



A NEW CLASS OF GENERALIZED POLYNOMIALS ASSOCIATED WITH MILNE-THOMSON-BASED POLY-BERNOULLI POLYNOMIALS

WASEEM AHMAD KHAN, DIVESH SRIVASTAVA,
AND KOTTAKKARAN SOOPPY NISAR

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Abstract. Motivated by their importance and potential for applications in certain problems in number theory, combinatorics, classical and numerical analysis, and other field of applied mathematics, a variety of polynomials and numbers with their variants and extensions have recently been introduced and investigated. In this sequel, we modify the known generating functions of polynomials, due to both Milne-Thomson and Dere and Simsek, to introduce a new class of generalized polynomials and present some of their involved properties. As obvious special cases of the newly introduced polynomials, we also called power sum-Laguerre-Hermite polynomials and generalized Laguerre and poly-Bernoulli polynomials and present some of their involved identities and formulas. The results presented here, being very general, are pointed out to be specialized to yield a number of known and new identities involving relatively simple and familiar polynomials.

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1. INTRODUCTION

Throughout this presentation, we use the following standard notions $\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{N}_0 = \{0, 1, 2, \dots\} = \mathbb{N} \cup \{0\}$, $\mathbb{Z}^- = \{-1, -2, \dots\}$. Also as usual \mathbb{Z} denotes the set of integers, \mathbb{R} denotes the set of real numbers and \mathbb{C} denotes the set of complex numbers.

The two variable Laguerre polynomials (2-VLP) $L_n(x; y)$ are defined by the following generating function [1, 5]:

$$\frac{1}{1-yt} \exp\left(\frac{-xt}{1-yt}\right) = \sum_{n=0}^{\infty} L_n(x; y) \frac{t^n}{n!} \quad (|yt| < 1).$$

Also, equivalently, the polynomials $L_n(x, y)$ are given by [4]

$$e^{yt} C_0(xt) = \sum_{n=0}^{\infty} L_n(x; y) \frac{t^n}{n!}, \quad (1.1)$$

where $C_0(x)$ denotes the 0th order Tricomi function. The n^{th} order Tricomi functions $C_n(x)$ are defined by [4]

$$\exp\left(t - \frac{x}{t}\right) = \sum_{n=0}^{\infty} C_n(x) t^n \quad (t \in \mathbb{C} \setminus \{0\}, x \in \mathbb{C}).$$

We have

$$C_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{r!(n+r)!}. \quad (1.2)$$

The Tricomi functions $C_n(x)$ are associated with the Bessel function of the first kind $J_n(x)$ [3, 5]

$$C_n(x) = x^{-\frac{n}{2}} J_n(2\sqrt{x}).$$

From (1.1) and (1.2), we note that

$$L_n(x, y) = n! \sum_{k=0}^n \frac{(-1)^k y^{n-k} x^k}{(k!)^2 (n-k)!} = y^n L_n(x, y), \quad (1.3)$$

where $L_n(x)$ are the ordinary Laguerre polynomials [25]. We have

$$L_n(x, 0) = \frac{(-1)^n x^n}{n!}, \quad L_n(0, y) = y^n \quad L_n(x, 1) = L_n(x).$$

For $s \in \mathbb{Z}$, the polylogarithm function is defined by a power series in z as

$$\text{Li}_s(z) = \sum_{j=1}^{\infty} \frac{z^j}{j^s} = z + \frac{z^2}{2^s} + \frac{z^3}{3^s}, \quad (|z| < 1).$$

It is notice that

$$\text{Li}_1(z) = \sum_{j=1}^{\infty} \frac{z^j}{j} = -\log(1-z).$$

In (2015), Jolany *et al.* [7] introduced the generalized poly-Bernoulli polynomials are defined by means of the following generating function:

$$\frac{\text{Li}_k(1 - (ab)^{-t})}{b^t - a^{-t}} e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x; a, b) \frac{t^n}{n!}; \quad |t| < \frac{2\pi}{|\ln a + \ln b|}.$$

In the case when $x = 0$, $B_n^{(k)}(a, b) = B_n^{(k)}(0; a, b)$ are called the generalized poly-Bernoulli numbers [8].

Milne-Thomson [27] defined polynomials $\Phi_n^{(\alpha)}(x)$ of degree n and order α by the following generating function:

$$f(t, \alpha) e^{xt+g(t)} = \sum_{n=0}^{\infty} \Phi_n^{(\alpha)}(x) \frac{t^n}{n!},$$

where $f(t, \alpha)$ is a function of t and $\alpha \in \mathbb{Z}$ and $g(t)$ is a function of t . Then, by choosing some explicit functions of $f(t, \alpha)$ and $g(t)$, he [27] presented several interesting properties for certain polynomials such as Bernoulli polynomials and Hermite polynomials.

Dere and Simsek [6] made a slight modification of the Milne-Thomson polynomials $\Phi_n^{(\alpha)}(x)$ to give polynomials $\Phi_n^{(\alpha)}(x, \nu)$ of degree n and order α by means of the following generating function:

$$G_1(t, x; \alpha, \nu) = f(t, \alpha)e^{xt+h(t, \nu)} = \sum_{n=0}^{\infty} \Phi_n^{(\alpha)}(x; \nu) \frac{t^n}{n!}, \tag{1.4}$$

where $f(t, \alpha)$ and $h(t, \nu)$ are functions of t and $\alpha \in \mathbb{Z}$ and t and $\nu \in \mathbb{N}_0$, respectively, which are analytic in a neighborhood of $t = 0$.

Observe that $\Phi_n^{(\alpha)}(\nu) = \Phi_n^{(\alpha)}(0; \nu)$, (see for details [26, 27]). In particular, choosing $f(t, \alpha) = \left(\frac{\text{Li}_k(1-e^{-t})}{1-e^{-t}}\right)^\alpha$ in (1.4), we obtain the following following polynomials given by the generating function:

$$G_2(t, x; k, \nu) = \frac{\text{Li}_k(1-e^{-t})}{1-e^{-t}} e^{xt+h(t, \nu)} = \sum_{n=0}^{\infty} B_n^{(k)}(x; \nu) \frac{t^n}{n!}, \tag{1.5}$$

Note that the polynomials $B_n^{(k)}(x; \nu)$ are related to both Bernoulli polynomials and Hermite polynomials. For example, if $h(t, 0) = 0$ in (1.5), we get

$$B_n^{(k)}(x; 0) = B_n^{(k)}(x),$$

where $B_n^{(k)}(x)$ are called the poly-Bernoulli polynomials [2].

The two-variable Hermite-Kampé de Fériet polynomials $H_n(x, y)$ [2, 3] are generated by

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!}. \tag{1.6}$$

Note that

$$H_n(x, y) = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} n! \frac{y^r x^{n-2r}}{r!(n-2r)!} \quad (\text{see [9, 10, 14, 16]}),$$

It is clear that

$$H_n(2x, -1) = H_n(x), \quad H_n(x, 0) = x^n.$$

Dere and Simsek [6] generalized the polynomials $H_n(x, y)$ in (1.6) to define two variable Hermite polynomials $H_n^{(l)}(x, y)$ by the following generating function:

$$e^{xt+yt^l} = \sum_{n=0}^{\infty} H_n^{(l)}(x, y) \frac{t^n}{n!}; \quad (l \in \mathbb{N} \setminus \{0\}).$$

Taking $h(t, y) = yt^2$ in (1.5), we get the generalized Hermite poly-Bernoulli polynomials of two variables ${}_H B_n^{(k)}(x, y)$ introduced by Pathan and Khan [21]:

$$\frac{\text{Li}_k(1 - e^{-t})}{1 - e^{-t}} e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_H B_n^{(k)}(x, y) \frac{t^n}{n!}. \quad (1.7)$$

In this paper, we introduce generalized Laguerre-based poly-Bernoulli polynomials by using Milne-Thomson polynomials and investigate some of their properties such as explicit summation formulas, addition formulas, implicit formula and symmetry identities.

2. MILNE-THOMSON BASED POLY-BERNOULLI POLYNOMIALS

In this section, we introduce generalized Laguerre-based poly-Bernoulli polynomials by using Milne-Thomson polynomials. By replacing x by y and v by z in (1.4) and $f(t, \alpha) = \frac{\text{Li}_k(1 - (ab)^{-t})}{b^t - a^{-t}} C_0(xt)$. First, we start with the following definition as.

Definition 1. Let $a, b \in \mathbb{R}^+$ with $a \neq b$. The generalized poly-Bernoulli polynomials $B_n^{(k)}(x, y, z; a, b, e)$ are defined by the following generating function:

$$\frac{\text{Li}_k(1 - (ab)^{-t})}{b^t - a^{-t}} e^{yt+h(t,z)} C_0(xt) = \sum_{n=0}^{\infty} B_n^{(k)}(x, y, z; a, b, e) \frac{t^n}{n!}, \quad (2.1)$$

$$\left(x, y, z \in \mathbb{R}; |t| < \frac{2\pi}{|\ln a + \ln b|} \right).$$

In particular, taking $h(t, z) = zt^2$ in (2.1), we get the following.

Definition 2. Let $a, b > 0$, $a \neq b$. The generalized Laguerre-based poly-Bernoulli polynomials ${}_L B_n^{(k)}(x, y, z; a, b)$ are defined by

$$\frac{\text{Li}_k(1 - (ab)^{-t})}{b^t - a^{-t}} \exp(yt + zt^2) C_0(xt) = \sum_{n=0}^{\infty} {}_L B_n^{(k)}(x, y, z; a, b) \frac{t^n}{n!}, \quad (2.2)$$

$$\left(x, y, z \in \mathbb{R}; |t| < \frac{2\pi}{|\ln a + \ln b|} \right).$$

Remark 1. On setting $z = 0$ in (2.2) reduces to the generalized Laguerre-based poly-Bernoulli polynomials in two variables defined as

$$\frac{\text{Li}_k(1 + (ab)^{-t})}{b^t - a^{-t}} e^{yt} C_0(xt) = \sum_{n=0}^{\infty} {}_L B_n^{(k)}(x, y; a, b) \frac{t^n}{n!}, \quad \left(|t| < \frac{2\pi}{|\ln a + \ln b|} \right).$$

Remark 2. Taking $k = 1$, $a = 1$, $b = e$ and $z = 0$ in (2.2) reduces to the known Laguerre Bernoulli polynomials defined as

$$\left(\frac{t}{e^t - 1} \right) e^{yt} C_0(xt) = \sum_{n=0}^{\infty} {}_L B_n(x, y) \frac{t^n}{n!}; \quad (|t| < 2\pi).$$

Remark 3. Letting $x = 0$ in (2.2) reduces to the known generalized Hermite poly-Bernoulli polynomials defined by [19]

$$\frac{\text{Li}_k(1 - (ab)^{-t})}{b^t - a^{-t}} e^{yt+zt^2} = \sum_{n=0}^{\infty} {}_H B_n^{(k)}(y, z; a, b) \frac{t^n}{n!}, \quad \left(|t| < \left| \frac{2\pi}{(\ln a + \ln b)} \right| \right).$$

By using generalized Laguerre-based poly-Bernoulli polynomials ${}_L B_n^{(k)}(x, y, z; a, b)$ defined by (2.2), we have the following properties which are stated as theorems below.

Theorem 1. For $n \geq 0$, we have

$${}_L B_n^{(k)}(x, y, z; a, b) = \sum_{m=0}^n \binom{n}{m} {}_H B_m^{(k)}(0, z; a, b) L_{n-m}(x, y). \tag{2.3}$$

Proof. Using (1.3), (1.7) and (2.2), we have

$$\begin{aligned} \sum_{n=0}^{\infty} {}_L H B_n^{(k)}(x, y, z; a, b) \frac{t^n}{n!} &= \frac{\text{Li}_k(1 - (ab)^{-t})}{b^t - a^{-t}} \exp(yt + zt^2) C_0(xt) \\ &= \sum_{m=0}^{\infty} {}_H B_m^{(k)}(0, z; a, b) \frac{t^m}{m!} \sum_{n=0}^{\infty} L_n(x, y) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \binom{n}{m} {}_H B_m^{(k)}(0, z; a, b) L_{n-m}(x, y) \right) \frac{t^n}{n!}. \end{aligned}$$

By comparing the coefficients of t^n in the above equation, we obtain the result (2.3). □

Theorem 2. For $n \geq 0$, we have

$${}_L B_n^{(k)}(x, y + u, z + v; a, b) = \sum_{m=0}^n \binom{n}{m} {}_L B_{n-m}^{(k)}(x, y, z; a, b) H_m(u, v). \tag{2.4}$$

$${}_L B_n^{(k)}(x, y + u, z; a, b) = \sum_{m=0}^n \binom{n}{m} {}_L B_{n-m}^{(k)}(x, y; a, b) H_m(u, z). \tag{2.5}$$

Proof. On replacing y by $y + u$ and z by $z + v$ in (2.2), we get

$$\begin{aligned} \sum_{n=0}^{\infty} {}_L B_n^{(k)}(x, y + u, z + v; a, b) \frac{t^n}{n!} &= \frac{\text{Li}_k(1 - (ab)^{-t})}{b^t - a^{-t}} \exp(yt + zt^2) C_0(xt) \exp(ut + vt^2) \\ &= \sum_{n=0}^{\infty} {}_L B_n^{(k)}(x, y, z; a, b) \frac{t^n}{n!} \sum_{m=0}^{\infty} H_m(u, v) \frac{t^m}{m!}, \\ \sum_{n=0}^{\infty} {}_L B_n^{(k)}(x, y + u, z + v; a, b) \frac{t^n}{n!} &= \sum_{n=0}^{\infty} \left[\sum_{m=0}^n \binom{n}{m} {}_L B_{n-m}^{(k)}(x, y, z; a, b) H_m(u, v) \right] \frac{t^n}{n!}. \end{aligned}$$

Now equating the coefficients of the like powers of t in the above equation, we get the result (2.4).

Again, from (2.2), we have

$$\begin{aligned} \sum_{n=0}^{\infty} {}_L B_n^{(k)}(x, y+u, z; a, b) \frac{t^n}{n!} &= \frac{\text{Lik}(1 - (ab)^{-t})}{b^t - a^{-t}} \exp(yt + zt^2) C_0(xt) \exp(ut) \\ &= \frac{\text{Lik}(1 - (ab)^{-t})}{b^t - a^{-t}} e^{yt} C_0(xt) \exp(ut + zt^2) \\ &= \sum_{n=0}^{\infty} {}_L B_n^{(k)}(x, y; a, b) \frac{t^n}{n!} \sum_{m=0}^{\infty} H_m(u, z) \frac{t^m}{m!}. \\ \sum_{n=0}^{\infty} {}_L B_n^{(k)}(x, y+u, z; a, b) \frac{t^n}{n!} &= \sum_{n=0}^{\infty} \left[\sum_{m=0}^n \binom{n}{m} {}_L B_{n-m}^{(k)}(x, y; a, b) H_m(u, z) \right] \frac{t^n}{n!}. \quad (2.6) \end{aligned}$$

On comparing the coefficients of equal powers of t^n in (2.6), we acquire the desired result (2.5). \square

Theorem 3. Let $p, q > 0$ and $n \geq 0$, we have

$${}_L B_n^{(k)}(x, py, qz; a, b) = \sum_{m=0}^n \binom{n}{m} {}_L B_{n-m}^{(k)}(x, (p-1)y, (q-1)z; a, b) H_m(y, z). \quad (2.7)$$

Proof. Rewriting the generating function (2.2), we have

$$\begin{aligned} \sum_{n=0}^{\infty} {}_L B_n^{(k)}(x, py, qz; a, b) \frac{t^n}{n!} &= \frac{\text{Lik}(1 - (ab)^{-t})}{b^t - a^{-t}} e^{yt+zt^2} e^{(p-1)yt+(q-1)z^2} C_0(xt) \\ &= \left(\sum_{n=0}^{\infty} {}_L B_n^{(k)}(x, (p-1)y, (q-1)z; a, b) \frac{t^n}{n!} \right) \left(\sum_{m=0}^{\infty} H_m(y, z) \frac{t^m}{m!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \binom{n}{m} {}_L B_{n-m}^{(k)}(x, (p-1)y, (q-1)z; a, b) H_m(y, z) \right) \frac{t^n}{n!}. \quad (2.8) \end{aligned}$$

On comparing the coefficients of $\frac{t^n}{n!}$ in (2.8), we obtain at the desired result (2.7). \square

3. SUMMATION FORMULAE FOR GENERALIZED LAGUERRE-BASED POLY-BERNOULLI POLYNOMIALS.

In this section, we derive some implicit summation formulae for generalized Laguerre-based poly-Bernoulli polynomials ${}_L B_n^{(k)}(x, y, z; a, b)$. We start with the following theorem as.

Theorem 4. The following relation holds true:

$${}_L B_{l+p}^{(k)}(x, v, z; a, b) = \sum_{m,n=0}^{l,p} \binom{l}{m} \binom{p}{n} (v-y)^{m+n} {}_L B_{l+p-m-n}^{(k)}(x, y, z; a, b). \quad (3.1)$$

Proof. Replace t by $t + u$ in (2.2) and rewrite the generating function (2.2), we have

$$\frac{Li_k(1 - (ab)^{-(t+u)})}{b^t - a^{-(t+u)}} \exp(z(t + u)^2) C_0(x(t + u)) = \exp(-y(t + u)) \sum_{l,p=0}^{\infty} {}_L B_{l+p}^{(k)}(x, y, z; a, b) \frac{t^l u^p}{l! p!}. \quad (3.2)$$

Upon setting y by v in (3.2), it is not difficult to show that

$$\exp((v - y)(t + u)) \sum_{l,p=0}^{\infty} {}_L B_{l+p}^{(k)}(x, y, z; a, b) \frac{t^l u^p}{l! p!} = \sum_{l,p=0}^{\infty} {}_L B_{l+p}^{(k)}(x, v, z; a, b) \frac{t^l u^p}{l! p!}. \quad (3.3)$$

Expanding the exponential part from the above equation (3.3) with the help of result [19], we get

$$\begin{aligned} & \sum_{N=0}^{\infty} \frac{[(v - y)(t + u)]^N}{N!} \sum_{l,p=0}^{\infty} {}_L B_{l+p}^{(k)}(x, y, z; a, b) \frac{t^l u^p}{l! p!} \\ &= \sum_{l,p=0}^{\infty} {}_L B_{l+p}^{(k)}(x, v, z; a, b) \frac{t^l u^p}{l! p!}. \\ & \sum_{m,n=0}^{\infty} \frac{(v - y)^{m+n} t^m u^n}{m! n!} \sum_{l,p=0}^{\infty} {}_L B_{l+p}^{(k)}(x, y, z; a, b) \frac{t^l u^p}{l! p!} \\ &= \sum_{l,p=0}^{\infty} {}_L B_{l+p}^{(k)}(x, v, z; a, b) \frac{t^l u^p}{l! p!}. \end{aligned} \quad (3.4)$$

On replacing l with $l - m$ and p with $p - n$ in L.H.S. of (3.4), yields

$$\begin{aligned} & \sum_{l,p=0}^{\infty} \sum_{m,n=0}^{l,p} \frac{(v - y)^{m+n}}{m! n!} {}_L B_{l+p-m-n}^{(k)}(x, y, z; a, b) \frac{t^l u^p}{(l - m)! (p - n)!} \\ &= \sum_{l,p=0}^{\infty} {}_L B_{l+p}^{(k)}(v, x, z; a, b) \frac{t^l u^p}{l! p!}. \end{aligned} \quad (3.5)$$

On comparing the coefficients of equal powers of t and u form (3.5), we obtain the result (3.1). □

Remark 4. For $l = 0$ in (3.1) reduces to

$${}_L B_p^{(k)}(x, v, z; a, b) = \sum_{n=0}^p \binom{p}{n} (v - y)^n {}_L B_{p-n}^{(k)}(x, y, z; a, b).$$

Remark 5. On taking $v = 0$ in (3.1), we get

$${}_L B_{l+p}^{(k)}(x, z; a, b) = \sum_{m,n=0}^{l,p} \binom{l}{m} \binom{p}{n} (-y)^{m+n} {}_L B_{l+p-m-n}^{(k)}(x, y, z; a, b).$$

Theorem 5. *The following relation holds true:*

$${}_L B_n^{(k)}(x, y+1, z; a, b) = \sum_{m=0}^{n-2j} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-2j}{m} {}_L B_{n-2j-m}^{(k)}(x, y; a, b) \frac{n!}{j!(n-2j)!} z^j. \quad (3.6)$$

Proof. On replacing y by $y+1$, (2.2) converts as

$$\begin{aligned} \sum_{n=0}^{\infty} {}_L B_n^{(k)}(x, y+1, z; a, b) \frac{t^n}{n!} &= \frac{\text{Li}_k(1 - (ab)^{-t})}{b^t - a^{-t}} e^{(y+1)t + zt^2} C_0(xt). \\ \sum_{n=0}^{\infty} {}_L B_n^{(k)}(x, y+1, z; a, b) \frac{t^n}{n!} &= \frac{\text{Li}_k(1 - (ab)^{-t})}{b^t - a^{-t}} e^{yt} C_0(xt) e^{t + zt^2} \\ &= \sum_{n=0}^{\infty} {}_L B_n^{(k)}(x, y; a, b) \frac{t^n}{n!} \sum_{m=0}^{\infty} \frac{t^m}{m!} \sum_{j=0}^{\infty} z^j \frac{t^{2j}}{j!}. \\ \sum_{n=0}^{\infty} {}_L B_n^{(k)}(x, y+1, z; a, b) \frac{t^n}{n!} &= \sum_{n=0}^{\infty} \sum_{m=0}^n \binom{n}{m} {}_L B_{n-m}^{(k)}(x, y; a, b) \frac{t^n}{n!} \sum_{j=0}^{\infty} z^j \frac{t^{2j}}{j!}. \\ L.H.S &= \sum_{n=0}^{\infty} \left[\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{m=0}^{n-2j} \binom{n-2j}{m} z^j {}_L B_{n-2j-m}^{(k)}(y; a, b) \frac{n!}{j!(n-2j)!} \right] \frac{t^n}{n!}. \end{aligned}$$

On comparing of the coefficients of $\frac{t^n}{n!}$, we obtain at the desired result (3.6). \square

Theorem 6. *The following relation holds true:*

$${}_L B_n^{(k)}(x, y+1, z; a, b) \frac{t^n}{n!} = \sum_{m=0}^n \binom{n}{m} {}_L B_m^{(k)}(x, y, z; a, b). \quad (3.7)$$

Proof. We start with the following relation

$$\begin{aligned} \sum_{n=0}^{\infty} {}_L B_n^{(k)}(x, y+1, z; a, b) \frac{t^n}{n!} - \sum_{n=0}^{\infty} {}_L B_n^{(k)}(x, y, z; a, b) \frac{t^n}{n!} \\ &= \frac{\text{Li}_k(1 - (ab)^{-t})}{b^t - a^{-t}} e^{yt + zt^2} C_0(xt) (e^t - 1) \\ &= \sum_{m=0}^{\infty} {}_L B_m^{(k)}(x, y, z; a, b) \frac{t^m}{m!} \sum_{n=0}^{\infty} \frac{t^n}{n!} - \sum_{n=0}^{\infty} {}_L B_n^{(k)}(x, y, z; a, b) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^n \binom{n}{m} {}_L B_m^{(k)}(x, y, z; a, b) \frac{t^n}{n!} - \sum_{n=0}^{\infty} {}_L B_n^{(k)}(x, y, z; a, b) \frac{t^n}{n!}. \end{aligned}$$

On comparing the coefficients of equal powers of t^n produce the result (3.7). \square

4. SYMMETRY IDENTITIES FOR THE GENERALIZED LAGUERRE-BASED POLY-BERNOULLI POLYNOMIALS

Recently, Khan *et al.* [11–13, 15, 17, 18] and Pathan and Khan [19–24] have established some interesting symmetry identities for various polynomials. Here, we present certain symmetry identities for the generalized Laguerre poly-Bernoulli polynomials ${}_L B_n^{(k)}(x, y, z; a, b)$ in the following form.

Theorem 7. *The following symmetry identities hold true:*

$$\begin{aligned} & \sum_{m=0}^n \binom{n}{m} a^{n-m} b^m {}_L B_{n-m}^{(k)}(bx_1, by, b^2z; A, B) {}_L B_m^{(k)}(ax_2, ay, a^2z; A, B) \\ &= \sum_{m=0}^n \binom{n}{m} b^{n-m} a^m {}_L B_{n-m}^{(k)}(ax_1, ay, a^2z; A, B) {}_L B_m^{(k)}(bx_2, by, b^2z; A, B). \end{aligned} \tag{4.1}$$

Proof. Consider the following function

$$A(t) = \left[\frac{(\text{Li}_k(1 - (ab)^{-t}))^2}{(B^{at} - A^{-at})(B^{bt} - A^{-bt})} \right] e^{abyt + a^2b^2z^2} C_0(abx_1t) C_0(afx_2t).$$

We see that the function $A(t)$ is symmetric in a and b and we can expand $A(t)$ into series in two ways to obtain

$$\begin{aligned} A(t) &= \left(\sum_{n=0}^{\infty} {}_L B_n^{(k)}(bx_1, by, b^2z; A, B) \frac{(at)^n}{n!} \right) \left(\sum_{m=0}^{\infty} {}_L B_m^{(k)}(ax_2, ay, a^2z; A, B) \frac{(bt)^m}{m!} \right). \\ A(t) &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \binom{n}{m} a^{n-m} b^m {}_L B_{n-m}^{(k)}(bx_1, by, b^2z; A, B) \right. \\ &\quad \left. \times {}_L B_m^{(k)}(ax_2, ay, a^2z; A, B) \right) \frac{t^n}{n!}. \end{aligned} \tag{4.2}$$

Similarly, we have

$$\begin{aligned} A(t) &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \binom{n}{m} b^{n-m} a^m {}_L B_{n-m}^{(k)}(ax_1, ay, a^2z; A, B) \right. \\ &\quad \left. \times {}_L B_m^{(k)}(bx_2, by, b^2z; A, B) \right) \frac{t^n}{n!}. \end{aligned} \tag{4.3}$$

On comparing the coefficients of t^n in (4.2) and (4.3), we obtain the result (4.1). \square

Remark 6. If $b = 1$, Theorem 7 reduces to

$$\begin{aligned} & \sum_{m=0}^n \binom{n}{m} a^{n-m} {}_L B_{n-m}^{(k)}(x_1, y, z; A, B) {}_L B_m^{(k)}(ax_2, ay, a^2z; A, B) \\ &= \sum_{m=0}^n \binom{n}{m} a^m {}_L B_{n-m}^{(k)}(ax_1, ay, a^2z; A, B) {}_L B_m^{(k)}(x_2, y, z; A, B). \end{aligned}$$

Theorem 8. *The following symmetry identities hold true:*

$$\begin{aligned} & \sum_{m=0}^n \binom{n}{m} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} b^m a^{n-m} {}_L B_n^{(k)}(bx_1, by + \frac{b}{a}i + j, b^2z; A, B) {}_L B_m^{(k)}(ax_2, av; A, B) \\ &= \sum_{m=0}^n \binom{n}{m} \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} a^m b^{n-m} {}_L B_n^{(k)}(ax_1, ay + \frac{a}{b}i + j, a^2z; A, B) \\ & \quad \times {}_L B_m^{(k)}(bx_2, bv; A, B). \end{aligned} \quad (4.4)$$

Proof. Let

$$\begin{aligned} B(t) &= \left[\frac{(\text{Li}_k(1 - (ab)^{-t}))^2}{(B^{at} - A^{-at})(B^{bt} - A^{-bt})} \right] \frac{(e^{abt} - 1)^2 e^{ab(y+v)t + a^2 b^2 z^2}}{(e^{at} - 1)(e^{bt} - 1)} C_0(abx_1 t) C_0(abx_2 t) \\ &= \frac{(\text{Li}_k(1 - (ab)^{-t}))}{(B^{at} - A^{-at})} e^{abyt + a^2 b^2 z^2} C_0(abx_1 t) \left(\frac{(e^{abt} - 1)}{(e^{bt} - 1)} \right) \frac{(\text{Li}_k(1 - (ab)^{-t}))}{(B^{bt} - A^{-bt})} \\ & \quad \times e^{abvt} C_0(abx_2 t) \left(\frac{(e^{abt} - 1)}{(e^{at} - 1)} \right) \\ &= \frac{(\text{Li}_k(1 - (ab)^{-t}))}{(B^{at} - A^{-at})} e^{abyt + a^2 b^2 z^2} C_0(abx_1 t) \\ & \quad \times \sum_{i=0}^{a-1} e^{bti} \frac{(\text{Li}_k(1 - (ab)^{-t}))}{(B^{bt} - A^{-bt})} e^{abvt} C_0(abx_2 t) \sum_{j=0}^{b-1} e^{atj} \\ &= \frac{(\text{Li}_k(1 - (ab)^{-t}))}{(B^{at} - A^{-at})} e^{a^2 b^2 z^2} C_0(abx_1 t) \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} e^{(by + \frac{b}{a}i + j)at} \\ & \quad \times \sum_{m=0}^{\infty} {}_L B_m^{(k)}(ax_2, av; A, B) \frac{(bt)^m}{m!} \\ &= \sum_{n=0}^{\infty} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} {}_L B_n^{(k)}(bx_1, by + \frac{b}{a}i + j, b^2z; A, B) \frac{(at)^n}{n!} \sum_{m=0}^{\infty} {}_L B_m^{(k)}(ax_2, av; A, B) \frac{(bt)^m}{m!}. \end{aligned}$$

$$\begin{aligned} B(t) &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \binom{n}{m} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} b^m a^{n-m} {}_L B_n^{(k)}(bx_1, by + \frac{b}{a}i + j, b^2z; A, B) \right. \\ & \quad \left. \times {}_L B_m^{(k)}(ax_2, av; A, B) \right) \frac{t^n}{n!}. \end{aligned} \quad (4.5)$$

On the other hand,

$$\begin{aligned} B(t) &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \binom{n}{m} \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} a^m b^{n-m} {}_L B_n^{(k)}(ax_1, ay + \frac{a}{b}i + j, a^2z; A, B) \right. \\ & \quad \left. \times {}_L B_m^{(k)}(bx_2, bv; A, B) \right) \frac{t^n}{n!}. \end{aligned} \quad (4.6)$$

Comparing the coefficients of t^n in (4.5) and (4.6), we get the result (4.4). \square

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Authors' addresses

Waseem Ahmad Khan

Prince Mohammad Bin Fahd University, Department of Electrical Engineering, P.O Box 1664, Al Khobar, Saudi Arabia

E-mail address: wkhan1@pmu.edu.sa

Divesh Srivastava

Allenhouse Business School, Allenhouse Institute of Management (Mathematics), Rooma, Kanpur, India

E-mail address: mba.divesh@allenhouse.ac.in

Kottakkaran Sooppy Nisar

(**Corresponding author**) Prince Sattam Bin Abdulaziz University, Department of Mathematics, College of Science and Humanities in Alkharj, Alkharj,, 11942, Saudi Arabia

Current address: Prince Sattam Bin Abdulaziz University, Department of Mathematics, College of Science and Humanities in Alkharj, Alkharj,, 11942, Saudi Arabia

E-mail address: n.sooppy@psau.edu.sa