Miskolc Mathematical Notes
Vol. 22 (2021), No. 1, pp. 91–106

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2021.2849

FUZZY SUB-HOOPS BASED ON FUZZY POINTS

R. A. BORZOOEI, M. MOHSENI TAKALLO, M. AALY KOLOGANI, AND Y.B. JUN

Received 12 February, 2019

Abstract. Using the *belongs to* relation $(\in)$ and *quasi-coincident with* relation (q) between fuzzy points and fuzzy sets, the notions of an $(\in,\in)$-fuzzy sub-hoop, an $(\in,\in\vee q)$-fuzzy sub-hoop and a $(q,\in\vee q)$-fuzzy sub-hoop are introduced, and several properties are investigated. Characterizations of an $(\in,\in)$-fuzzy sub-hoop and an $(\in,\in\vee q)$-fuzzy sub-hoop are displayed. Relations between an $(\in,\in)$-fuzzy sub-hoop, an $(\in,\in\vee q)$-fuzzy sub-hoop and a $(q,\in\vee q)$-fuzzy sub-hoop are discussed. Conditions for a fuzzy set to be a $(q,\in\vee q)$-fuzzy sub-hoop are considered, and condition for an $(\in,\in\vee q)$-fuzzy sub-hoop to be a $(q,\in\vee q)$-fuzzy sub-hoop are provided.

2010 *Mathematics Subject Classification:* 03G25, 06A12, 06B99, 06D72.

Keywords: Sub-hoop, $(\in,\in)$-fuzzy sub-hoop, $(\in,\in\vee q)$-fuzzy sub-hoop, $(q,\in\vee q)$-fuzzy sub-hoop.

1. INTRODUCTION

After the introduction of the concept of a fuzzy set by Zadeh [18], several researches were conducted on the generalizations of the concept of a fuzzy set. One of the least satisfactory areas in the early development of fuzzy topology has been that surrounding the concept of fuzzy point. In the original classical theory, where values are taken in the closed unit interval I, it soon became apparent that, in order to build up a reasonable theory, points should be defined as fuzzy singletons while membership requires strict inequality. So crisp points, taking value 1, are excluded, and fuzzy topology would seem not to include general topology. This disturbing state of affairs was to some extent overcome by [15] who replaced membership by quasi-coincidence (not belonging to the complement, where belonging is taken as $\leq$), thus reinstating crisp points. More recently [11] has drawn attention to a duality between quasi-coincidence and strict inequality membership. The duality, however, is only partial [17].

Hoop, which is introduced by B. Bosbach in [9], is naturally ordered commutative residuated integral monoids. Several properties of hoops are displayed in [3–5, 8, 10, 13, 16, 19]. For example, Blok [3, 4], investigated structure of hoops and their applicational reducts. Borzooei and Aaly Kologani in [5] defined (implicative, positive implicative, fantastic) filters in a hoop and discussed their relations and properties. Using filter, they considered a congruence relation on a hoop, and induced the quotient structure which is a hoop. They also provided conditions for the quotient structure to be Brouwerian semilattice, Heyting algebra and Wajesberg hoop. After that in [2], they studied these notions in pseudo-hoops. The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [15], played a vital role to generate some different types of fuzzy subalgebras in of BCK/BCI-algebras. On (α,β)-fuzzy subalgerbas of BCK/BCI-algebras, introduced by Jun [12]. In particular, $(\in,\in\vee q)$-fuzzy subalgebra is an important and useful generalization of a fuzzy subalgebra in BCK/BCI-algebras. It is now natural to investigate similar type of generalizations of the existing fuzzy subsystems of other algebraic structures.

In this paper, we introduce the notions of an $(\in,\in)$-fuzzy sub-hoop, an $(\in,\in\vee q)$-fuzzy sub-hoop and a $(q,\in\vee q)$-fuzzy sub-hoop, and investigate several properties. We discuss characterizations of an $(\in,\in)$-fuzzy sub-hoop and an $(\in,\in\vee q)$-fuzzy sub-hoop. We find relations between an $(\in,\in)$-fuzzy sub-hoop, an $(\in,\in\vee q)$-fuzzy sub-hoop and a $(q,\in\vee q)$-fuzzy sub-hoop. We consider conditions for a fuzzy set to be a $(q,\in\vee q)$-fuzzy sub-hoop of H. We provide a condition for an $(\in,\in\vee q)$-fuzzy sub-hoop to be a $(q,\in\vee q)$-fuzzy sub-hoop.

2. Preliminaries

By a *hoop* we mean an algebra $(H,\odot,\to,1)$ in which $(H,\odot,1)$ is a commutative monoid and the following assertions are valid.

(H1) $(\forall x\in H)(x\to x=1)$,
(H2) $(\forall x,y\in H)(x\odot(x\to y)=y\odot(y\to x))$,
(H3) $(\forall x,y,z\in H)(x\to(y\to z)=(x\odot y)\to z)$.

By a *sub-hoop* of a hoop H we mean a subset S of H which satisfies the condition:

$$(\forall x,y\in H)(x,y\in S\ \Rightarrow\ x\odot y\in S,\ x\to y\in S). \tag{2.1}$$

Note that every non-empty sub-hoop contains the element 1.

Every hoop H satisfies the following conditions (see [9]).

$$(\forall x,y\in H)(x\odot y\leq z\ \Leftrightarrow\ x\leq y\to z). \tag{2.2}$$

$$(\forall x,y\in H)(x\odot y\leq x,y). \tag{2.3}$$

$$(\forall x,y\in H)(x\leq y\to x). \tag{2.4}$$

$$(\forall x\in H)(x\to 1=1). \tag{2.5}$$

$$(\forall x\in H)(1\to x=x). \tag{2.6}$$

A fuzzy set λ in a set X of the form

$$\lambda(y):=\begin{cases} t\in(0,1] & \text{if } y=x,\\ 0 & \text{if } y\neq x,\end{cases}$$

is said to be a *fuzzy point* with support x and value t and is denoted by x_t.

For a fuzzy point x_t and a fuzzy set λ in a set X, Pu and Liu [15] gave meaning to the symbol $x_t\alpha\lambda$, where $\alpha \in \{\in, q, \in\vee q, \in\wedge q\}$.

To say that $x_t \in \lambda$ (resp. $x_t q\lambda$) means that $\lambda(x) \geq t$ (resp. $\lambda(x) + t > 1$), and in this case, x_t is said to *belong to* (resp. *be quasi-coincident with*) a fuzzy set λ.

To say that $x_t \in\vee q\lambda$ (resp. $x_t \in\wedge q\lambda$) means that $x_t \in \lambda$ or $x_t q\lambda$ (resp. $x_t \in \lambda$ and $x_t q\lambda$).

3. (α, β)-FUZZY SUB-HOOPS FOR $(\alpha, \beta) \in \{(\in, \in), (\in, \in\vee q), (q, \in\vee q)\}$

In what follows, let H be a hoop unless otherwise specified.

Definition 1. A fuzzy set λ in H is called an $(\in, \in)$*-fuzzy sub-hoop* of H if the following assertion is valid.

$$(\forall x, y \in H)(\forall t, k \in (0, 1]) \left(x_t \in \lambda,\ y_k \in \lambda \Rightarrow \begin{cases} (x \odot y)_{\min\{t,k\}} \in \lambda \\ (x \to y)_{\min\{t,k\}} \in \lambda) \end{cases} \right). \quad (3.1)$$

Example 1. Let $H = \{0, a, b, c, d, 1\}$ be a set with binary operations $\odot$ and $\to$ in Table 1 and Table 2, respectively.

TABLE 1. Cayley table for the binary operation "$\odot$"

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	d	0	d	a
b	0	d	c	c	0	b
c	0	0	c	c	0	c
d	0	d	0	0	0	d
1	0	a	b	c	d	1

TABLE 2. Cayley table for the binary operation "$\to$"

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	c	b	1
b	d	a	1	b	a	1
c	a	a	1	1	a	1
d	b	1	1	b	1	1
1	0	a	b	c	d	1

Then $(H, \odot, \rightarrow, 1)$ is a hoop. Define a fuzzy set λ in H as follows:

$$\lambda : H \rightarrow [0,1],\ x \mapsto \begin{cases} 0.5 & \text{if } x = 0, \\ 0.7 & \text{if } x = a, \\ 0.3 & \text{if } x = b, \\ 0.5 & \text{if } x = c, \\ 0.3 & \text{if } x = d, \\ 0.8 & \text{if } x = 1 \end{cases}$$

It is routine to verify that λ is an $(\in,\in)$-fuzzy sub-hoop of H.

We consider characterizations of an $(\in,\in)$-fuzzy sub-hoop.

Theorem 1. *A fuzzy set λ in H is an $(\in,\in)$-fuzzy sub-hoop of H if and only if the following assertion is valid.*

$$(\forall x,y \in H) \left(\begin{array}{l} \lambda(x \odot y) \geq \min\{\lambda(x), \lambda(y)\} \\ \lambda(x \rightarrow y) \geq \min\{\lambda(x), \lambda(y)\} \end{array} \right). \tag{3.2}$$

Proof. Assume that λ is an $(\in,\in)$-fuzzy sub-hoop of H. Note that $x_{\lambda(x)} \in \lambda$ and $y_{\lambda(y)} \in \lambda$ for all $x,y \in H$. It follows from (3.1) that $(x \odot y)_{\min\{\lambda(x),\lambda(y)\}} \in \lambda$ and $(x \rightarrow y)_{\min\{\lambda(x),\lambda(y)\}} \in \lambda$. Hence

$$\lambda(x \odot y) \geq \min\{\lambda(x), \lambda(y)\}$$

and

$$\lambda(x \rightarrow y) \geq \min\{\lambda(x), \lambda(y)\}$$

for all $x,y \in H$.

Conversely, suppose that λ satisfies the condition (3.2). Let $x,y \in H$ and $t,k \in (0,1]$ such that $x_t \in \lambda$ and $y_k \in \lambda$. Then $\lambda(x) \geq t$ and $\lambda(y) \geq k$, which implies from (3.2) that

$$\lambda(x \odot y) \geq \min\{\lambda(x), \lambda(y)\} \geq \min\{t,k\}$$

and

$$\lambda(x \rightarrow y) \geq \min\{\lambda(x), \lambda(y)\} \geq \min\{t,k\}$$

for all $x,y \in H$. Hence $(x \odot y)_{\min\{t,k\}} \in \lambda$ and $(x \rightarrow y)_{\min\{t,k\}} \in \lambda$. Therefore λ is an $(\in,\in)$-fuzzy sub-hoop of H. □

Given a fuzzy set λ in H, we consider the set

$$U(\lambda;t) := \{x \in H \mid \lambda(x) \geq t\},$$

which is called an $\in$-*level set* of λ (related to t).

Theorem 2. *A fuzzy set λ in H is an $(\in,\in)$-fuzzy sub-hoop of H if and only if the non-empty $\in$-level set $U(\lambda;t)$ of λ is a sub-hoop of H for all $t \in [0,1]$.*

Proof. Let λ be a fuzzy set in H such that $U(\lambda;t)$ is a non-empty sub-hoop of H for all $t \in [0,1]$. Let $x,y \in H$ and $t,k \in (0,1]$ such that $x_t \in \lambda$ and $y_k \in \lambda$. Then $\lambda(x) \geq t$ and $\lambda(y) \geq k$, and so $x,y \in U(\lambda;\min\{t,k\})$. By hypothesis, we have $x \odot y \in U(\lambda;\min\{t,k\})$ and $x \to y \in U(\lambda;\min\{t,k\})$. Hence $(x \odot y)_{\min\{t,k\}} \in \lambda$ and $(x \to y)_{\min\{t,k\}} \in \lambda$. Therefore λ is an $(\in,\in)$-fuzzy sub-hoop of H.

Conversely, assume that λ is an $(\in,\in)$-fuzzy sub-hoop of H. Let $x,y \in U(\lambda;t)$ for all $t \in [0,1]$. Then $\lambda(x) \geq t$ and $\lambda(y) \geq t$, that is, $x_t \in \lambda$ and $y_t \in \lambda$. It follows from (3.1) that $(x \odot y)_t \in \lambda$ and $(x \to y)_t \in \lambda$, that is, $x \odot y \in U(\lambda;t)$ and $x \to y \in U(\lambda;t)$. Therefore $U(\lambda;t)$ of λ is a sub-hoop of H for all $t \in [0,1]$. □

Theorem 3. *Let λ be an $(\in,\in)$-fuzzy sub-hoop of H such that $|\mathrm{Im}(\lambda)| \geq 3$. Then λ can be expressed as the union of two fuzzy sets μ and ν where μ and ν are $(\in,\in)$-fuzzy sub-hoops of H such that*

(1) $\mathrm{Im}(\mu)$ *and* $\mathrm{Im}(\nu)$ *have at least two elements.*
(2) μ *and* ν *have no same family of* $\in$*-level sub-hoops.*

Proof. Let λ be an $(\in,\in)$-fuzzy sub-hoop of H with $\mathrm{Im}(\lambda) = \{t_0, t_1, \ldots, t_n\}$ where $t_0 > t_1 > \cdots > t_n$ and $n \geq 2$. Then

$$U(\lambda;t_0) \subseteq U(\lambda;t_1) \subseteq \cdots \subseteq U(\lambda;t_n) = H$$

is a chain of $\in$-level sub-hoops of λ. Define two fuzzy sets μ and ν in H by

$$\mu(x) = \begin{cases} k_1 & \text{if } x \in U(\lambda;t_1), \\ t_r & \text{if } x \in U(\lambda;t_r) \setminus U(\lambda;t_{r-1}) \text{ for } r = 2,3,\cdots,n, \end{cases}$$

and

$$\nu(x) = \begin{cases} t_0 & \text{if } x \in U(\lambda;t_0), \\ t_1 & \text{if } x \in U(\lambda;t_1) \setminus U(\lambda;t_0), \\ k_2 & \text{if } x \in U(\lambda;t_3) \setminus U(\lambda;t_1), \\ t_r & \text{if } x \in U(\lambda;t_r) \setminus U(\lambda;t_{r-1}) \text{ for } r = 4,5\cdots,n, \end{cases}$$

respectively, where $k_1 \in (t_2,t_1)$ and $k_2 \in (t_4,t_2)$. Then μ and ν are $(\in,\in)$-fuzzy sub-hoops of H, and their $\in$-level sub-hoops are chains as follows:

$$U(\mu;t_1) \subseteq U(\mu;t_2) \subseteq \cdots \subseteq U(\mu;t_n) = H$$

and

$$U(\nu;t_0) \subseteq U(\nu;t_1) \subseteq U(\nu;t_3) \subseteq \cdots \subseteq U(\mu;t_n) = H$$

It is clear that $\mu \subseteq \lambda$, $\nu \subseteq \lambda$ and $\mu \cup \nu = \lambda$. This completes the proof. □

Definition 2. A fuzzy set λ in H is called an $(\in, \in\vee q)$-*fuzzy sub-hoop* of H if the following assertion is valid.

$$(\forall x,y \in H)(\forall t,k \in (0,1])\left(x_t \in \lambda,\ y_k \in \lambda \Rightarrow \begin{cases} (x \odot y)_{\min\{t,k\}} \in\vee q\lambda \\ (x \to y)_{\min\{t,k\}} \in\vee q\lambda) \end{cases} \right). \tag{3.3}$$

Example 2. Consider the hoop $(H,\odot,\rightarrow,1)$ which is described in Example 1.
(1) Define a fuzzy set λ in H as follows:

$$\lambda : H \rightarrow [0,1],\ x \mapsto \begin{cases} 0.5 & \text{if } x = 1, \\ 0.3 & \text{if } x = c, \\ 0.2 & \text{if } x = b, \\ 0.1 & \text{if } x \in \{0,a,d\}. \end{cases}$$

It is routine to verify that λ is an $(\in, \in\vee q)$-fuzzy sub-hoop of H.
(2) Define a fuzzy set μ in H as follows:

$$\mu : H \rightarrow [0,1],\ x \mapsto \begin{cases} 0.8 & \text{if } x = 0, \\ 0.7 & \text{if } x = a, \\ 0.3 & \text{if } x = b, \\ 0.4 & \text{if } x = c, \\ 0.3 & \text{if } x = d, \\ 0.5 & \text{if } x = 1 \end{cases}$$

It is routine to verify that μ is an $(\in, \in\vee q)$-fuzzy sub-hoop of H.

We consider characterizations of $(\in, \in\vee q)$-fuzzy sub-hoop.

Theorem 4. *A fuzzy set λ in H is an $(\in, \in\vee q)$-fuzzy sub-hoop of H if and only if the following assertion is valid.*

$$(\forall x, y \in H) \left(\begin{array}{l} \lambda(x\odot y) \geq \min\{\lambda(x), \lambda(y), 0.5\} \\ \lambda(x \rightarrow y) \geq \min\{\lambda(x), \lambda(y), 0.5\} \end{array} \right). \tag{3.4}$$

Proof. Assume that λ is an $(\in, \in\vee q)$-fuzzy sub-hoop of H and let $x, y \in H$. Suppose that $\min\{\lambda(x), \lambda(y)\} < 0.5$.

If $\lambda(x\odot y) < \min\{\lambda(x), \lambda(y)\}$ or $\lambda(x \rightarrow y) < \min\{\lambda(x), \lambda(y)\}$, then $\lambda(x\odot y) < t \leq \min\{\lambda(x), \lambda(y)\}$ or $\lambda(x \rightarrow y) < k \leq \min\{\lambda(x), \lambda(y)\}$ for some $t, k \in (0,1]$. It follows that

$$x_t \in \lambda \text{ and } y_t \in \lambda$$

or

$$x_k \in \lambda \text{ and } y_k \in \lambda.$$

But $(x\odot y)_{\min\{t,t\}} = (x\odot y)_t \overline{\in\vee q} \lambda$ or $(x \rightarrow y)_{\min\{k,k\}} = (x \rightarrow y)_k \overline{\in\vee q} \lambda$. This is a contradiction, and so $\lambda(x\odot y) \geq \min\{\lambda(x), \lambda(y)\}$ and $\lambda(x \rightarrow y) \geq \min\{\lambda(x), \lambda(y)\}$ whenever $\min\{\lambda(x), \lambda(y)\} < 0.5$.

Assume that $\min\{\lambda(x), \lambda(y)\} \geq 0.5$. Then $x_{0.5} \in \lambda$ and $y_{0.5} \in \lambda$. It follows from (3.3) that $(x\odot y)_{0.5} = (x\odot y)_{\min\{0.5,0.5\}} \in\vee q \lambda$ and $(x \rightarrow y)_{0.5} = (x \rightarrow y)_{\min\{0.5,0.5\}} \in\vee q \lambda$. Thus $\lambda(x\odot y) \geq 0.5$ and $\lambda(x \rightarrow y) \geq 0.5$. Consequently, $\lambda(x\odot y) \geq \min\{\lambda(x), \lambda(y), 0.5\}$ and $\lambda(x \rightarrow y) \geq \min\{\lambda(x), \lambda(y), 0.5\}$.

Conversely, suppose that λ satisfies the condition (3.4). Let $x, y \in H$ and $t, k \in (0,1]$ such that $x_t \in \lambda$ and $y_k \in \lambda$. Then $\lambda(x) \geq t$ and $\lambda(y) \geq k$. If $\lambda(x\odot y) < \min\{t, k\}$, then

$\min\{\lambda(x),\lambda(y)\} \geq 0.5$ because if $\min\{\lambda(x),\lambda(y)\} < 0.5$, then

$$\lambda(x\odot y) \geq \min\{\lambda(x),\lambda(y),0.5\} \geq \min\{\lambda(x),\lambda(y)\} \geq \min\{t,k\}$$

which is a contradiction. Similarly, if $\lambda(x \to y) < \min\{t,k\}$, then $\min\{\lambda(x),\lambda(y)\} \geq 0.5$. It follows that

$$\lambda(x\odot y) + \min\{t,k\} > 2\lambda(x\odot y) \geq 2\min\{\lambda(x),\lambda(y),0.5\} = 1$$

and

$$\lambda(x \to y) + \min\{t,k\} > 2\lambda(x \to y) \geq 2\min\{\lambda(x),\lambda(y),0.5\} = 1.$$

Hence $(x\odot y)_{\min\{t,k\}} q\lambda$ and $(x \to y)_{\min\{t,k\}} q\lambda$, and so $(x\odot y)_{\min\{t,k\}} \in\vee q\lambda$ and $(x \to y)_{\min\{t,k\}} \in\vee q\lambda$. Therefore λ is an $(\in,\in\vee q)$-fuzzy sub-hoop of H. □

Theorem 5. *A fuzzy set λ in H is an $(\in,\in\vee q)$-fuzzy sub-hoop of H if and only if the non-empty $\in$-level set $U(\lambda;t)$ of λ is a sub-hoop of H for all $t \in (0,0.5]$.*

Proof. Assume that λ is an $(\in,\in\vee q)$-fuzzy sub-hoop of H. Let $x,y \in U(\lambda;t)$ for $t \in (0,0.5]$. Then $\lambda(x) \geq t$ and $\lambda(y) \geq t$.

It follows from Theorem 4 that $\lambda(x\odot y) \geq \min\{\lambda(x),\lambda(y),0.5\} \geq \min\{t,0.5\} = t$ and

$\lambda(x \to y) \geq \min\{\lambda(x),\lambda(y),0.5\} \geq \min\{t,0.5\} = t$. Hence $x\odot y \in U(\lambda;t)$ and $x \to y \in U(\lambda;t)$. Therefore $U(\lambda;t)$ is a sub-hoop of H.

Conversely, suppose that the non-empty $\in$-level set $U(\lambda;t)$ of λ is a sub-hoop of H for all $t \in (0,0.5]$. If there exists $x,y \in H$ such that $\lambda(x\odot y) < \min\{\lambda(x),\lambda(y),0.5\}$ or $\lambda(x \to y) < \min\{\lambda(x),\lambda(y),0.5\}$, then $\lambda(x\odot y) < t \leq \min\{\lambda(x),\lambda(y),0.5\}$ or $\lambda(x \to y) < t \leq \min\{\lambda(x),\lambda(y),0.5\}$ for some $t \in (0,1]$. Hence $t \leq 0.5$ and $x,y \in U(\lambda;t)$, and so $x\odot y \in U(\lambda;t)$ and $x \to y \in U(\lambda;t)$. This is a contradiction, and therefore $\lambda(x\odot y) \geq \min\{\lambda(x),\lambda(y),0.5\}$ and $\lambda(x \to y) \geq \min\{\lambda(x),\lambda(y),0.5\}$. Using Theorem 4, we conclude that λ is an $(\in,\in\vee q)$-fuzzy sub-hoop of H. □

Theorem 6. *Every $(\in,\in)$-fuzzy sub-hoop is an $(\in,\in\vee q)$-fuzzy sub-hoop.*

Proof. Straightforward. □

The converse of Theorem 6 is not true in general as seen in the following example.

Example 3. The $(\in,\in\vee q)$-fuzzy sub-hoop μ in Example 2(2) is not an $(\in,\in)$-fuzzy sub-hoop of H since $a_{0.55} \in \mu$ and $0_{0.75} \in \mu$, but $(a \to 0)_{\min\{0.55,0.75\}} \overline{\in} \mu$.

We provide a condition for an $(\in,\in\vee q)$-fuzzy sub-hoop to be an $(\in,\in)$-fuzzy sub-hoop.

Theorem 7. *If an $(\in,\in\vee q)$-fuzzy sub-hoop λ of H satisfies the condition*

$$(\forall x \in H)(\lambda(x) < 0.5), \tag{3.5}$$

then λ is an $(\in,\in)$-fuzzy sub-hoop of H.

Proof. Let $x,y \in H$ and $t,k \in (0,1]$ such that $x_t \in \lambda$ and $y_k \in \lambda$. Then $\lambda(x) \geq t$ and $\lambda(y) \geq k$. Using (3.5) and Theorem 4, we have

$$\lambda(x \odot y) \geq \min\{\lambda(x), \lambda(y), 0.5\} = \min\{\lambda(x), \lambda(y)\} \geq \min\{t,k\}$$

and

$$\lambda(x \to y) \geq \min\{\lambda(x), \lambda(y), 0.5\} = \min\{\lambda(x), \lambda(y)\} \geq \min\{t,k\}.$$

Hence $(x \odot y)_{\min\{t,k\}} \in \lambda$ and $(x \to y)_{\min\{t,k\}} \in \lambda$. Therefore λ is an $(\in,\in)$-fuzzy sub-hoop of H. □

Proposition 1. *If λ is a non-zero $(\in, \in \vee q)$-fuzzy sub-hoop of H, then $\lambda(1) > 0$.*

Proof. Assume that $\lambda(1) = 0$. Since λ is non-zero, there exists $x \in H$ such that $\lambda(x) = t \neq 0$, and so $x_t \in \lambda$. Then $\lambda(x \to x) = \lambda(1) = 0$ and $\lambda(x \to x) + t = \lambda(1) + t = t \leq 1$, that is, $(x \to x)_t \overline{\in} \lambda$ and $(x \to x)_t \overline{q} \lambda$. Thus $(x \to x)_t \overline{\in \vee q} \lambda$, which is a contradiction. Therefore $\lambda(1) > 0$. □

Corollary 1. *If λ is a non-zero $(\in,\in)$-fuzzy sub-hoop of H, then $\lambda(1) > 0$.*

Theorem 8. *If λ is a non-zero $(\in,\in)$-fuzzy sub-hoop of H, then the set*

$$H_0 := \{x \in H \mid \lambda(x) \neq 0\} \tag{3.6}$$

is a sub-hoop of H.

Proof. Let $x,y \in H_0$. Then $\lambda(x) > 0$ and $\lambda(y) > 0$. Note that $x_{\lambda(x)} \in \lambda$ and $y_{\lambda(y)} \in \lambda$. If $\lambda(x \odot y) = 0$ or $\lambda(x \to y) = 0$, then $\lambda(x \odot y) = 0 < \min\{\lambda(x), \lambda(y)\}$ or $\lambda(x \to y) = 0 < \min\{\lambda(x), \lambda(y)\}$, that is, $(x \odot y)_{\min\{\lambda(x),\lambda(y)\}} \overline{\in} \lambda$ or $(x \to y)_{\min\{\lambda(x),\lambda(y)\}} \overline{\in} \lambda$. This is a contradiction, and so $\lambda(x \odot y) \neq 0$ and $\lambda(x \to y) \neq 0$. Hence $x \odot y \in H_0$ and $x \to y \in H_0$. Therefore H_0 is a sub-hoop of H. □

Theorem 9. *For any sub-hoop S of H and $t \in (0, 0.5]$, there exists an $(\in, \in \vee q)$-fuzzy sub-hoop λ of H such that $U(\lambda;t) = S$.*

Proof. Let λ be a fuzzy set in H defined by

$$\lambda : H \to [0,1],\ x \mapsto \begin{cases} t & \text{if } x \in S, \\ 0 & \text{otherwise,} \end{cases} \tag{3.7}$$

where $t \in (0, 0.5]$. It is clear that $U(\lambda;t) = S$. Suppose that $\lambda(x \odot y) < \min\{\lambda(x), \lambda(y), 0.5\}$ or $\lambda(x \to y) < \min\{\lambda(x), \lambda(y), 0.5\}$ for some $x,y \in H$. Since $|\mathrm{Im}(\lambda)| = 2$, it follows that $\lambda(x \odot y) = 0$ or $\lambda(x \to y) = 0$, and $\min\{\lambda(x), \lambda(y), 0.5\} = t$. Since $t \leq 0.5$, we have $\lambda(x) = t = \lambda(y)$ and so $x,y \in S$. Then $x \odot y \in S$ and $x \to y \in S$, which imply that $\lambda(x \odot y) = t$ and $\lambda(x \to y) = t$. This is a contradiction, and so $\lambda(x \odot y) \geq \min\{\lambda(x), \lambda(y), 0.5\}$ and $\lambda(x \to y) \geq \min\{\lambda(x), \lambda(y), 0.5\}$. Using Theorem 4, we know that λ is an $(\in, \in \vee q)$-fuzzy sub-hoop of H. □

For any fuzzy set λ in H and $t \in (0,1]$, we consider the following sets so called *q-set* and $\in\vee q$*-set*, respectively.

$$\lambda_q^t := \{x \in H \mid x_t\, q\, \lambda\} \text{ and } \lambda_{\in\vee q}^t := \{x \in H \mid x_t \in\vee q \lambda\}$$

It is clear that $\lambda_{\in\vee q}^t = U(\lambda;t) \cup \lambda_q^t$.

Theorem 10. *A fuzzy set λ in H is an $(\in, \in\vee q)$-fuzzy sub-hoop of H if and only if $\lambda_{\in\vee q}^t$ is a sub-hoop of H for all $t \in (0,1]$.*

We call $\lambda_{\in\vee q}^t$ an $\in\vee q$*-level sub-hoop* of λ.

Proof. Assume that λ is an $(\in, \in\vee q)$-fuzzy sub-hoop of H. Let $x, y \in \lambda_{\in\vee q}^t$ for $t \in (0,1]$. Then $x_t \in\vee q\lambda$ and $y_t \in\vee q\lambda$, i.e., $\lambda(x) \geq t$ or $\lambda(x)+t > 1$, and $\lambda(y) \geq t$ or $\lambda(y)+t > 1$. It follows from (3.4) that $\lambda(x\odot y) \geq \min\{t, 0.5\}$ and $\lambda(x \to y) \geq \min\{t, 0.5\}$. In fact, if $\lambda(x\odot y) < \min\{t, 0.5\}$ or $\lambda(x \to y) < \min\{t, 0.5\}$, then $x_t \overline{\in\vee q}\lambda$ or $y_t \overline{\in\vee q}\lambda$, a contradiction.

If $t \leq 0.5$, then $\lambda(x\odot y) \geq \min\{t, 0.5\} = t$ and $\lambda(x \to y) \geq \min\{t, 0.5\} = t$. Hence $x\odot y \in U(\lambda;t) \subseteq \lambda_{\in\vee q}^t$ and $x \to y \in U(\lambda;t) \subseteq \lambda_{\in\vee q}^t$.

If $t > 0.5$, then $\lambda(x\odot y) \geq \min\{t, 0.5\} = 0.5$ and $\lambda(x \to y) \geq \min\{t, 0.5\} = 0.5$. Hence $\lambda(x\odot y)+t > 0.5+0.5 = 1$ and $\lambda(x \to y)+t > 0.5+0.5 = 1$, that is, $(x\odot y)_t\, q\, \lambda$ and $(x \to y)_t\, q\, \lambda$. It follows that $x\odot y \in \lambda_q^t \subseteq \lambda_{\in\vee q}^t$ and $x \to y \in \lambda_q^t \subseteq \lambda_{\in\vee q}^t$. Therefore $\lambda_{\in\vee q}^t$ is a sub-hoop of H for all $t \in (0,1]$.

Conversely, let λ be a fuzzy set in H and $t \in (0,1]$ such that $\lambda_{\in\vee q}^t$ is a sub-hoop of H. Suppose that $\lambda(x\odot y) < \min\{\lambda(x), \lambda(y), 0.5\}$ or $\lambda(x \to y) < \min\{\lambda(x), \lambda(y), 0.5\}$ for some $x, y \in H$. Then $\lambda(x\odot y) < t < \min\{\lambda(x), \lambda(y), 0.5\}$ or $\lambda(x \to y) < t < \min\{\lambda(x), \lambda(y), 0.5\}$ for some $t \in (0, 0.5)$. Hence $x, y \in U(\lambda;t) \subseteq \lambda_{\in\vee q}^t$, and so $x\odot y \in \lambda_{\in\vee q}^t$ and $x \to y \in \lambda_{\in\vee q}^t$. Thus $\lambda(x\odot y) \geq t$ or $\lambda(x\odot y)+t > 1$, and $\lambda(x \to y) \geq t$ or $\lambda(x \to y)+t > 1$. This is a contradiction, and therefore $\lambda(x\odot y) \geq \min\{\lambda(x), \lambda(y), 0.5\}$ and $\lambda(x \to y) \geq \min\{\lambda(x), \lambda(y), 0.5\}$ for all $x, y \in H$. Consequently, λ is an $(\in, \in\vee q)$-fuzzy sub-hoop of H by Theorem 4. □

Theorem 11. *If λ is an $(\in, \in\vee q)$-fuzzy sub-hoop of H, then the q-set λ_q^t is a sub-hoop of H for all $t \in (0.5,1]$.*

Proof. Let $x, y \in \lambda_q^t$ for $t \in (0.5,1]$. Then $\lambda(x)+t > 1$ and $\lambda(y)+t > 1$, which imply from Theorem 4 that

$$\begin{aligned}\lambda(x\odot y)+t &\geq \min\{\lambda(x), \lambda(y), 0.5\}+t \\ &= \min\{\lambda(x)+t, \lambda(y)+t, 0.5+t\} > 1,\end{aligned}$$

and

$$\begin{aligned}\lambda(x \to y)+t &\geq \min\{\lambda(x), \lambda(y), 0.5\}+t \\ &= \min\{\lambda(x)+t, \lambda(y)+t, 0.5+t\} > 1,\end{aligned}$$

that is, $(x \odot y)_t\, q\, \lambda$ and $(x \to y)_t\, q\, \lambda$. Hence $x \odot y \in \lambda_q^t$ and $x \to y \in \lambda_q^t$. Therefore λ_q^t is a sub-hoop of H for all $t \in (0.5, 1]$. □

Theorem 12. *Let $f : H \to K$ be a homomorphism of hoops. If λ and μ are $(\in, \in \vee q)$-fuzzy sub-hoops of H and K, respectively, then*

(1) *$f^{-1}(\mu)$ is an $(\in, \in \vee q)$-fuzzy sub-hoop of H.*
(2) *If f is onto and λ satisfies the condition*

$$(\forall T \subseteq H)(\exists x_0 \in T)\left(\lambda(x_0) = \sup_{x \in T} \lambda(x)\right), \tag{3.8}$$

then $f(\lambda)$ is an $(\in, \in \vee q)$-fuzzy sub-hoop of K.

Proof. (1) Let $x, y \in H$ and $t, k \in (0, 1]$ such that $x_t \in f^{-1}(\mu)$ and $y_k \in f^{-1}(\mu)$. Then $(f(x))_t \in \mu$ and $(f(y))_k \in \mu$. Since μ is an $(\in, \in \vee q)$-fuzzy sub-hoop of K, we have

$$(f(x \odot y))_{\min\{t,k\}} = (f(x) \odot f(y))_{\min\{t,k\}} \in \vee\, q\, \mu$$

and

$$(f(x \to y))_{\min\{t,k\}} = (f(x) \to f(y))_{\min\{t,k\}} \in \vee\, q\, \mu.$$

Hence $(x \odot y)_{\min\{t,k\}} \in \vee\, q\, f^{-1}(\mu)$ and $(x \to y)_{\min\{t,k\}} \in \vee\, q\, f^{-1}(\mu)$. Therefore $f^{-1}(\mu)$ is an $(\in, \in \vee q)$-fuzzy sub-hoop of H.

(2) Let $a, b \in K$ and $t, k \in (0, 1]$ such that $a_t \in f(\lambda)$ and $b_k \in f(\lambda)$. Then $(f(\lambda))(a) \geq t$ and $(f(\lambda))(b) \geq k$. Using the condition (3.8), there exist $x \in f^{-1}(a)$ and $y \in f^{-1}(b)$ such that

$$\lambda(x) = \sup_{z \in f^{-1}(a)} \lambda(z) \text{ and } \lambda(y) = \sup_{w \in f^{-1}(b)} \lambda(w).$$

Then $x_t \in \lambda$ and $y_k \in \lambda$, which imply that $(x \odot y)_{\min\{t,k\}} \in \vee\, q\, \lambda$ and $(x \to y)_{\min\{t,k\}} \in \vee\, q\, \lambda$ since λ is an $(\in, \in \vee q)$-fuzzy sub-hoop of H. Now $x \odot y \in f^{-1}(a \odot b)$ and $x \to y \in f^{-1}(a \to b)$, and so $(f(\lambda))(a \odot b) \geq \lambda(x \odot y)$ and $(f(\lambda))(a \to b) \geq \lambda(x \to y)$. Hence

$$(f(\lambda))(a \odot b) \geq \min\{t, k\} \text{ or } (f(\lambda))(a \odot b) + \min\{t, k\} > 1$$

and

$$(f(\lambda))(a \to b) \geq \min\{t, k\} \text{ or } (f(\lambda))(a \to b) + \min\{t, k\} > 1,$$

that is, $(a \odot b)_{\min\{t,k\}} \in \vee\, q\, f(\lambda)$ and $(a \to b)_{\min\{t,k\}} \in \vee\, q\, f(\lambda)$. Therefore $f(\lambda)$ is an $(\in, \in \vee q)$-fuzzy sub-hoop of K. □

Theorem 13. *Let λ be an $(\in, \in \vee q)$-fuzzy sub-hoop of H such that $|\{\lambda(x) \mid \lambda(x) < 0.5\}| \geq 2$. Then there exist two $(\in, \in \vee q)$-fuzzy sub-hoops μ and ν of H such that*

(1) *$\lambda = \mu \cup \nu$.*
(2) *$\mathrm{Im}(\mu)$ and $\mathrm{Im}(\nu)$ have at least two elements.*
(3) *μ and ν have no the same family of $\in \vee q$-level sub-hoops.*

Proof. Let $\{\lambda(x) \mid \lambda(x) < 0.5\} = \{t_1, t_2, \ldots, t_r\}$ where $t_1 > t_2 > \cdots > t_r$ and $r \geq 2$. Then the chain of $\in\vee q$-level sub-hoops of λ is

$$\lambda^{0.5}_{\in\vee q} \subseteq \lambda^{t_1}_{\in\vee q} \subseteq \lambda^{t_2}_{\in\vee q} \subseteq \cdots \subseteq \lambda^{t_r}_{\in\vee q} = H.$$

Define two fuzzy sets μ and ν in H by

$$\mu(x) = \begin{cases} t_1 & \text{if } x \in \lambda^{t_1}_{\in\vee q}, \\ t_n & \text{if } x \in \lambda^{t_n}_{\in\vee q} \setminus \lambda^{t_{n-1}}_{\in\vee q} \text{ for } n = 2, 3, \cdots, r, \end{cases}$$

and

$$\nu(x) = \begin{cases} \lambda(x) & \text{if } x \in \lambda^{0.5}_{\in\vee q}, \\ k & \text{if } x \in \lambda^{t_2}_{\in\vee q} \setminus \lambda^{0.5}_{\in\vee q}, \\ t_n & \text{if } x \in \lambda^{t_n}_{\in\vee q} \setminus \lambda^{t_{n-1}}_{\in\vee q} \text{ for } n = 3, 4, \cdots, r, \end{cases}$$

respectively, where $k \in (t_3, t_2)$. Then μ and ν are $(\in, \in\vee q)$-fuzzy sub-hoops of H, and $\mu \subseteq \lambda$ and $\nu \subseteq \lambda$. The chains of $\in\vee q$-level sub-hoops of μ and ν are given by

$$\mu^{t_1}_{\in\vee q} \subseteq \mu^{t_2}_{\in\vee q} \subseteq \cdots \subseteq \mu^{t_r}_{\in\vee q} \text{ and } \nu^{0.5}_{\in\vee q} \subseteq \nu^{t_2}_{\in\vee q} \subseteq \cdots \subseteq \nu^{t_r}_{\in\vee q},$$

respectively. It is clear that $\mu \cup \nu = \lambda$. This completes the proof. □

Definition 3. A fuzzy set λ in H is called a $(q, \in\vee q)$-*fuzzy sub-hoop* of H if the following assertion is valid.

$$(\forall x, y \in H)(\forall t, k \in (0,1]) \left(x_t\, q\lambda,\ y_k\, q\lambda \Rightarrow \begin{cases} (x \odot y)_{\min\{t,k\}} \in\vee q\lambda \\ (x \to y)_{\min\{t,k\}} \in\vee q\lambda) \end{cases} \right). \quad (3.9)$$

Example 4. Let $H = \{0, a, b, 1\}$ be a set with binary operations $\odot$ and $\to$ in Table 3 and Table 4, respectively.

TABLE 3. Cayley table for the binary operation "$\odot$"

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

TABLE 4. Cayley table for the binary operation "$\to$"

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Define a fuzzy set λ in H as follows:

$$\lambda : H \to [0,1],\ x \mapsto \begin{cases} 0.8 \text{ if } x = 1, \\ 0.6 \text{ if } x = b, \\ 0.55 \text{ if } x = a, \\ 0.7 \text{ if } x = 0. \end{cases}$$

It is routine to verify that λ is a $(q, \in\vee q)$-fuzzy sub-hoop of H.

Question 1. Let λ be a fuzzy set in H such that

(1) $0 \neq \lambda(a) \leq 0.5$ for some $a \in H$,
(2) $(\forall x \in H)\ (x \neq a \ \Rightarrow\ \lambda(x) \geq 0.5)$.

Then is λ a $(q, \in\vee q)$-fuzzy sub-hoop of H?

The answer to this question is negative as seen in the following example.

Example 5. Consider the hoop $(H, \odot, \to, 1)$ which is described in Example 4. Let λ be a fuzzy set in H defined by $\lambda(0) = 0.6$, $\lambda(a) = 0.4$, $\lambda(b) = 0.55$ and $\lambda(1) = 0.8$. Then λ is not a $(q, \in\vee q)$-fuzzy sub-hoop of H since $a_{0.7}\, q\, \lambda$ and $b_{0.46}\, q\, \lambda$, but $(a\odot b)_{\min\{0.7,0.46\}}\, \overline{\in\vee q}\, \lambda$ and/or $(b \to a)_{\min\{0.7,0.46\}}\, \overline{\in\vee q}\, \lambda$.

We consider conditions for a fuzzy set to be a $(q, \in\vee q)$-fuzzy sub-hoop of H.

Theorem 14. *Let S be a sub-hoop of H and let λ be a fuzzy set in H such that*

$$(\forall x \in H) \left(\begin{array}{l} \lambda(x) = 0 \text{ if } x \notin S \\ \lambda(x) \geq 0.5 \text{ if } x \in S \end{array} \right). \tag{3.10}$$

Then λ is a $(q, \in\vee q)$-fuzzy sub-hoop of H.

Proof. Let $x, y \in H$ and $t, k \in (0,1]$ such that $x_t\, q\, \lambda$ and $y_k\, q\, \lambda$, that is, $\lambda(x) + t > 1$ and $\lambda(y) + k > 1$. Then $x\odot y \in S$ and $x \to y \in S$ because if $x\odot y \notin S$, then $x \in H \setminus S$ or $y \in H \setminus S$. Thus $\lambda(x) = 0$ or $\lambda(y) = 0$, and so $t > 1$ or $k > 1$. This is contradiction. Similarly, if $x \to y \notin S$, then we arrive at a contradiction. If $\min\{t,k\} > 0.5$, then $\lambda(x\odot y) + \min\{t,k\} > 1$ and $\lambda(x \to y) + \min\{t,k\} > 1$, and so $(x\odot y)_{\min\{t,k\}}\, q\, \lambda$ and $(x \to y)_{\min\{t,k\}}\, q\, \lambda$. If $\min\{t,k\} \leq 0.5$, then $\lambda(x\odot y) \geq 0.5 \geq \min\{t,k\}$ and $\lambda(x \to y) \geq 0.5 \geq \min\{t,k\}$. Thus $(x\odot y)_{\min\{t,k\}} \in \lambda$ and $(x \to y)_{\min\{t,k\}} \in \lambda$. Therefore $(x\odot y)_{\min\{t,k\}} \in\vee q \lambda$ and $(x \to y)_{\min\{t,k\}} \in\vee q \lambda$. Consequently, λ is a $(q, \in\vee q)$-fuzzy sub-hoop of H. □

Corollary 2. *If a fuzzy set λ in H satisfies $\lambda(x) \geq 0.5$ for all $x \in H$, then λ is a $(q, \in\vee q)$-fuzzy sub-hoop of H.*

Theorem 15. *If λ is a $(q, \in\vee q)$-fuzzy sub-hoop of H such that λ is not constant on H_0, then there exists $x \in H$ such that $\lambda(x) \geq 0.5$. Moreover $\lambda(x) \geq 0.5$ for all $x \in H_0$.*

Proof. If $\lambda(x) < 0.5$ for all $x \in H$, then there exists $a \in H_0$ such that $t_a = \lambda(a) \neq \lambda(1) = t_1$ since λ is not constant on H_0. Then $t_a < t_1$ or $t_a > t_1$. If $t_1 < t_a$, then we can take $\delta > 0.5$ such that $t_1 + \delta < 1 < t_a + \delta$. It follows that $a_\delta\, q\,\lambda$, $\lambda(a \to a) = \lambda(1) = t_1 < \delta = \min\{\delta, \delta\}$ and $\lambda(a \to a) + \min\{\delta, \delta\} = \lambda(1) + \delta = t_1 + \delta < 1$. Hence $(a \to a)_{\min\{\delta,\delta\}} \overline{\in \vee q} \lambda$, which is a contradiction. If $t_1 > t_a$, then $t_a + \delta < 1 < t_1 + \delta$ for some $\delta > 0.5$. It follows that $1_\delta\, q\,\lambda$ and $a_1\, q\,\lambda$, but $(1 \to a)_{\min\{1,\delta\}} = a_\delta \overline{\in \vee q} \lambda$ since $\lambda(a) < 0.5 < \delta$ and $\lambda(a) + \delta = t_a + \delta < 1$. This leads a contradiction, and therefore $\lambda(x) \geq 0.5$ for some $x \in H$. We now prove that $\lambda(1) \geq 0.5$. Suppose that $\lambda(1) = t_1 < 0.5$. Since $\lambda(x) = t_x \geq 0.5$ for some $x \in H$, it follows that $t_1 < t_x$. Choose $t_0 > t_1$ such that $t_1 + t_0 < 1 < t_x + t_0$. Then $\lambda(x) + t_0 = t_x + t_0 > 1$, i.e., $x_{t_0}\, q\,\lambda$. Also we have

$$\lambda(x \to x) = \lambda(1) = t_1 < t_0 = \min\{t_0, t_0\}$$

and

$$\lambda(x \to x) + \min\{t_0, t_0\} = \lambda(1) + t_0 = t_1 + t_0 < 1.$$

Thus $(x \to x)_{\min\{t_0,t_0\}} \overline{\in \vee q} \lambda$, a contradiction. Hence $\lambda(1) \geq 0.5$. Finally, assume that $t_a = \lambda(a) < 0.5$ for some $a \in H_0$. Take $t \in (0, 1]$ such that $t_a + t < 0.5$. Then $\lambda(a) + 1 = t_a + 1 > 1$ and $\lambda(1) + (0.5 + t) > 1$, which imply that $a_1\, q\,\lambda$ and $1_{0.5+t}\, q\,\lambda$. But $(1 \to a)_{\min\{1,0.5+t\}} = a_{\min\{1,0.5+t\}} \overline{\in \vee q} \lambda$ since $\lambda(1 \to a) = \lambda(a) < 0.5 + t < \min\{1, 0.5 + t\}$ and

$$\lambda(1 \to a) + \min\{1, 0.5 + t\} = \lambda(a) + 0.5 + t = t_a + 0.5 + t < 0.5 + 0.5 = 1.$$

This is a contradiction. Therefore $\lambda(x) \geq 0.5$ for all $x \in H_0$. This completes the proof. □

Theorem 16. *If λ is a $(q, \in \vee q)$-fuzzy sub-hoop of H, then the set H_0 in (3.6) is a sub-hoop of H.*

Proof. Let $x, y \in H_0$. Then $\lambda(x) + 1 > 1$ and $\lambda(y) + 1 > 1$, that is, $x_1\, q\,\lambda$ and $y_1\, q\,\lambda$. Assume that $\lambda(x \odot y) = 0$ or $\lambda(x \to y) = 0$. Then

$$\lambda(x \odot y) < 1 = \min\{1, 1\} \text{ and } \lambda(x \odot y) + \min\{1, 1\} = 1$$

or

$$\lambda(x \to y) < 1 = \min\{1, 1\} \text{ and } \lambda(x \to y) + \min\{1, 1\} = 1,$$

that is, $(x \odot y)_{\min\{1,1\}} \overline{\in \vee q} \lambda$ or $(x \to y)_{\min\{1,1\}} \overline{\in \vee q} \lambda$. This is a contradiction, and so $\lambda(x \odot y) \neq 0$ and $\lambda(x \to y) \neq 0$, i.e., $x \odot y \in H_0$ and $x \to y \in H_0$. Consequently, H_0 is a sub-hoop of H. □

Theorem 17. *If λ is a $(q, \in \vee q)$-fuzzy sub-hoop of H, then the q-set λ_q^t is a sub-hoop of H for all $t \in (0.5, 1]$.*

Proof. Let $x, y \in \lambda_q^t$ for $t \in (0.5, 1]$. Then $x_t\, q\,\lambda$ and $y_t\, q\,\lambda$. Since λ is a $(q, \in \vee q)$-fuzzy sub-hoop of H, we have $(x \odot y)_t \in \vee q \lambda$ and $(x \to y)_t \in \vee q \lambda$. If $(x \odot y)_t\, q\,\lambda$ (and $(x \to y)_t\, q\,\lambda$), then $x \odot y \in \lambda_q^t$ (and $x \to y \in \lambda_q^t$). If $(x \odot y)_t \in \lambda$ (and $(x \to y)_t \in \lambda$), then $\lambda(x \odot y) \geq t > 1 - t$ (and $\lambda(x \to y) \geq t > 1 - t$) since $t > 0.5$. Thus $(x \odot y)_t\, q\,\lambda$ (and $(x \to y)_t\, q\,\lambda$), that is, $x \odot y \in \lambda_q^t$ (and $x \to y \in \lambda_q^t$). Therefore λ_q^t is a sub-hoop of H for all $t \in (0.5, 1]$. □

We consider relations between $(\in, \in\vee q)$-fuzzy sub-hoop and $(q, \in\vee q)$-fuzzy sub-hoop.

Theorem 18. *Every $(q, \in\vee q)$-fuzzy sub-hoop is an $(\in, \in\vee q)$-fuzzy sub-hoop.*

Proof. Let λ be a $(q, \in\vee q)$-fuzzy sub-hoop of H. Let $x, y \in H$ and $t, k \in (0,1]$ such that $x_t \in \lambda$ and $y_k \in \lambda$. Suppose that $(x\odot y)_{\min\{t,k\}} \overline{\in\vee q}\lambda$ or $(x \to y)_{\min\{t,k\}} \overline{\in\vee q}\lambda$. Then

$$\lambda(x\odot y) < \min\{t,k\} \text{ and } \lambda(x\odot y) + \min\{t,k\} \leq 1 \tag{3.11}$$

or

$$\lambda(x \to y) < \min\{t,k\} \text{ and } \lambda(x \to y) + \min\{t,k\} \leq 1. \tag{3.12}$$

It follows that $\lambda(x\odot y) < \min\{t,k,0.5\}$ or $\lambda(x \to y) < \min\{t,k,0.5\}$. Hence

$$\begin{aligned} 1 - \lambda(x\odot y) &> 1 - \min\{t,k,0.5\} = \max\{1-t, 1-k, 0.5\} \\ &\geq \max\{1-\lambda(x), 1-\lambda(y), 0.5\} \end{aligned}$$

or

$$\begin{aligned} 1 - \lambda(x \to y) &> 1 - \min\{t,k,0.5\} = \max\{1-t, 1-k, 0.5\} \\ &\geq \max\{1-\lambda(x), 1-\lambda(y), 0.5\}. \end{aligned}$$

Therefore there exist $\delta_1, \delta_2 \in (0,1]$ such that

$$1 - \lambda(x\odot y) \geq \delta_1 > \max\{1-\lambda(x), 1-\lambda(y), 0.5\} \tag{3.13}$$

or

$$1 - \lambda(x \to y) \geq \delta_2 > \max\{1-\lambda(x), 1-\lambda(y), 0.5\}. \tag{3.14}$$

From the right inequalities in (3.13) and (3.14), we have

$$\lambda(x) + \delta_1 > 1 \text{ and } \lambda(y) + \delta_1 > 1, \text{ i.e., } x_{\delta_1}\, q\, \lambda \text{ and } y_{\delta_1}\, q\, \lambda$$

or

$$\lambda(x) + \delta_2 > 1 \text{ and } \lambda(y) + \delta_2 > 1, \text{ i.e., } x_{\delta_2}\, q\, \lambda \text{ and } y_{\delta_2}\, q\, \lambda.$$

Since λ is a $(q, \in\vee q)$-fuzzy sub-hoop of H, it follows that $(x\odot y)_{\delta_1} \in\vee q\, \lambda$ or $(x \to y)_{\delta_2} \in\vee q\, \lambda$. From the left inequalities in (3.13) and (3.14), we have $\lambda(x\odot y) + \delta_1 \leq 1$ or $\lambda(x \to y) + \delta_2 \leq 1$, that is, $(x\odot y)_{\delta_1} \overline{q}\, \lambda$ or $(x \to y)_{\delta_2} \overline{q}\, \lambda$. Also $\lambda(x\odot y) \leq 1 - \delta_1 < 1 - 0.5 = 0.5 < \delta_1$ or $\lambda(x \to y) \leq 1 - \delta_2 < 1 - 0.5 = 0.5 < \delta_2$. Hence $(x\odot y)_{\delta_1} \overline{\in\vee q}\lambda$ or $(x \to y)_{\delta_2} \overline{\in\vee q}\lambda$. This is a contradiction, and so $(x\odot y)_{\min\{t,k\}} \in\vee q\, \lambda$ and $(x \to y)_{\min\{t,k\}} \in\vee q\, \lambda$. Therefore λ is an $(\in, \in\vee q)$-fuzzy sub-hoop of H. □

The following example shows that any $(\in, \in\vee q)$-fuzzy sub-hoop may not be a $(q, \in\vee q)$-fuzzy sub-hoop.

Example 6. In Example 1, the fuzzy set λ is an $(\in, \in\vee q)$-fuzzy sub-hoop of H. But it is not a $(q, \in\vee q)$-fuzzy sub-hoop of H since $a_{0.4}\, q\, \lambda$ and $b_{0.8}\, q\, \lambda$. But $(a\odot b)_{\min\{0.4,0.8\}} \overline{\in\vee q}\lambda$ and/or $(a \to b)_{\min\{0.4,0.8\}} \overline{\in\vee q}\lambda$.

We provide a condition for an $(\in, \in\vee q)$-fuzzy sub-hoop to be a $(q, \in\vee q)$-fuzzy sub-hoop.

Theorem 19. *Let λ be an $(\in, \in\vee q)$-fuzzy sub-hoop of H. If every fuzzy point has the value in $(0, 0.5]$, then λ is a $(q, \in\vee q)$-fuzzy sub-hoop of H.*

Proof. Let $x, y \in H$ and $t, k \in (0, 0.5]$ such that $x_t\, q\, \lambda$ and $y_k\, q\, \lambda$. Then $\lambda(x) > 1 - t \geq t$ and $\lambda(y) > 1 - k \geq k$, that is, $x_t \in \lambda$ and $y_k \in \lambda$. Since λ is an $(\in, \in\vee q)$-fuzzy sub-hoop of H, it follows that $(x \odot y)_{\min\{t,k\}} \in\vee q\, \lambda$ and $(x \to y)_{\min\{t,k\}} \in\vee q\, \lambda$. Therefore λ is a $(q, \in\vee q)$-fuzzy sub-hoop of H. □

4. Conclusion

Our aim was to define the concepts of an $(\in, \in)$-fuzzy sub-hoop, an $(\in, \in\vee q)$-fuzzy sub-hoop and a $(q, \in\vee q)$-fuzzy sub-hoop, and we discussed some properties and found some equivalent definitions of them. Then, we discussed characterizations of an $(\in, \in)$-fuzzy sub-hoop and an $(\in, \in\vee q)$-fuzzy sub-hoop. Also, we found relations between an $(\in, \in)$-fuzzy sub-hoop, an $(\in, \in\vee q)$-fuzzy sub-hoop and a $(q, \in\vee q)$-fuzzy sub-hoop and considered conditions for a fuzzy set to be a $(q, \in\vee q)$-fuzzy sub-hoop of H, and provided a condition for an $(\in, \in\vee q)$-fuzzy sub-hoop to be a $(q, \in\vee q)$-fuzzy sub-hoop. By [1, 6, 7, 14] we defined the concept of $(\in, \in)$-fuzzy filters (fuzzy implicative filters, fuzzy positive implicative filters, fuzzy fantastic filters) of hoop and $(\in, \in\vee q)$-fuzzy filters (fuzzy implicative filters, fuzzy positive implicative filters, fuzzy fantastic filters) of hoop and have investigated some equivalent definitions and properties of them.

References

[1] M. Aaly Kologani, M. Mohseni Takallo, and H. Kim, "Fuzzy filters of hoops based on fuzzy points," *Mathematics*, vol. 7, no. 5, p. 430, 2019, doi: 10.3390/math7050430.

[2] S. Z. Alavia, R. A. Borzooei, and M. Aaly Kologani, "Fuzzy filters in pseudo hoops." *J. Intell. Fuzzy Syst*, vol. 3, no. 32, pp. 1997–2007, 2017, doi: 10.3233/JIFS-161586.

[3] W. Blok and I. Ferreirim, " Hoops and their implicational reducts," *Logic in Computer Science*, vol. 1, no. 28, pp. 219–230, 1993.

[4] W. Blok and I. Ferreirim, "On the structure of hoops," *Algebra Universalis*, no. 43, pp. 233–257, 2000.

[5] R. Borzooei and M. Aaly Kologani, " Filter theory of hoop-algebras," *J. Adv. Res. Pure Math*, vol. 6, no. 4, pp. 72–86, 2014.

[6] R. Borzooei, M. Aaly Kologani, Y. B. Jun, and M. Mohseni Takallo, "Fuzzy fantastic filters of hoops based on fuzzy points," *Submitted to Fuzzy Information and Engineering*.

[7] R. Borzooei, M. Aaly Kologani, M. Sabetkish, and Y. B. Jun, "Fuzzy positive implicative filters of hoops based on fuzzy points," *Mathematics*, vol. 7, no. 6, p. 566, 2019, doi: 10.3390/math7060566.

[8] R. Borzooei, H. R. Varasteh, and K. Borna, " Fundamental hoop-algebras," *Ratio Mathematica*, vol. 29, pp. 25–40, 2015.

[9] B. Bosbach, " Komplementäre halbgruppen, axiomatik und arithmetik," *Fundamenta Mathematicae*, vol. 64, no. -, pp. 257–287, 1969.

[10] S. Ghorbani, "Localization of hoop algebras ," *J. Adv. Res. Pure Math.*, vol. 5, no. 3, pp. 1–13, 2013.
[11] J. M. Hu, " Fuzzy topological spaces," *J. Math. Anal. Appl.*, vol. 110, pp. 141–178, 1985.
[12] Y. B. Jun, "On (α,β)-fuzzy subalgerbas of BCK/BCI-algebras," *Bull. Korean Math. Soc*, vol. 42, no. 4, pp. 703–711, 2005.
[13] C. Luo, X. Xin, and P. He, "*n*-fold (positive) implicative filters of hoops," *Ital. J. Pure Appl. Math.*, vol. 38, pp. 631–642, 2017.
[14] M. Mohseni Takallo, M. Aaly Kologani, X. Xin, Y. Jun, and R. Borzooei, "Fuzzy implicative filters of hoops based on fuzzy points," *Submitted*.
[15] P. M. Pu and Y. M. Liu, "Fuzzy topology I, Neighborhood structure of a fuzzy point and Moore-Smith convergence," *J. Math. Anal. Appl.* , vol. 76, pp. 571–599, 1980.
[16] R. Tayebi Khorami and A. Borumand Saeid, "Some unitary operators on hoop-algebras," *Fuzzy Inf. Eng.*, vol. 9, no. 2, pp. 205–223, 2017.
[17] M. W. Warner, "A note on the relation between G- and Q-theories for first countability and sequentiality," *Fuzzy Sets and Systems*, vol. 33, pp. 321–327, 1989.
[18] L. A. Zadeh, " Fuzzy sets ," *Inform. Control*, vol. 8, pp. 338–353, 1965.
[19] M. A. Zander, "Decomposability of the finitely generated free hoop residuation algebra," *Studia Logica* , vol. 88, no. 2, pp. 233–246, 2008.

Authors' addresses

R. A. Borzooei
Shahid Beheshti University, Department of Mathematics, 1983963113 Tehran, Iran
E-mail address: borzooei@sbu.ac.ir

M. Mohseni Takallo
Shahid Beheshti University, Department of Mathematics, 1983963113 Tehran, Iran
E-mail address: mohammad.mohseni1122@gmail.com

M. Aaly Kologani
Hatef Higher Education Institute, Zahedan, Iran
E-mail address: mona4011@gmail.com

Y.B. Jun
Gyeongsang National University, Department of Mathematics Education, 52828, Jinju, Korea
E-mail address: skywine@gmail.com