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ON INTEGRAL INEQUALITIES OF HERMITE–HADAMARD TYPE FOR COORDINATED r-MEAN CONVEX FUNCTIONS

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Abstract. In the paper, the authors first introduce a concept "r-mean convex function on coordinates" and then establish several integral inequalities of the Hermite–Hadamard type for r-convex functions and r-mean convex functions on coordinates.

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1. Introduction

The following definition is well known in the literature.

Definition 1. A function $h : I \subseteq \mathbb{R} \to \mathbb{R}$ is called to be convex if the inequality

$$h(\lambda x + (1-\lambda)y) \le \lambda h(x) + (1-\lambda)h(y)$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$. If the above inequality is reversed, then we call f a concave function.

If $h : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function on $[a,b]$ with $a, b \in I$ and $a < b$, then

$$h\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t \le \frac{h(a)+h(b)}{2}. \tag{1.1}$$

This inequality is called Hermite-Hardamard's integral inequality in the literature.

Hermite–Hadamard's integral inequality (1.1) has been refined, generalized, and applied by a number of mathematicians. For more information, please refer to [7, 12, 25], for example.

Definition 2 ([2, 3])**.** A real function h defined on a convex set $\mathbb{D} \subseteq \mathbb{R}^n$ is said to be r-convex if

$$\phi(q_1 x + q_2 y) \le \begin{cases} \ln\left(q_1 e^{rh(x)} + q_2 e^{rh(y)}\right)^{1/r}, & r \ne 0 \\ q_1 h(x) + q_2 h(y), & r = 0 \end{cases}$$

for all $x, y \in \mathbb{D}$ and $q_1, q_2 \ge 0$ with $q_1 + q_2 = 1$.

Definition 3 ([14])**.** For $r \in \mathbb{R}$, a function $h : I \subseteq \mathbb{R} \to \mathbb{R}_+ = (0, \infty)$ is called to be r-convex if

$$h(\lambda x + (1-\lambda)y) \le \begin{cases} \left\{\lambda[h(x)]^r + (1-\lambda)[h(y)]^r\right\}^{1/r}, & r \ne 0 \\ [h(x)]^\lambda [h(y)]^{1-\lambda}, & r = 0 \end{cases} \tag{1.2}$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$. If the inequality (1.2) is reversed, then we call h an r-concave function.

Definition 4 ([22, 24])**.** For $r \in \mathbb{R}$, a function $h : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ is called to be r-mean convex if

$$h\big([\lambda x^r + (1-\lambda)y^r]^{1/r}\big) \le \left\{\lambda[h(x)]^r + (1-\lambda)[h(y)]^r\right\}^{1/r}, \quad r \ne 0$$

and

$$h\big(x^\lambda y^{1-\lambda}\big) \le [h(x)]^\lambda [h(y)]^{1-\lambda}, \quad r = 0$$

hold for $x, y \in I$ and $\lambda \in [0,1]$. If the above inequality is reversed, then we call h an r-mean concave function.

Definition 5 ([27])**.** For $r \in \mathbb{R}$, a function $h : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ is called to be geometrically r-convex if

$$h\big(x^\lambda y^{1-\lambda}\big) \le \begin{cases} \left\{\lambda[h(x)]^r + (1-\lambda)[h(y)]^r\right\}^{1/r}, & r \ne 0 \\ [h(x)]^\lambda [h(y)]^{1-\lambda}, & r = 0 \end{cases}$$

holds for $x, y \in I$ and $\lambda \in [0,1]$.

For properties and inequalities of the Hermite–Hadamard type relating to r-convex functions, r-mean convex functions, and geometrically r-convex functions, please read the papers [8, 11, 14, 17, 18, 21, 22, 24, 27, 29, 30] and closely related references.

Definition 6 ([5, 7])**.** A function $h : \Delta = [a,b] \times [c,d] \subseteq \mathbb{R}^2 \to \mathbb{R}$ is called to be convex on coordinates on Δ with $a < b$ and $c < d$ if the partial functions

$$h_y : [a,b] \to \mathbb{R}, \quad h_y(u) = h(u,y) \quad \text{and} \quad h_x : [c,d] \to \mathbb{R}, \quad h_x(v) = h(x,v)$$

are convex for all $x \in [a,b]$ and $y \in [c,d]$.

Definition 7 ([5, 7])**.** A function $h : \Delta = [a,b] \times [c,d] \subseteq \mathbb{R}^2 \to \mathbb{R}$ is called to be convex on coordinates on Δ with $a < b$ and $c < d$ if the inequality

$$\begin{aligned} & h(tx + (1-t)z, \lambda y + (1-\lambda)w) \\ & \quad \le t\lambda h(x,y) + t(1-\lambda)h(x,w) + (1-t)\lambda h(z,y) + (1-t)(1-\lambda)h(z,w) \end{aligned}$$

holds for all $t, \lambda \in [0,1]$ and $(x,y), (z,w) \in \Delta$.

Definition 8 ([1])**.** A function $h : \Delta = [a,b] \times [c,d] \subseteq \mathbb{R}^2 \to \mathbb{R}_+$ is called co-ordinated logarithmically convex on Δ with $a < b$ and $c < d$ for all $t, \lambda \in [0,1]$ and $(x,y), (z,w) \in \Delta$ if

$$h(tx+(1-t)z,\lambda y+(1-\lambda)w) \le [h(x,y)]^{t\lambda}[h(x,w)]^{t(1-\lambda)}[h(z,y)]^{(1-t)\lambda}[h(z,w)]^{(1-t)(1-\lambda)}.$$

Definition 9 ([13])**.** For $r\in\mathbb{R}$, a function $h:\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}^2\to\mathbb{R}_+$ is called coordinated r-convex on Δ with $a<b$ and $c<d$ for all $t,\lambda\in[0,1]$ and $(x,y),(z,w)\in\Delta$ if

$$h(tx+(1-t)z,\lambda y+(1-\lambda)w) \le \begin{cases} \big\{t\lambda[h(x,y)]^r+t(1-\lambda)[h(x,w)]^r+(1-t)\lambda[h(z,y)]^r \\ \qquad +(1-t)(1-\lambda)[h(z,w)]^r\big\}^{1/r}, & r\neq 0; \\ [h(x,y)]^{t\lambda}[h(x,w)]^{t(1-\lambda)}[h(z,y)]^{(1-t)\lambda}[h(z,w)]^{(1-t)(1-\lambda)}, & r=0. \end{cases}$$

Remark 1. Obviously, if putting $r=0$ in Definition 9, then h is just the ordinary coordinated logarithmically convex function on Δ.

Stolarsky's mean $E(u,v;r,s)$ for $(u,v;r,s)\in\mathbb{R}_+^2\times\mathbb{R}^2$ is defined by

$$\begin{aligned} E(u,v;r,s)&=\left[\frac{r(v^s-u^s)}{s(v^r-u^r)}\right]^{1/(s-r)}, && rs(r-s)(u-v)\neq 0;\\ E(u,v;0,s)&=\left[\frac{v^s-u^s}{s(\ln v-\ln u)}\right]^{1/s}, && s(u-v)\neq 0;\\ E(u,s;r,r)&=\frac{1}{e^{1/r}}\left(\frac{u^{u^r}}{v^{v^r}}\right)^{1/(u^r-v^r)}, && r(u-v)\neq 0;\\ E(u,v;0,0)&=\sqrt{uv}, && u\neq v;\\ E(u,u;r,s)&=u, && u=v. \end{aligned}$$

The quantities $L(u,v)=E(u,v;0,1)$ and $L_r(u,v)=E(u,v;r,r+1)$ are respectively called the logarithmic mean and generalized logarithmic mean of two real positive numbers u,v. For more information on Stolarsky's mean, please refer to the papers [9, 10, 15, 16, 19, 20] and closely related references therein.

Theorem 1 ([8, Theorem 2.1])**.** *Suppose that $h:[a,b]\subseteq\mathbb{R}\to\mathbb{R}_+$ is a logarithmically convex function with $a<b$. Then*

$$\frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\le L(h(a),h(b)),$$

where $L(x,y)$ is the logarithmic mean.

Theorem 2 ([8, Theorem 3.1])**.** *Suppose that $h:[a,b]\subseteq\mathbb{R}\to\mathbb{R}_+$ is an r-convex function for $r\in\mathbb{R}$ with $a<b$. Then*

$$\frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\le L_r\big(h(a),h(b)\big),$$

where $L_r(x,y)$ is the generalized logarithmic mean.

Theorem 3 ([5,7]). *Let $h:\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}^2$ be convex on coordinates on Δ with $a<b$ and $c<d$. Then*

$$\begin{aligned}
h\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b h\left(x,\frac{c+d}{2}\right)\mathrm{d}x+\frac{1}{d-c}\int_c^d h\left(\frac{a+b}{2},y\right)\mathrm{d}y\right]\\
&\le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d h(x,y)\,\mathrm{d}y\,\mathrm{d}x\\
&\le \frac{1}{4}\left[\frac{1}{b-a}\int_a^b[h(x,c)+h(x,d)]\,\mathrm{d}x+\frac{1}{d-c}\int_c^d[h(a,y)+h(b,y)]\,\mathrm{d}y\right]\\
&\le \frac{1}{4}[h(a,c)+h(b,c)+h(a,d)+h(b,d)].
\end{aligned}$$

Theorem 4 ([1, Theorem 3.3]). *Suppose that $h:\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}^2\to\mathbb{R}_+$ is logarithmically convex on coordinates Δ for $a<b$ and $c<d$. Let*

$$A=\frac{h(a,c)h(b,d)}{h(b,c)h(a,d)},\quad B=\frac{h(a,d)}{h(b,d)},\quad C=\frac{h(b,c)}{h(b,d)}.$$

Then

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d h(x,y)\,\mathrm{d}y\,\mathrm{d}x\le M_h(\Delta),$$

where

$$M_h(\Delta)=\begin{cases}
1, & A=B=C=1;\\
\frac{B-1}{\ln B}\frac{C-1}{\ln C}, & A=1;\\
H(C), & B=1;\\
H(B), & C=1;\\
\frac{C-1}{\ln C}, & A=B=1;\\
\frac{B-1}{\ln B}, & A=C=1;\\
\frac{\gamma+\ln(-\ln A)+Ei(1,-\ln A)}{\ln A}, & B=C=1;\\
\frac{1}{2}\left[\frac{B-1}{\ln B}+\frac{AB-1}{\ln(AB)}\right], & A,B,C>0;\\
\int_0^1 C^\beta\frac{AB-1}{\ln(AB)}\,\mathrm{d}\beta, & \textit{otherwise},
\end{cases}\tag{1.3}$$

$$H(x)=\frac{Ei(1,-\ln A)+\ln\ln x-Ei(1,-\ln(Ax))-\ln\ln(Ax)}{\ln A}+\begin{cases}\frac{2\ln(\ln A)-\ln(-\ln A)}{\ln A}, & -1<\frac{\ln x}{\ln A}<0;\\ 0, & \textit{otherwise},\end{cases}$$

$$Ei(x)=V.P.\int_{-x}^{\infty}\frac{e^{-t}}{t}\,\mathrm{d}t,$$

is the exponential integral function, and γ is the Euler constant.

In very recent years, some other kinds of inequalities of the Hermite–Hadamard type were created in, for example, [1, 4, 6, 13, 23, 26, 28, 30] and closely related references therein.

In this paper, by combining the definition of convex functions with the definition of coordinated convex functions, we introduce the concept "r-mean convex function on coordinates" and establish integral inequalities of the Hermite–Hadamard type for r-mean convex functions on coordinates.

2. A definition and a lemma

In this section, we define a concept "r-mean convex function on coordinates" and prepare a lemma necessary for establishing new inequalities of the Hermite–Hadamard type for r-mean convex function on coordinates.

Definition 10. For $r \in \mathbb{R}$, a function $h : \Delta = [a,b] \times [c,d] \subseteq \mathbb{R}_+^2 \to \mathbb{R}_+$ is called a coordinated r-mean convex on Δ with $a < b$ and $c < d$ if

$$h\big([tx^r + (1-t)z^r]^{1/r}, [\lambda y^r + (1-\lambda)w^r]^{1/r}\big) \le \big\{t\lambda[h(x,y)]^r + t(1-\lambda)[h(x,w)]^r + (1-t)\lambda[h(z,y)]^r + (1-t)(1-\lambda)[h(z,w)]^r\big\}^{1/r}$$

for $r \neq 0$ and

$$h\big(x^t z^{1-t}, y^\lambda w^{1-\lambda}\big) \le [h(x,y)]^{t\lambda}[h(x,w)]^{t(1-\lambda)}[h(z,y)]^{(1-t)\lambda}[h(z,w)]^{(1-t)(1-\lambda)}$$

for $r = 0$, where $t, \lambda \in [0,1]$ and $(x,y), (z,w) \in \Delta$.

Remark 2. In Definition 10, if $r = 0$, then we call h a coordinated geometrically convex function on Δ; if $r = -1$, then we call f a coordinated harmonically convex function on Δ.

In order to prove our main theorems, we need the following lemma.

Lemma 1. *Let $r, R, T, S \in \mathbb{R}$ and $r \neq 0$ such that $R + T + S > 0$, $R + S > 0$, $T + S > 0$, and $S > 0$. Then*

$$F(R,T,S,r) \triangleq \int_0^1 \int_0^1 \big(Rt + T\lambda + S\big)^{1/r} \,\mathrm{d}\lambda \,\mathrm{d}t$$

$$
= \begin{cases} \dfrac{r^2\big[(R+T+S)^{1/r+2}-(R+S)^{1/r+2} -(T+S)^{1/r+2}+S^{1/r+2}\big]}{(r+1)(2r+1)RT}, & (r+1)\left(r+\frac{1}{2}\right)RT \neq 0; \\ \dfrac{r\big[(R+T+S)^{1/r+1}-S^{1/r+1}\big]}{(r+1)(R+T)}, & (r+1)(R+T)\neq 0, \quad RT=0; \\ \dfrac{(R+T+S)\ln(R+T+S)-(R+S)\ln(R+S) -(T+S)\ln(T+S)+S\ln S}{RT}, & r=-1, \quad RT\neq 0; \\ \dfrac{\ln(R+T+S)-\ln S}{R+T}, & r=-1, \quad RT=0, \quad R+T\neq 0; \\ \dfrac{\ln(R+S)+\ln(T+S)-\ln(R+T+S)-\ln S}{RT}, & r=-\frac{1}{2}, \quad RT\neq 0; \\ \dfrac{1}{S(R+T+S)}, & r=-\frac{1}{2}, \quad RT=0; \\ S^{1/r}, & r\neq 0, \quad R=T=0. \end{cases}
$$

Proof. This follows from a straightforward computation. □

3. Main results

In this section, we establish some integral inequalities of the Hermite–Hadamard type for r-mean convex functions on coordinates.

Theorem 5. *Suppose that $h:[a,b]\subseteq \mathbb{R}_+\to\mathbb{R}_+$ is an r-mean convex function on $[a,b]$ for all $r\in\mathbb{R}$ with $r\neq 0$. If $h\in L_1[a,b]$, then*

$$
h\left(\left(\frac{a^r+b^r}{2}\right)^{1/r}\right)\le\left(\frac{r}{b^r-a^r}\int_a^b\frac{[h(x)]^r}{x^{1-r}}\,\mathrm{d}x\right)^{1/r} \tag{3.1}
$$

and

$$
\frac{r}{b^r-a^r}\int_a^b\frac{h(x)}{x^{1-r}}\,\mathrm{d}x\le L_r\big(h(a),h(b)\big),
$$

where $L_r(x,y)$ is the generalized logarithmic mean.

Proof. For $t\in[0,1]$, by the r-mean convexity of h on $[a,b]$, we have

$$
\begin{aligned}
h\left(\left(\frac{a^r+b^r}{2}\right)^{1/r}\right)&=h\left(\left[\frac{ta^r+(1-t)b^r+(1-t)a^r+tb^r}{2}\right]^{1/r}\right)\\
&\le\left(\frac{1}{2}\right)^{1/r}\left\{\big[h([ta^r+(1-t)b^r]^{1/r})\big]^r+\big[h([(1-t)a^r+tb^r]^{1/r})\big]^r\right\}^{1/r}.
\end{aligned}
$$

If $r>0$, then

$$
\begin{aligned}
&\left[h\left(\left(\frac{a^r+b^r}{2}\right)^{1/r}\right)\right]^r\\
&\qquad\le\frac{1}{2}\left\{\big[h\big([ta^r+(1-t)b^r]^{1/r}\big)\big]^r+\big[h\big([(1-t)a^r+tb^r]^{1/r}\big)\big]^r\right\}.
\end{aligned}
$$

Integrating with respect to t over $[0,1]$ and putting $x = ta^r + (1-t)b^r$ for $t \in [0,1]$ yield

$$h\left(\left(\frac{a^r+b^r}{2}\right)^{1/r}\right) \le \left(\frac{r}{b^r-a^r}\int_a^b \frac{[h(x)]^r}{x^{1-r}}\,\mathrm{d}x\right)^{1/r}. \tag{3.2}$$

Similarly, if $r<0$, then we obtain the inequality (3.1). Taking $x^r = ta^r + (1-t)b^r$ for $t \in [0,1]$ and utilizing the r-mean convexity of f on $[a,b]$ lead to

$$\begin{aligned}\frac{r}{b^r-a^r}\int_a^b \frac{h(x)}{x^{1-r}}\,\mathrm{d}x &= \int_0^1 h\big([ta^r+(1-t)b^r]^{1/r}\big)\,\mathrm{d}t\\ &\le \int_0^1 \big\{t[h(a)]^r+(1-t)[h(b)]^r\big\}^{1/r}\,\mathrm{d}t = L_r\big(h(a),h(b)\big).\end{aligned}$$

Theorem 5 is thus proved. □

Theorem 6. *Let $h : \Delta = [a,b]\times[c,d] \subseteq \mathbb{R}_+^2 \to \mathbb{R}_+$ and let $a<b$, $r\in\mathbb{R}$ with $r \ne 0$, and $h \in L_1(\Delta)$. If h is a coordinated r-mean convex function on Δ, then*

$$\begin{aligned}&h\left(\left(\frac{a^r+b^r}{2}\right)^{1/r},\left(\frac{c^r+d^r}{2}\right)^{1/r}\right)\\ &\qquad\le \left[\frac{r^2}{(b^r-a^r)(d^r-c^r)}\int_a^b\int_c^d \frac{[h(x,y)]^r}{(xy)^{1-r}}\,\mathrm{d}y\,\mathrm{d}x\right]^{1/r}\end{aligned}$$

and

$$\begin{aligned}&\frac{r^2}{(b^r-a^r)(d^r-c^r)}\int_a^b\int_c^d \frac{h(x,y)}{(xy)^{1-r}}\,\mathrm{d}y\,\mathrm{d}x\\ &\qquad\le \begin{cases}\frac{F(R,T,S,r)}{2^{1/r}}, & r>0;\\ \min\big\{L_r\big(h(a,c),h(a,d)\big)+L_r\big(h(b,c),h(b,d)\big), & \\ \quad\frac{L_r\big(h(a,c),h(b,c)\big)+L_r\big(h(a,d),h(b,d)\big)\big\}}{2}, & r<0,\end{cases}\end{aligned}$$

where $L_r(x,y)$ is the generalized logarithmic mean, $M_h(\Delta)$ and $F(R,T,S,r)$ are respectively defined as in (1.3) and Lemma 1, and

$$\begin{aligned}R &= h^r(a,c)+h^r(a,d)-h^r(b,c)-h^r(b,d),\\ T &= h^r(a,c)+h^r(b,c)-h(a,d)]^r-h^r(b,d),\\ S &= h^r(b,c)+h^r(a,d)+2h^r(b,d).\end{aligned}$$

Proof. From the coordinated r-mean convexity of h on Δ, we have

$$\begin{aligned}h\left(\left(\frac{a^r+b^r}{2}\right)^{1/r},\left(\frac{c^r+d^r}{2}\right)^{1/r}\right) = h\Bigg(&\left[\frac{ta^r+(1-t)b^r+(1-t)a^r+tb^r}{2}\right]^{1/r},\\ &\left[\frac{\lambda c^r+(1-\lambda)d^r+(1-\lambda)c^r+\lambda d^r}{2}\right]^{1/r}\Bigg)\end{aligned}$$

$$\leq \left(\frac{1}{4}\right)^{1/r} \big\{\big[h\big([ta^r+(1-t)b^r]^{1/r},[\lambda c^r+(1-\lambda)d^r]^{1/r}\big)\big]^r + \big[h([ta^r+(1-t)b^r]^{1/r},[(1-\lambda)c^r+\lambda d^r]^{1/r})\big]^r + \big[h([(1-t)a^r+tb^r]^{1/r},[\lambda c^r+(1-\lambda)d^r]^{1/r})\big]^r + \big[h([(1-t)a^r+tb^r]^{1/r},[(1-\lambda)c^r+\lambda d^r]^{1/r})\big]^r\big\}^{1/r}$$

for all $t,\lambda \in [0,1]$. As did in the proof of the inequality (3.2), we can obtain

$$h\left(\left[\frac{a^r+b^r}{2}\right]^{1/r},\left[\frac{c^r+d^r}{2}\right]^{1/r}\right) \leq \left[\frac{r^2}{(b^r-a^r)(d^r-c^r)}\int_a^b\int_c^d \frac{[h(x,y)]^r}{(xy)^{1-r}}\,\mathrm{d}y\,\mathrm{d}x\right]^{1/r}.$$

Letting $x^r = ta^r+(1-t)b^r$ and $y^r = tc^r+(1-t)d^r$ for $t,\lambda \in [0,1]$ and using the coordinated r-mean convexity of h on Δ lead to

$$\frac{r^2}{(b^r-a^r)(d^r-c^r)}\int_a^b\int_c^d \frac{h(x,y)}{(xy)^{1-r}}\,\mathrm{d}y\,\mathrm{d}x = \int_0^1\int_0^1 h\big([ta^r+(1-t)b^r]^{1/r}, [\lambda c^r+(1-\lambda)d^r]^{1/r}\big)\,\mathrm{d}t\,\mathrm{d}\lambda \leq \int_0^1\int_0^1 \{t\lambda h^r(a,c)+t(1-\lambda)h^r(a,d) +(1-t)\lambda h^r(b,c)+(1-t)(1-\lambda)h^r(b,d)\}^{1/r}\,\mathrm{d}t\,\mathrm{d}\lambda. \tag{3.3}$$

(1) When $r>0$, since $2\lambda t \leq \lambda + t$ for all $\lambda,t \in [0,1]$, by Lemma 1 and the inequality (3.3), we have

$$\frac{r^2}{(b^r-a^r)(d^r-c^r)}\int_a^b\int_c^d \frac{h(x,y)}{(xy)^{1-r}}\,\mathrm{d}y\,\mathrm{d}x \leq \int_0^1\int_0^1 \{t\lambda h^r(a,c) +t(1-\lambda)h^r(a,d)+(1-t)\lambda h^r(b,c)+(1-t)(1-\lambda)h^r(b,d)\}^{1/r}\,\mathrm{d}t\,\mathrm{d}\lambda$$
$$\leq \frac{1}{2^{1/r}}\int_0^1\int_0^1 \{(t+\lambda)h^r(a,c)+(t+(1-\lambda))h^r(a,d)+((1-t)+\lambda)h^r(b,c) +((1-t)+(1-\lambda))h^r(b,d)\}^{1/r}\,\mathrm{d}t\,\mathrm{d}\lambda = \frac{1}{2^{1/r}}F(R,T,S,r).$$

(2) When $r<0$, for $x,y>0$, using the convexity of $g(\lambda) = [\lambda x+(1-\lambda)y]^{1/r}$ on $[0,1]$ and by the inequality (3.3), we obtain

$$\frac{r^2}{(b^r-a^r)(d^r-c^r)}\int_a^b\int_c^d \frac{h(x,y)}{(xy)^{1-r}}\,\mathrm{d}y\,\mathrm{d}x \leq \int_0^1\int_0^1 \{t[\lambda h^r(a,c) +(1-\lambda)h^r(a,d)]+(1-t)[\lambda h^r(b,c)+(1-\lambda)h^r(b,d)]\}^{1/r}\,\mathrm{d}t\,\mathrm{d}\lambda$$
$$\leq \int_0^1\int_0^1 \{t[\lambda h^r(a,c)+(1-\lambda)h^r(a,d)]^{1/r}+(1-t)[\lambda h^r(b,c)$$

$$+(1-\lambda)h^r(b,d)]^{1/r}\big\}\,\mathrm{d}t\,\mathrm{d}\lambda = \frac{1}{2}\int_0^1 \big[\big\{\lambda h^r(a,c)+(1-\lambda)h^r(a,d)\big\}^{1/r}$$
$$+\big\{\lambda h^r(b,c)+(1-\lambda)h^r(b,d)\big\}^{1/r}\big]\,\mathrm{d}\lambda$$
$$= \frac{L_r(h(a,c),h(a,d))+L_r(h(b,c),h(b,d))}{2}.$$

Similarly, we have

$$\frac{r^2}{(b^r-a^r)(d^r-c^r)}\int_a^b\int_c^d \frac{h(x,y)}{(xy)^{1-r}}\,\mathrm{d}y\,\mathrm{d}x \le \int_0^1\int_0^1 \{\lambda[th^r(a,c)$$
$$+(1-t)h^r(b,c)]+(1-\lambda)[th^r(a,d)+(1-t)h^r(b,d)]\}^{1/r}\,\mathrm{d}t\,\mathrm{d}\lambda$$
$$\le \frac{L_r(h(a,c),h(b,c))+L_r(h(a,d),h(b,d))}{2}.$$

Theorem 6 is thus proved. □

Theorem 7. *Suppose that $h : \Delta = [a,b]\times[c,d] \subseteq \mathbb{R}^2_+ \to \mathbb{R}_+$ is geometrically convex on coordinates Δ for $a<b$ and $c<d$. Then*

$$h\big(\sqrt{ab},\sqrt{cd}\big) \le \frac{1}{(\ln b-\ln a)(\ln d-\ln c)}\int_a^b\int_c^d \frac{h(x,y)}{xy}\,\mathrm{d}y\,\mathrm{d}x \le M_h(\Delta),$$

where $M_h(\Delta)$ is defined as in (1.3).

Proof. Since h is geometrically convex on coordinates Δ, we have

$$h\big(\sqrt{ab},\sqrt{cd}\big) = \int_0^1\int_0^1 h\big((a^t b^{1-t})^{1/2}(a^{1-t}b^t)^{1/2},$$
$$(c^\lambda d^{1-\lambda})^{1/2}(c^{1-\lambda}d^\lambda)^{1/2}\big)\,\mathrm{d}t\,\mathrm{d}\lambda$$
$$\le \int_0^1\int_0^1 \big[h\big(a^t b^{1-t},c^\lambda d^{1-\lambda}\big)h\big(a^t b^{1-t},c^{1-\lambda}d^\lambda\big)$$
$$\times h\big(a^{1-t}b^t,c^\lambda d^{1-\lambda}\big)h\big(a^{1-t}b^t,c^{1-\lambda}d^\lambda\big)\big]^{1/4}\,\mathrm{d}t\,\mathrm{d}\lambda$$
$$\le \frac{1}{4}\int_0^1\int_0^1 \big[h\big(a^t b^{1-t},c^\lambda d^{1-\lambda}\big)+h\big(a^t b^{1-t},c^{1-\lambda}d^\lambda\big)$$
$$+h\big(a^{1-t}b^t,c^\lambda d^{1-\lambda}\big)+h\big(a^{1-t}b^t,c^{1-\lambda}d^\lambda\big)\big]\,\mathrm{d}t\,\mathrm{d}\lambda$$
$$= \frac{1}{(\ln b-\ln a)(\ln d-\ln c)}\int_a^b\int_c^d \frac{h(x,y)}{xy}\,\mathrm{d}y\,\mathrm{d}x.$$

Putting $x=a^t b^{1-t}$ and $y=c^\lambda d^{1-\lambda}$ for $0\le t$ and $\lambda\le 1$, utilizing the coordinated geometric convexity of h, and employing Theorem 4 result in

$$\frac{1}{(\ln b-\ln a)(\ln d-\ln c)}\int_a^b\int_c^d \frac{h(x,y)}{xy}\,\mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^1 \int_0^1 h\big(a^t b^{1-t}, c^\lambda d^{1-\lambda}\big)\,\mathrm{d}\lambda\,\mathrm{d}t$$

$$\le \int_0^1 \int_0^1 [h(a,c)]^{t\lambda}[h(a,d)]^{t(1-\lambda)}[h(b,c)]^{(1-t)\lambda}[h(b,d)]^{(1-t)(1-\lambda)}\,\mathrm{d}\lambda\,\mathrm{d}t$$

$$= M_h(\Delta).$$

Theorem 7 is thus proved. □

Corollary 1. *Suppose that $h:\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}_+^2\to\mathbb{R}_+$ is coordinated geometrically convex on Δ for $a<b$ and $c<d$. Then*

$$h\big(\sqrt{ab},\sqrt{cd}\big)\le \frac{1}{(\ln b-\ln a)(\ln d-\ln c)}\int_a^b\int_c^d \frac{h(x,y)}{xy}\,\mathrm{d}y\,\mathrm{d}x \le \frac{h(a,c)+h(a,d)+h(b,c)+h(b,d)}{4}.$$

Theorem 8. *Suppose that $h:\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}^2\to\mathbb{R}_+$ is logarithmically convex on coordinates Δ for $a<b$ and $c<d$. Then*

$$h\left(\frac{a+b}{2},\frac{c+d}{2}\right)\le\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d h(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$

Proof. Since h is geometrically convex on coordinates Δ, we have

$$h\left(\frac{a+b}{2},\frac{c+d}{2}\right)=\int_0^1\int_0^1 h\left(\frac{ta+(1-t)b+(1-t)a+tb}{2},\frac{\lambda c+(1-\lambda)d+(1-\lambda)c+\lambda d}{2}\right)\mathrm{d}t\,\mathrm{d}\lambda$$

$$\le\int_0^1\int_0^1\big[h\big(ta+(1-t)b,\lambda c+(1-\lambda)d\big)h\big(ta+(1-t)b,(1-\lambda)c+\lambda d\big)\times h\big((1-t)a+tb,\lambda c+(1-\lambda)d\big)h\big((1-t)a+tb,(1-\lambda)c+\lambda d\big)\big]^{1/4}\,\mathrm{d}t\,\mathrm{d}\lambda$$

$$\le\frac14\int_0^1\int_0^1\big[h\big(ta+(1-t)b,\lambda c+(1-\lambda)d\big)+h\big(ta+(1-t)b,(1-\lambda)c+\lambda d\big)+h\big((1-t)a+tb,\lambda c+(1-\lambda)d\big)+h\big((1-t)a+tb,(1-\lambda)c+\lambda d\big)\big]\,\mathrm{d}t\,\mathrm{d}\lambda$$

$$=\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d h(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$

Theorem 8 is thus proved. □

Corollary 2. *Under conditions of Theorems 4 and 8, we have*

$$h\left(\frac{a+b}{2},\frac{c+d}{2}\right)\le\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d h(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

$$\leq M_h(\Delta) \leq \frac{h(a,c)+h(a,d)+h(b,c)+h(b,d)}{4}.$$

where $M_h(\Delta)$ is defined as in (1.3).

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