^{p_2}) = \lambda^{\alpha_2+p_2 s_2}. \end{cases}$$

We cannot obtain all solutions of (1.3) from known solutions of (1.4). However, when f_1 and f_2 have power growth in u and v, we are able to do that by using another transformation. Indeed, let us consider the following Kirchhoff type system

$$\begin{cases} M_1\left([u]_{s_1,p_1}^{p_1}\right)(-\Delta)_{p_1}^{s_1} u = w_1(x) u^{q_1} v^{r_1} & \text{in } \mathbb{R}^N, \\ M_2\left([v]_{s_2,p_2}^{p_2}\right)(-\Delta)_{p_2}^{s_2} v = w_2(x) u^{r_2} v^{q_2} & \text{in } \mathbb{R}^N \end{cases} \tag{1.5}$$

and its corresponding local system

$$\begin{cases} (-\Delta)_{p_1}^{s_1} u = w_1(x) u^{q_1} v^{r_1} & \text{in } \mathbb{R}^N, \\ (-\Delta)_{p_2}^{s_2} v = w_2(x) u^{r_2} v^{q_2} & \text{in } \mathbb{R}^N. \end{cases} \tag{1.6}$$

We have the following.

Theorem 3. *Assume that* $0 < s_1, s_2 \le 1 < p_1, p_2$*,* $M_1, M_2 : [0,+\infty) \to [0,+\infty)$*,* $w_1, w_2 : \mathbb{R}^N \to (0,+\infty)$ *are measurable and* $q_1, q_2, r_1, r_2 \in \mathbb{R}$ *such that* $(q_1 - p_1 + 1)(q_2 - p_2 + 1) \neq r_1 r_2$*. Then* (u,v) *is a positive solution of* (1.5) *if and only if* $(u,v) = (\lambda_1 u_0, \lambda_2 v_0)$*, where* (u_0, v_0) *is a positive solution of* (1.6) *and* (λ_1, λ_2) *is a positive solution of the algebraic system*

$$\begin{cases} M_1(\lambda_1^{p_1}[u_0]_{s_1,p_1}^{p_1}) = \lambda_1^{q_1-p_1+1}\lambda_2^{r_1}, \\ M_2(\lambda_2^{p_2}[v_0]_{s_2,p_2}^{p_2}) = \lambda_1^{r_2}\lambda_2^{q_2-p_2+1}. \end{cases} \tag{1.7}$$

Remark 1. It should be noted that in our results we do not assume any condition on M, M_1, M_2 except their non-negativity. Our method can also be applied to fractional p-Kirchhoff type inequalities and poly-harmonic equations of p-Kirchhoff type.

The proofs of our theorems are given in Section 3. In the next section, we utilize these theorems to classify all positive solutions of some fractional p-Kirchhoff equations and systems having power growth and Hénon-Hardy potentials.

2. Classification of positive solutions

We consider the p-Laplace equation of Hénon-Lane-Emden type

$$-\Delta_p u = |x|^\alpha u^q \quad \text{in } \mathbb{R}^N. \tag{2.1}$$

Before stating our classification results, let us introduce the following exponents

$$q_S(p,\alpha) = \begin{cases} \frac{(p-1)N+p\alpha+p}{N-p} & \text{if } N > p, \\ \infty & \text{if } N \le p \end{cases}$$

and

$$q_S(p) = q_S(p,0).$$

When $N > p > -\alpha$ and $q = q_S(p,\alpha)$, solutions of (2.1) may be found by minimizing $\int_{\mathbb{R}^N} |\nabla u|^p \, dx$ over the manifold $\mathcal{N}(p,\alpha) = \{u \in W^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |x|^\alpha |u|^{\frac{p(N+\alpha)}{N-p}} dx = 1\}$. Let us denote

$$S(p,\alpha) = \inf_{u \in \mathcal{N}(p,\alpha)} \left(\int_{\mathbb{R}^N} |\nabla u|^p \, dx \right)^{\frac{N+\alpha}{p+\alpha}}$$

and

$$S(p) = S(p,0).$$

By a standard argument, solution w of (2.1) obtained from minimizers of $S(p,\alpha)$ satisfies $\int_{\mathbb{R}^N} |\nabla w|^p\,dx = \int_{\mathbb{R}^N} |x|^\alpha |w|^{\frac{p(N+\alpha)}{N-p}}\,dx = S(p,\alpha)$.

We consider equation (2.1) in the autonomous case $\alpha = 0$. Serrin and Zou [23, Corollary II] proved that (2.1) has no positive solution if $\alpha = 0$, $p > 1$ and $q < q_S(p)$. Therefore, utilizing Theorem 1, we have the following nonexistence result.

Proposition 1. *If $p > 1$ and $q < q_S(p)$, then the equation*

$$-M\left(\int_{\mathbb{R}^N} |\nabla u|^p\,dx\right)\Delta_p u = u^q \quad \text{in } \mathbb{R}^N$$

has no positive $C^1(\mathbb{R}^N)$ solution.

In critical case $q = q_S(p) = \frac{(p-1)N+p}{N-p}$ where $1 < p < N$ and $\alpha = 0$, all positive $\mathcal{D}^{1,p}(\mathbb{R}^N)$ solutions of (2.1) have been recently classified by Vétois [26] (for $1 < p < 2$) and Sciunzi [20] (for $2 < p < N$). See also [2] for classical result in the case $p = 2$. They proved that all positive solutions of (2.1) under these assumptions are of the form

$$U_{p,\mu,x_0}(x) = \left(\frac{\mu^{\frac{1}{p-1}} N^{\frac{1}{p}} \left(\frac{N-p}{p-1}\right)^{\frac{p-1}{p}}}{\mu^{\frac{p}{p-1}} + |x-x_0|^{\frac{p}{p-1}}}\right)^{\frac{N-p}{p}},$$

for $\mu > 0$ and $x_0 \in \mathbb{R}^N$, see [20, Theorem 1.1]. We also know that $\int_{\mathbb{R}^N} |\nabla U_{p,\mu,x_0}|^p\,dx = S(p)$. From this fact and Theorem 1, we have the following result, which extends [14, Theorem 1.1 and 1.2] to p-Kirchhoff case.

Proposition 2. *If $1 < p < N$, then every positive $\mathcal{D}^{1,p}(\mathbb{R}^N)$ solution u of the equation*

$$-M\left(\int_{\mathbb{R}^N} |\nabla u|^p\,dx\right)\Delta_p u = u^{\frac{(p-1)N+p}{N-p}} \quad \text{in } \mathbb{R}^N$$

must have the form $u(x) = U_{p,\mu,x_0}(\frac{x}{\lambda})$, where $\mu > 0, x_0 \in \mathbb{R}^N$ and $\lambda > 0$ is a solution of

$$M(\lambda^{N-p} S(p)) = \lambda^p. \tag{2.2}$$

Therefore, if (2.2) has no positive solution, then the above p-Kirchhoff equation has no positive solution.

We now turn our attention to the Hénon case $\alpha \geq 0$. When $p = 2$ and $1 < q < q_S(2,\alpha)$, the nonexistence of positive $H^1_{loc}(\mathbb{R}^N) \cap L^\infty_{loc}(\mathbb{R}^N)$ solutions of (2.1) was recently obtained in [7, Theorem 1]. The optimal nonexistence result for the case $p \neq 2$ is still an open problem. There are some partial results anyway. This type of nonexistence result was proved in [9, Section 3] for radial positive solutions and

$q < q_S(p,\alpha)$, in [16, Theorem 12.4] for $q \le \frac{(p-1)(N+\alpha)}{N-p}$ ($q < \infty$ if $N \le p$), in [17, Theorem 1.2] for $q < q_S(p,\alpha)$, $N < p+1$ and in [17, Theorem 1.3] for $q < q_S(p)$. Combining these facts with Theorem 1, we have the following.

Proposition 3. *Let $\alpha \ge 0$ and $q > p-1$, the p-Kirchhoff equation*

$$-M\left(\int_{\mathbb{R}^N} |\nabla u|^p \, dx\right) \Delta_p u = |x|^\alpha u^q \quad \text{in } \mathbb{R}^N$$

has no positive $C^1(\mathbb{R}^N)$ solution u under one of the following assumptions

(i) $p = 2$ *and* $q < q_S(2,\alpha)$,
(ii) $q \le \frac{(p-1)(N+\alpha)}{N-p}$ ($q < \infty$ *if* $N \le p$),
(iii) $q < q_S(p,\alpha)$ *and u is radial,*
(iv) $q < q_S(p,\alpha)$ *and* $N < p+1$,
(v) $q < q_S(p)$.

Recently, all positive solutions of class $H^1_{loc}(\mathbb{R}^N) \cap L^\infty_{loc}(\mathbb{R}^N)$ to equation (2.1) when $p = 2$, $-2 < \alpha < 0$, $N \ge 3$ and $q = q_S(2,\alpha) = \frac{N+2\alpha+2}{N-2}$ was classified in [7, Theorem 2]. It was proved that all such solutions must have the form

$$V_{\alpha,\mu}(x) = \left(\frac{N(N-2)(\alpha+2)^2}{4}\right)^{\frac{N-2}{4}} \left(\frac{\mu}{\mu^2 + |x|^{\alpha+2}}\right)^{\frac{N-2}{\alpha+2}}$$

for $\mu > 0$. It is also clear that $\int_{\mathbb{R}^N} |\nabla V_{\alpha,\mu}|^2 \, dx = S(2,\alpha)$. From this and Theorem 1, we have the following proposition, which extends [14, Theorem 1.1 and 1.2] to Hardy case.

Proposition 4. *If $N \ge 3$ and $-2 < \alpha < 0$, then every positive $H^1_{loc}(\mathbb{R}^N) \cap L^\infty_{loc}(\mathbb{R}^N)$ solution u of the equation*

$$-M\left(\int_{\mathbb{R}^N} |\nabla u|^2 \, dx\right) \Delta u = |x|^\alpha u^{\frac{N+2\alpha+2}{N-2}} \quad \text{in } \mathbb{R}^N$$

must have the form $u(x) = V_{\alpha,\mu}(\frac{x}{\lambda})$ where $\mu > 0$ and $\lambda > 0$ is a solution of

$$M(\lambda^{N-2} S(2,\alpha)) = \lambda^{\alpha+2}. \tag{2.3}$$

Therefore, if (2.3) has no positive solution, then the above Kirchhoff equation has no positive solution.

By similar arguments, we may obtain some classification results for fractional Kirchhoff problems. Let us denote

$$\mathcal{L}_\alpha = \left\{ u : \mathbb{R}^n \to \mathbb{R} \,\middle|\, \int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+s}} dx < \infty \right\}.$$

The nonexistence of nontrivial nonnegative $C^{1,1}_{\text{loc}} \cap \mathcal{L}_\alpha$ solutions of $(-\Delta)^s u = u^q$ in $\mathbb{R}^N$ when $0 < s < 1$ and $1 < q < \frac{N+2s}{N-2s}$ was proved recently in [3, Theorem 4].

Moreover, if $q = \frac{N+2s}{N-2s}$, then nonnegative $C^{1,1}_{\mathrm{loc}} \cap \mathcal{L}_\alpha$ solutions of this equation are radially symmetric and hence must assume an explicit form. We may utilize these results and Theorem 1 to obtain

Proposition 5. *Assume $0 < s < 1 < q$ and $u \in C^{1,1}_{\mathrm{loc}} \cap \mathcal{L}_\alpha$ is a nonegative solution of the equation*

$$M\left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dxdy\right)(-\Delta)^s u = u^q \quad \text{in } \mathbb{R}^N.$$

Then

(i) *In the critical case $q = \frac{N+2s}{N-2s}$, $u \equiv 0$ or u assumes the form*

$$u(x) = c\left(\frac{\mu}{\mu^2 + |x-x_0|^2}\right)^{\frac{N-2s}{2}}, \quad c,\mu > 0, x_0 \in \mathbb{R}^N.$$

(ii) *In the subcritical case $1 < q < \frac{N+2s}{N-2s}$, $u \equiv 0$.*

To demonstrate an application of Theorem 3, let us now consider the Lane-Emden system

$$\begin{cases} -\Delta u = v^p & \text{in } \mathbb{R}^N, \\ -\Delta v = u^q & \text{in } \mathbb{R}^N. \end{cases} \tag{2.4}$$

The famous Lane-Emden conjecture states that: *If the positive pair (p,q) lies below the Sobolev critical hyperbola, i.e. if*

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N}, \tag{2.5}$$

then there is no classical positive solution to (2.4).

Up to now, the conjecture is proved to be true for radial functions by Mitidieri [15] and Serrin-Zou [21]. For the full conjecture, Souto [25], Mitidieri [15] and Serrin-Zou [22] proved that there is no supersolution to (2.4) if $pq \le 1$ or $\max\{\frac{2(p+1)}{pq-1}, \frac{2(q+1)}{pq-1}\} \ge N-2$. This solves the Lane-Emden conjecture in dimensions $N = 1,2$. More recently, the conjecture is proved in dimensions $N = 3,4$ by Souplet and his collaborators, see [24]. For $N \ge 5$, the conjecture is prove to be true for (p,q) verifying (2.5) and one of the following extra conditions:

- If $p,q < \frac{N+2}{N-2}$, see Felmer-de Figuereido [4].
- If $\max(p,q) \ge N-3$, see Souplet [24].
- If $\min\{\frac{2(p+1)}{pq-1}, \frac{2(q+1)}{pq-1}\} \ge \frac{N-2}{2}$, see Busca-Manásevich [1].
- If $p = 1$ or $q = 1$, see Lin [12].

We may utilize Theorem 3 and above facts to obtain an analogous result for the Kirchhoff-Lane-Emden system.

Proposition 6. *Assume that* $p,q>0$ *verify* (2.5) *and* $M_1, M_2 : [0,+\infty) \to [0,+\infty)$. *Then the Kirchhoff-Lane-Emden system*

$$\begin{cases} -M_1(\int_{\mathbb{R}^N} |\nabla u|^2\,dx)\Delta u = v^p & \text{in } \mathbb{R}^N, \\ -M_2(\int_{\mathbb{R}^N} |\nabla v|^2\,dx)\Delta v = u^q & \text{in } \mathbb{R}^N \end{cases}$$

has no classical positive solution (u,v) *under one of the following extra conditions*

(i) u,v *are radial,*
(ii) $N \le 4$,
(iii) $p,q < \frac{N+2}{N-2}$,
(iv) $\max\{p,q\} \ge N-3$,
(v) $\min\{\frac{2(p+1)}{pq-1}, \frac{2(q+1)}{pq-1}\} \ge \frac{N-2}{2}$,
(vi) $p=1$ *or* $q=1$.

Certainly, we can state the following open problem which is equivalent to the Lane-Emden conjecture by Theorem 3.

Open problem. Assume that $p,q>0$ verify (2.5) and $M_1, M_2 : [0,+\infty) \to [0,+\infty)$. Then the Kirchhoff-Lane-Emden system has no classical positive solution.

3. Proofs of main results

Proof of Theorem 1. For any u,φ and $\lambda>0$, we have

$$\langle u^\lambda, \varphi\rangle_{s,p} = \lambda^{N-ps}\langle u, \varphi^{1/\lambda}\rangle_{s,p}. \tag{3.1}$$

Indeed, if $s=1$, then

$$\begin{aligned} \langle u^\lambda, \varphi\rangle_{1,p} &= \lambda^{-p+1}\int_{\mathbb{R}^N} \left|\nabla u\left(\frac{x}{\lambda}\right)\right|^{p-2} \nabla u\left(\frac{x}{\lambda}\right)\cdot \nabla\varphi(x)\,dx \\ &= \lambda^{N-p+1}\int_{\mathbb{R}^N} |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla\varphi(\lambda x)\,dx \\ &= \lambda^{N-p}\langle u, \varphi^{1/\lambda}\rangle_{1,p}, \end{aligned}$$

while if $0<s<1$, then

$$\begin{aligned} \langle u^\lambda, \varphi\rangle_{s,p} &= \iint_{\mathbb{R}^{2N}} \frac{|u\left(\frac{x}{\lambda}\right) - u\left(\frac{y}{\lambda}\right)|^{p-2}[u\left(\frac{x}{\lambda}\right) - u\left(\frac{y}{\lambda}\right)][\varphi(x)-\varphi(y)]}{|x-y|^{N+ps}}\,dxdy \\ &= \lambda^{N-ps}\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}[u(x)-u(y)][\varphi(\lambda x)-\varphi(\lambda y)]}{|x-y|^{N+ps}}\,dxdy \\ &= \lambda^{N-ps}\langle u, \varphi^{1/\lambda}\rangle_{s,p}. \end{aligned}$$

Note that (3.1) also implies $[u^\lambda]^p_{s,p} = \lambda^{N-ps}[u]^p_{s,p}$.

Assume that u_0 is a solution of (1.2) and $\lambda > 0$ satisfies $M(\lambda^{N-ps}[u_0]_{s,p}^p) = \lambda^{\alpha+ps}$. Then $u = u_0^\lambda$ verifies $[u]_{s,p}^p = \lambda^{N-ps}[u_0]_{s,p}^p$ and for all $\varphi \in C_c^\infty(\mathbb{R}^N)$,

$$\begin{aligned} M\left([u]_{s,p}^p\right)\langle u,\varphi\rangle_{s,p} &= M\left(\lambda^{N-ps}[u_0]_{s,p}^p\right)\langle u_0^\lambda,\varphi\rangle_{s,p} \\ &= M\left(\lambda^{N-ps}[u_0]_{s,p}^p\right)\lambda^{N-ps}\langle u_0,\varphi^{1/\lambda}\rangle_{s,p} \\ &= M\left(\lambda^{N-ps}[u_0]_{s,p}^p\right)\lambda^{N-ps}\int_{\mathbb{R}^N} f(x,u_0(x))\varphi(\lambda x)\,dx \\ &= \lambda^\alpha \int_{\mathbb{R}^N} f\left(\frac{x}{\lambda},u_0\left(\frac{x}{\lambda}\right)\right)\varphi(x)\,dx \\ &= \int_{\mathbb{R}^N} f(x,u(x))\varphi(x)\,dx, \end{aligned}$$

where we have used the definition of λ and assumption (F) in the last two lines. Therefore, u is a solution of (1.1). On the other hand, if $M\left([u]_{s,p}^p\right) = 0$ and $f(x,u) = 0$ a.e. in $\mathbb{R}^N$, then clearly u is a solution of (1.1).

Conversely, assume that u is a solution of (1.1) and either $M\left([u]_{s,p}^p\right) > 0$ or $f(x,u) \neq 0$ in a subset of $\mathbb{R}^N$ with positive measure. Clearly, $M\left([u]_{s,p}^p\right) > 0$ in both cases. We define $u_0 = u^{1/\lambda}$ where $\lambda = M([u]_{s,p}^p)^{1/(\alpha+ps)}$. Then $u = u_0^\lambda$ and $[u]_{s,p}^p = \lambda^{N-ps}[u_0]_{s,p}^p$. Therefore, λ satisfies $M(\lambda^{N-ps}[u_0]_{s,p}^p) = \lambda^{\alpha+ps}$. For all $\varphi \in C_c^\infty(\mathbb{R}^N)$,

$$\begin{aligned} \langle u_0,\varphi\rangle_{s,p} = \langle u^{1/\lambda},\varphi\rangle_{s,p} &= \lambda^{-N+ps}\langle u,\varphi^\lambda\rangle_{s,p} \\ &= \lambda^{-N+ps}M^{-1}\left([u]_{s,p}^p\right)\int_{\mathbb{R}^N} f(x,u(x))\varphi\left(\frac{x}{\lambda}\right)dx \\ &= \lambda^{\alpha+ps}M^{-1}\left([u]_{s,p}^p\right)\int_{\mathbb{R}^N} f(x,u_0(x))\varphi(x)\,dx \\ &= \int_{\mathbb{R}^N} f(x,u_0(x))\varphi(x)\,dx, \end{aligned}$$

which means that u_0 is a solution of (1.2). □

Proof of Theorem 2. For all $\varphi \in C_c^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned} M_1\left([u_0^\lambda]_{s_1,p_1}^{p_1}\right)\langle u_0^\lambda,\varphi\rangle_{s_1,p_1} &= M_1\left(\lambda^{N-p_1s_1}[u_0]_{s_1,p_1}^{p_1}\right)\lambda^{N-p_1s_1}\langle u_0,\varphi^{1/\lambda}\rangle_{s_1,p_1} \\ &= \lambda^{N+\alpha_1}\int_{\mathbb{R}^N} f_1(x,u_0(x),v_0(x))\varphi(\lambda x)\,dx \\ &= \lambda^{\alpha_1}\int_{\mathbb{R}^N} f_1\left(\frac{x}{\lambda},u_0\left(\frac{x}{\lambda}\right),v_0\left(\frac{x}{\lambda}\right)\right)\varphi(x)\,dx \\ &= \int_{\mathbb{R}^N} f_1(x,u_0^\lambda(x),v_0^\lambda(x))\varphi(x)\,dx. \end{aligned}$$

Similarly,

$$M_2\left([v_0^\lambda]_{s_2,p_2}^{p_2}\right)\langle v_0^\lambda,\varphi\rangle_{s_2,p_2}=\int_{\mathbb{R}^N} f_2(x,u_0^\lambda(x),v_0^\lambda(x))\varphi(x)\,dx.$$

Therefore, $(u_0^\lambda, v_0^\lambda)$ is a solution of (1.3). □

Proof of Theorem 3. Assume that $(u,v)=(\lambda_1 u_0,\lambda_2 v_0)$, where (u_0,v_0) is a positive solution of (1.6) and $\lambda_1,\lambda_2>0$ satisfy (1.7). For all $\varphi\in C_c^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned}M_1\left([u]_{s_1,p_1}^{p_1}\right)\langle u,\varphi\rangle_{s_1,p_1}&=M_1\left(\lambda_1^{p_1}[u_0]_{s_1,p_1}^{p_1}\right)\lambda_1^{p_1-1}\langle u_0,\varphi\rangle_{s_1,p_1}\\&=\lambda_1^{q_1}\lambda_2^{r_1}\int_{\mathbb{R}^N} w_1 u_0^{q_1}v_0^{r_1}\varphi\,dx\\&=\int_{\mathbb{R}^N} w_1 u^{q_1}v^{r_1}\varphi\,dx.\end{aligned}$$

Similarly,

$$M_2\left([v]_{s_2,p_2}^{p_2}\right)\langle v,\varphi\rangle_{s_2,p_2}=\int_{\mathbb{R}^N} w_2 u^{r_2}v^{q_2}\varphi\,dx.$$

Therefore, (u,v) is a solution of (1.5).

Conversely, assume that (u,v) is a positive solution of (1.5). We define $(u_0,v_0)=(u/\lambda_1, v/\lambda_2)$, where

$$\lambda_1=\left(\frac{M_1^{q_2-p_2+1}([u]_{s_1,p_1}^{p_1})}{M_2^{r_1}([v]_{s_2,p_2}^{p_2})}\right)^{\frac{1}{(q_1-p_1+1)(q_2-p_2+1)-r_1r_2}},$$

$$\lambda_2=\left(\frac{M_2^{q_1-p_1+1}([v]_{s_2,p_2}^{p_2})}{M_1^{r_2}([u]_{s_1,p_1}^{p_1})}\right)^{\frac{1}{(q_1-p_1+1)(q_2-p_2+1)-r_1r_2}},$$

Then $(u,v)=(\lambda_1u_0,\lambda_2v_0)$ and $[u]_{s_1,p_1}^{p_1}=\lambda_1^{p_1}[u_0]_{s_1,p_1}^{p_1}$, $[v]_{s_2,p_2}^{p_2}=\lambda_2^{p_2}[v_0]_{s_2,p_2}^{p_2}$. Therefore, λ_1,λ_2 satisfy (1.7). For all $\varphi\in C_c^\infty(\mathbb{R}^N)$,

$$\begin{aligned}\langle u_0,\varphi\rangle_{s_1,p_1}&=\lambda_1^{1-p_1}\langle u,\varphi\rangle_{s_1,p_1}\\&=\lambda_1^{1-p_1}M_1^{-1}\left([u]_{s_1,p_1}^{p_1}\right)\int_{\mathbb{R}^N} w_1u^{q_1}v^{r_1}\varphi\,dx\\&=\lambda_1^{-q_1}\lambda_2^{-r_1}\int_{\mathbb{R}^N} w_1u^{q_1}v^{r_1}\varphi\,dx\\&=\int_{\mathbb{R}^N} w_1u_0^{q_1}v_0^{r_1}\varphi\,dx.\end{aligned}$$

Similarly,

$$\langle v_0,\varphi\rangle_{s_2,p_2}=\int_{\mathbb{R}^N} w_2u_0^{r_2}v_0^{q_2}\varphi\,dx.$$

Hence, (u_0,v_0) is a solution of (1.2). □

REFERENCES

[1] J. Busca and R. Manásevich, "A Liouville-type theorem for Lane-Emden system," *Indiana Univ. Math. J.*, vol. 51, pp. 37–51, 2002, doi: 10.1512/iumj.2002.51.2160.

[2] W. Chen and C. Li, "Classification of solutions of some nonlinear elliptic equations," *Duke Math. J.*, vol. 63, pp. 615–622, 1991, doi: 10.1215/S0012-7094-91-06325-8.

[3] W. Chen, C. Li, and Y. Li, "A direct method of moving planes for the fractional Laplacian," *Adv. Math.*, vol. 308, pp. 404–437, 2017, doi: 10.1016/j.aim.2016.11.038.

[4] D. G. de Figueiredo and P. L. Felmer, "A Liouville-type theorem for systems," *Ann. Sc. Norm. Sup. Pisa*, vol. XXI, pp. 387–397, 1994.

[5] G. M. Figueiredo, N. Ikoma, and J. R. Santos Júnior, "Existence and concentration result for the Kirchhoff type equations with general nonlinearities," *Arch. Ration. Mech. Anal.*, vol. 213, pp. 931–979, 2014, doi: 10.1007/s00205-014-0747-8.

[6] A. Fiscella and P. Pucci, "p-fractional Kirchhoff equations involving critical nonlinearities," *Nonlinear Anal. Real World Appl.*, vol. 35, pp. 350–378, 2017, doi: 10.1016/j.nonrwa.2016.11.004.

[7] J. García-Melián, "Nonexistence of positive solutions for Henon equation," *arXiv:1703.04353*, 2017.

[8] J. Guo, S. Ma, and G. Zhang, "Solutions of the autonomous Kirchhoff type equations in $\mathbb{R}^N$," *Appl. Math. Lett.*, vol. 82, pp. 14–17, 2018, doi: 10.1016/j.aml.2018.02.011.

[9] N. Kawano, W. M. Ni, and S. Yotsutani, "A generalized Pohozaev identity and its applications," *J. Math. Soc. Japan*, vol. 42, pp. 541–564, 1990, doi: 10.2969/jmsj/04230541.

[10] G. Kirchhoff, *Mechanik*. Leipzig: Teubner, 1883.

[11] P. Le, N. V. Huynh, and V. Ho, "Classification results for Kirchhoff equations in $\mathbb{R}^N$," *Complex Var. Elliptic Equ.*, vol. 64, pp. 1146–1157, 2019, doi: 10.1080/17476933.2018.1505874.

[12] C. S. Lin, "A classification of solutions of a conformally invariant fourth order equation in $\mathbb{R}^n$," *Comment. Math. Helv.*, vol. 73, pp. 206–231, 1998, doi: 10.1007/s000140050.

[13] J.-L. Lions, "On some questions in boundary value problems of mathematical physics," *North-Holl. Math. Stud.*, vol. 30, pp. 284–346, 1978, doi: 10.1016/S0304-0208(08)70870-3.

[14] J. Liu, J. F. Liao, and C. L. Tang, "Positive solutions for Kirchhoff-type equations with critical exponent in $\mathbb{R}^N$," *J. Math. Anal. Appl.*, vol. 429, pp. 1153–1172, 2015, doi: 10.1016/j.jmaa.2015.04.066.

[15] E. Mitidieri, "Nonexistence of positive solutions of semilinear elliptic systems in $\mathbb{R}^N$," *Diff. Inte. Equations*, vol. 9, pp. 465–479, 1996.

[16] E. Mitidieri and S. I. Pohozaev, "A priori estimates and the absence of solutions of nonlinear partial differential equations and inequalities," *Tr. Mat. Inst. Steklova*, vol. 234, pp. 1–384, 2001.

[17] Q. H. Phan and A. T. Duong, "Liouville-type theorems for a quasilinear elliptic equation of the Hénon-type," *Nonlinear Differ. Equ. Appl.*, vol. 22, pp. 1817–1829, 2015, doi: 10.1007/s00030-015-0345-y.

[18] M. A. Ragusa and A. Tachikawa, "On continuity of minimizers for certain quadratic growth functionals," *J. Math. Soc. Japan*, vol. 57, no. 3, pp. 691–700, 2005, doi: 10.2969/jmsj/1158241929.

[19] M. A. Ragusa and A. Tachikawa, "Correction and addendum to Boundary regularity of minimizers of $p(x)$-energy functionals," *Ann. Inst. H. Poincaré Anal. Non Linéaire*, vol. 34, no. 6, pp. 1633–1637, 2017, doi: 10.1016/j.anihpc.2017.09.004.

[20] B. Sciunzi, "Classification of positive $\mathcal{D}^{1,p}(\mathbb{R}^N)$-solutions to the critical p-Laplace equation in $\mathbb{R}^N$," *Adv. Math.*, vol. 291, pp. 12–23, 2016, doi: 10.1016/j.aim.2015.12.028.

[21] J. Serrin and H. Zou, "Non-existence of positive solutions of semilinear elliptic systems," *Discourses Math. Appl.*, vol. 3, pp. 55–68, 1994.

[22] J. Serrin and H. Zou, "Existence of positive solutions of the Lane-Emden system," *Atti Semin. Mat. Fis. Univ. Modena*, vol. 46, pp. 369–380, 1998.

[23] J. Serrin and H. Zou, "Cauchy–Liouville and universal boundedness theorems for quasilinear elliptic equations and inequalities," *Acta Math.*, vol. 189, pp. 79–142, 2002, doi: 10.1007/BF02392645.

[24] P. Souplet, "The proof of the Lane-Emden conjecture in four space dimensions," *Adv. Math.*, vol. 221, pp. 1409–1427, 2009, doi: 10.1016/j.aim.2009.02.014.

[25] M. Souto, "A priori estimates and existence of positive solutions of non-linear cooperative elliptic systems," *Diff. Inte. Equations*, vol. 8, pp. 1245–1258, 1995.

[26] J. Vétois, "A priori estimates and application to the symmetry of solutions for critical p-Laplace equations," *J. Differential Equations*, vol. 260, pp. 149–161, 2016, doi: 10.1016/j.jde.2015.08.041.

[27] M. Xiang, B. Zhang, and V. Rădulescu, "Multiplicity of solutions for a class of quasilinear Kirchhoff system involving the fractional p-Laplacian," *Nonlinearity*, vol. 29, pp. 3186–3205, 2016, doi: 10.1088/0951-7715/29/10/3186.

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