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A NOTE ON FARTHEST POINT PROBLEM IN BANACH SPACES

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Abstract. Farthest point problem states that "Must every uniquely remotal set in a Banach space be singleton?" In this paper we introduce the notion of partial ideal statistical continuity of a function which is way weaker than continuity of a function. We give an example to show that partial ideal statistical continuity is weaker than continuity. In this paper we use Ideal summability to give some answers to FPP problem which improves the result in [13]. We prove that if E is a non-empty, bounded, uniquely remotal subset in a real Banach space X such that E has a Chebyshev center c and the farthest point map $F : X \to E$ restricted to $[c, F(c)]$ is partially ideal statistically continuous at c then E is singleton.

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1. Introduction

Let $\mathbb{X}$ be a real Banach space and E be a nonempty, bounded subset of $\mathbb{X}$. For any $x \in \mathbb{X}$, the farthest distance of x from the set E is denoted by $D(x, E)$ and is defined by

$$D(x, E) = \sup \{\|x - e\| : e \in E\}.$$

The farthest distance from x to E may or may not attained by some elements of E. If the distance is attained then the collection of all such points of E is denoted by $F(x, E)$ i.e

$$F(x, E) = \{e \in E : \|x - e\| = D(x, E)\}.$$

We say that E is remotal if $F(x, E) \neq \phi \ \forall \ x \in \mathbb{X}$ and E is said to be uniquely remotal if $F(x, E)$ is singleton for each $x \in \mathbb{X}$.

The FPP was proposed by Motzkin, Starus and Valentine [11] in context of the Euclidean space E^n. The problem was considered in the setting of Banach spaces by Klee [9] where he proved that every compact uniquely remotal subset of a Banach space is singleton. In [1], Asplund solved the FPP in the affirmation in any finite dimensional Banach space with respect to a norm which is not necessarily symmetric. The study of remotal and uniquely remotal sets has attracted many researchers in the last few decades due to its connection to the geometry of Banach spaces. One can see [2, 12, 15] for further details.

Chebyshev centers of sets have played a major role in the study of uniquely remotal sets. Recall that a chebyshev center of a subset E of a normed space $\mathbb{X}$ is an element $c \in \mathbb{X}$ such that $D(c,E) = \inf_{x\in\mathbb{X}} D(x,E)$. Astaneh in 1983 proved that in an inner product space $\mathbb{X}$, every closed and bounded set has a Chebyshev center. But it is still unknown that whether every closed and bounded subset of a normed space $\mathbb{X}$ has a center or not.

The idea of convergence of a real sequence has been extended to statistical convergence by Fast [6] and Steinhaus [19] and later on re-introduced by Schoenberg [18] independently and is based on the notion of asymptotic density of the subset of natural numbers. However, the first idea of statistical convergence (by different name) was given by Zygmund [20] in the first edition of his monograph published in Warsaw in 1935. Later on it was further investigated from the sequence space point of view and linked with summability theorem by Fridy [7], Connor [3], Šalát [16], Das et. al. [4, 5], Fridy and Orhan [8]. If $\mathbb{N}$ denotes the set of natural numbers and $K \subset \mathbb{N}$ then $K(m,n)$ denotes the cardinality of the set $K \cap [m,n]$. The upper and lower natural density of the subset K is defined by

$$\bar{d}(K) = \lim_{n\to\infty} \sup \frac{K(1,n)}{n} \text{ and } \underline{d}(K) = \lim_{n\to\infty} \inf \frac{K(1,n)}{n}.$$

If $\bar{d}(K) = \underline{d}(K)$, then we say that the natural density of K exists, and it is denoted simply by

$$d(K) = \lim_{n\to\infty} \frac{K(1,n)}{n}.$$

A sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be statistically convergent to a real number x if for each $\varepsilon > 0$, the set $K = \{n \in \mathbb{N} : |x_n - x| \geq \varepsilon\}$ has natural density zero and we write $x_n \xrightarrow{S} x$.

The following definitions and notions will be needed.

Definition 1 ([10]). A family $\mathcal{I} \subset 2^{\mathbb{N}}$ of subsets of $\mathbb{N}$ (where $\mathbb{N}$ denotes the set of all non-negative integers) is said to be an ideal in $\mathbb{N}$ provided that the following conditions holds:
(i) $\phi \in \mathcal{I}$,
(ii) $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$,
(iii) $A \in \mathcal{I}, B \subseteq A \Rightarrow B \in \mathcal{I}$.

Definition 2 ([10]). An ideal $\mathcal{I}$ is said to be non-trivial if $\mathcal{I} \neq \{\phi\}$ and proper if $\mathbb{N} \notin \mathcal{I}$. A proper ideal $\mathcal{I}$ in $\mathbb{N}$ is said to be admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

Throughout this paper $\mathcal{I}$ will denote a non-trivial, proper, admissible ideal.

Definition 3 ([17]). Let $\mathcal{I}$ be a non-trivial, proper, admissible ideal. A sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be $\mathcal{I}$-statistically convergent to a real number x if for every $\varepsilon, \delta > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : |x_k - x| \geq \varepsilon\}| \geq \delta\right\} \in \mathcal{I}.$$

In [13] it was proved that if E is a uniquely remotal subset of a normed space, admitting a center c and if the farthest point map $F : X \to E$ restricted to $[c, F(c)]$ is continuous at c then E is singleton.

In a natural way, in this paper we first introduce the notion of partial ideal statistical continuity of a function via ideal summability and we give an example to show that this notion of partial ideal statistical continuity is much weaker than continuity and also weaker than partial continuity introduced by Sababheh et al. in [14]. We prove that if E is a non-empty, bounded, uniquely remotal subset in a real Banach space $\mathbb{X}$ such that E has a Chebyshev center c and the farthest point map $F : \mathbb{X} \to E$ restricted to $[c, F(c)]$ is partially ideal statistically continuous at c then E is singleton.

2. RESULTS

We first recall the definition of partial continuity of a function from [14] as follows.

Definition 4. Let $\mathbb{X}$ be real Banach space and A be a nonempty subset of $\mathbb{X}$. The function $F : A \to \mathbb{X}$ is said to be partially continuous at $a \in \mathbb{X}$ if there exists a non constant sequence $\{x_n\}_{n\in\mathbb{N}} \subset A$ such that $x_n \to a$ and $F(x_n) \to F(a)$ as $n \to \infty$.

Now we like to introduce the definition of partial ideal statistical continuity of a function as follows.

Definition 5. Let $\mathcal{I}$ be a non-trivial, proper, admissible ideal. Let $\mathbb{X}$ be a real Banach space and A be a nonempty subset of $\mathbb{X}$. The function $F : A \to \mathbb{X}$ is said to be partially ideal statistically continuous at $a \in \mathbb{X}$ if there exists a non constant sequence $\{x_n\}_{n\in\mathbb{N}} \subset A$ such that $\{x_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-statistically convergent to a and $\{F(x_n)\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-statistically convergent to $F(a)$.

Now we give an example to show that this notion of partial ideal statistical continuity is much weaker than partial continuity introduced by Sababheh et al. [14].

Example 1. Let $\mathcal{I}$ be a non-trivial, proper, admissible ideal. A function $f : [-1, 0] \to \mathbb{R}$ be defined by $f(x) = [x], x \in [-1, 0]$. It is easy to check that this function is not partially continuous (also not continuous) at the point $x = 0$. Now we show that this function is partially ideal statistically continuous at the point $x = 0$.

Let us define a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[-1, 0]$ by

$$x_n = \begin{cases} 0 \text{ if } n \neq m^2 \ \forall \ m \in \mathbb{N} \\ -\frac{1}{n} \text{ if } n = m^2 \text{ for some } m \in \mathbb{N}. \end{cases}$$

Now the natural density of the set $K = \{m^2 : m \in \mathbb{N}\}$ is 0. Let $\varepsilon > 0$.

As $\Big\{n \in \mathbb{N} : |x_n - 0| \geq \varepsilon\Big\} \subset \Big\{n \in \mathbb{N} : n = m^2 \text{ for some } m \in \mathbb{N}\Big\}$, so

$$\lim_{n\to\infty} \frac{1}{n}\Big|\{k \leq n : |x_k - x| \geq \varepsilon\}\Big| = 0.$$

This shows that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is statistically convergent to 0. Let $\delta > 0$. Then

$$\Big\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : |x_k - x| \geq \varepsilon\}| \geq \delta\Big\} \in \mathcal{I}$$

since $\mathcal{I}$ is admissible. So $\{x_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-statistically convergent to 0.

Now the sequence $\{f(x_n)\}_{n\in\mathbb{N}}$ is defined by

$$f(x_n) = \begin{cases} 0 \text{ if } n \neq m^2 \ \forall\ m \in \mathbb{N} \\ -1 \text{ if } n = m^2 \text{ for some } m \in \mathbb{N}, \end{cases}$$

Now it is easy to check that the sequence $\{f(x_n)\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-statistically convergent to $f(0) = 0$. This shows that this function is partially ideal statistically continuous at the point $x = 0$. So the notion of partial ideal statistical continuity is much weaker than partial continuity.

Theorem 1. *Let $\mathcal{I}$ be a non-trivial, proper, admissible ideal. Let E be nonempty, bounded subset of a real Banach space $\mathbb{X}$. If E is uniquely remotal such that E has Chebyshev center c and the farthest point map $F : X \to E$ restricted to $[c, F(c)]$ is partially ideal statistically continuous at c then E is singleton.*

Proof. Since E is uniquely remotal so for each $x \in \mathbb{X}$ there exists unique $e \in E$ such that $\|x - e\| = D(x, E)$ and the farthest point map $F : \mathbb{X} \to E$ defined by $F(x) = F(x, E),\ \forall\ x \in \mathbb{X}$ is well defined. We assume that E has Chebyshev center at $c = 0$.

Now suppose E is not singleton. So we have $F(0) \neq 0$.

It is given that the farthest point map $F : X \to E$ restricted to $[0, F(0)]$ is partially ideal statistically continuous at 0. So there exist a non constant sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[0, F(0)]$ such that $\{x_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-statistically convergent to 0 and $\{F(x_n)\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-statistically convergent to $F(0)$.

So we have $x_n = \mu_n F(0)$ with $\mu_n > 0,\ \forall\ n \in \mathbb{N}$ and μ_n is $\mathcal{I}$-statistically convergent to 0. Now for each $n \in \mathbb{N}$ there exists $\psi_n \in \mathbb{X}^*$ such that $\psi_n(F(x_n) - x_n) = \|F(x_n) - x_n\|$ and $\|\psi_n\| = 1$.

$$\begin{aligned} \psi_n(x_n) &= \psi_n(F(x_n)) - \psi_n(F(x_n) - x_n) \\ &\leq \|\psi_n\|\|F(x_n)\| - \|F(x_n) - x_n\| \\ &= \|F(x_n)\| - \|F(x_n) - x_n\| \\ &= \|F(x_n) - 0\| - \|F(x_n) - x_n\| \\ &\leq D(0, E) - D(x_n, E) \end{aligned}$$

$$\leq 0. \text{ (As 0 is the Chebyshev center)}$$

So

$$\begin{aligned}
\psi_n(x_n) &\leq 0 \ \forall\ n \in \mathbb{N} \\
\Rightarrow \psi_n(\mu_n F(0)) &\leq 0 \ \forall\ n \in \mathbb{N} \\
\Rightarrow \mu_n \psi_n(F(0)) &\leq 0 \ \forall\ n \in \mathbb{N} \\
\Rightarrow \psi_n(F(0)) &\leq 0 \ \forall\ n \in \mathbb{N} \text{ as } \mu_n > 0.
\end{aligned}$$

Now the sequence $\{F(x_n) - x_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-statistically convergent to $F(0)$. So for each $\varepsilon, \delta > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \|F(x_k) - x_k - F(0)\| \geq \varepsilon\}| \geq \delta\right\} \in \mathcal{I}.$$

Now we have

$$\Big| \|F(x_k) - x_k\| - \|F(0)\| \Big| \leq \|F(x_k) - x_k - F(0)\|.$$

Let $\varepsilon > 0$. So we have

$$\left\{k \leq n : \Big|\|F(x_k) - x_k\| - \|F(0)\|\Big| \geq \varepsilon\right\} \subset \left\{k \leq n : \|F(x_k) - x_k - F(0)\| \geq \varepsilon\right\}.$$

This implies that for every $\delta > 0$,

$$\begin{gathered}
\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \Big|\|F(x_k) - x_k\| - \|F(0)\|\Big| \geq \varepsilon\right\}\right| \geq \delta\right\} \subset \\
\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \|F(x_k) - x_k - F(0)\| \geq \varepsilon\right\}\right| \geq \delta\right\}.
\end{gathered}$$

Since the set on the right belongs to the ideal $\mathcal{I}$ so

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \Big|\|F(x_k) - x_k\| - \|F(0)\|\Big| \geq \varepsilon\right\}\right| \geq \delta\right\} \in \mathcal{I}.$$

So the sequence $\left\{\|F(x_k) - x_k\|\right\}_{k\in\mathbb{N}}$ is $\mathcal{I}$-statistically convergent to $\|F(0)\|$.
Now we have

$$\begin{aligned}
\psi_n(F(x_n) - x_n) - \psi_n(F(0)) &= \psi_n(F(x_n) - x_n - F(0)) \\
&\leq \|\psi_n\| \|F(x_n) - x_n - F(0)\| \\
&= \|F(x_n) - x_n - F(0)\|.
\end{aligned}$$

The sequence in the right is $\mathcal{I}$-statistically convergent to 0. So the sequence $\{\psi_n(F(0))\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-statistically convergent to $\|F(0)\|$. But this is possible only when $F(0) = 0$. This is a contradiction. This proves that the uniquely remotal set E is singleton. □

Corollary 1. *Let $\mathcal{I}$ be a non-trivial, proper, admissible ideal. Let E be nonempty, bounded subset of a real Banach space $\mathbb{X}$. If E is remotal such that E has Chebyshev center c and the extracted farthest point map $F : X \to E$ restricted to $[c, F(c)]$ is partially ideal statistically continuous at c then E is singleton.*

Theorem 2. *Let $\mathcal{I}$ be a non-trivial, proper, admissible ideal in $\mathbb{N}$. Let $\mathbb{X}$ be a real Banach space and E be non-empty, bounded, uniquely remotal set admitting a Chebyshev center c. If E is not singleton then the farthest point map F, restricted to $(c, F(c)]$ is not partially ideal statistically continuous at c.*

Proof. The uniquely remotal set E has a Chebyshev center c. Let $x \in (c, F(c)]$. Then $x = tc + (1-t)F(c)$ for some $t \in [0,1)$.

Now

$$\begin{aligned}\|x - F(x)\| &= \|tc + (1-t)F(c) - F(x)\| \\ &= \|tc - tF(x) + (1-t)F(c) - (1-t)F(x)\| \\ &\leq t\|c - F(x)\| + (1-t)\|F(c) - F(x)\| \\ &\leq t\|c - F(c)\| + (1-t)\|F(c) - F(x)\| \\ &\leq t\|x - F(x)\| + (1-t)\|F(c) - F(x)\|\end{aligned}$$

$$\Rightarrow \|F(x) - F(c)\| \geq \|x - F(x)\| \geq \|c - F(c)\| = r.$$

If we choose $||c - F(c)|| = r > 0$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $(c, F(c)]$, $\mathcal{I}$-statistically convergent to c but we have

$$\frac{1}{n}\left|\{k \leq n : \|F(x_k) - F(c)\| \geq r\}\right| = 1 \text{ for all } n \in \mathbb{N}.$$

Let $0 < \delta < 1$. Then we have

$$\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \|F(x_k) - F(c)\| \geq r\}| \geq \delta\} = \mathbb{N} \notin \mathcal{I}.$$

This shows that The farthest point map F is not partially ideal statistically continuous at c. This completes the proof. □

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