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$\oplus-$SUPPLEMENTED LATTICES

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Abstract. In this work, $\oplus-$supplemented and strongly $\oplus-$supplemented lattices are defined and investigated some properties of these lattices. Let L be a lattice and $1 = a_1 \oplus a_2 \oplus ... \oplus a_n$ with $a_1, a_2, ..., a_n \in L$. If $a_i/0$ is $\oplus-$supplemented for each $i = 1, 2, ..., n$, then L is also $\oplus-$supplemented. Let L be a distributive lattice and $1 = a_1 \oplus a_2 \oplus ... \oplus a_n$ with $a_1, a_2, ..., a_n \in L$. If $a_i/0$ is strongly $\oplus-$supplemented for each $i = 1, 2, ..., n$, then L is also strongly $\oplus-$supplemented. A lattice L has $(D1)$ property if and only if L is amply supplemented and strongly $\oplus-$supplemented.

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1. INTRODUCTION

Throughout this paper, all lattices are complete modular lattices with the smallest element 0 and the greatest element 1. Let L be a lattice, $a, b \in L$ and $a \leq b$. A sublattice $\{x \in L | a \leq x \leq b\}$ is called a *quotient sublattice*, denoted by b/a. An element a' of a lattice L is called a *complement* of a in L if $a \wedge a' = 0$ and $a \vee a' = 1$, this case we denote $1 = a \oplus a^{'}$ (a and $a^{'}$ also is called *direct summands* of L). L is called a *complemented lattice* if each element has at least one complement in L. An element a of L is said to be *small* or *superfluous* and denoted by $a \ll L$ if $b = 1$ for every element b of L such that $a \vee b = 1$. The meet of all the maximal elements ($\neq 1$) of a lattice L is called the *radical* of L and denoted by $r(L)$. An element c of L is called a *supplement* of b in L if it is minimal for $b \vee c = 1$. a is a supplement of b in a lattice L if and only if $a \vee b = 1$ and $a \wedge b \ll a/0$. A lattice L is said to be *supplemented* if every element of L has a supplement in L. We say that an element b of L *lies above* an element a of L if $a \leq b$ and $b \ll 1/a$. L is said to be *hollow* if every element ($\neq 1$) is superfluous in L, and L is said to be *local* if L has the greatest element ($\neq 1$). An element a of L is called a *weak supplement* of b in L if $a \vee b = 1$ and $a \wedge b \ll L$. A lattice L is said to be *weakly supplemented*, if every element of L has a weak supplement in L. We say that an element $a \in L$ has *ample supplements* in L if for every $b \in L$ with $a \vee b = 1$, a has a supplement b' in L with $b' \leq b$. L is called an *amply supplemented lattice*, if every element of L has ample supplements in L. It is clear that every supplemented lattice is weakly supplemented and every amply supplemented lattice is supplemented. A lattice L is said to be *distributive* if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for every $a, b, c \in L$. Let L be a lattice. It is defined β_* relation on the elements of L by $a\beta_* b$ with $a, b \in L$ if and only if for each $t \in L$ such that $a \vee t = 1$ then $b \vee t = 1$ and for each $k \in L$ such that $b \vee k = 1$ then $a \vee k = 1$.

Let L be a lattice. Consider the following conditions.

$(D1)$ For every element a of L, there exist $a_1, a_2 \in L$ such that $1 = a_1 \oplus a_2$, $a_1 \in a/0$ and $a_2 \wedge a \ll a_2/0$.

$(D3)$ If a_1 and a_2 are direct summands of L and $1 = a_1 \vee a_2$, then $a_1 \wedge a_2$ is also a direct summand of L.

More details about (amply) supplemented lattices are in [1,2] and [5]. More results about (amply) supplemented modules are in [4] and [9]. Some important properties of $\oplus-$supplemented modules are in [6] and [7]. The definition of β_* relation on lattices and some properties of this relation are in [8]. The definition of β^* relation on modules and some properties of this relation are in [3].

In this paper, we generalize some properties of $\oplus-$supplemented modules to lattices. We constitute relationships between $\oplus-$supplemented quotient sublattices and $\oplus-$supplemented lattices by Lemma 11 and Corollary 2. We also constitute relationships between lattices which has $(D1)$ property and strongly $\oplus-$supplemented lattices by Proposition 4. We give some examples at the end of this paper.

Lemma 1. *Let L be a lattice and $a, b, c \in L$ with $a \leq b$. If c is a supplement of b in L, then $a \vee c$ is a supplement of b in $1/a$.*

Proof. Similar to proof of [5, Proposition 12.2(7)]. □

Lemma 2 ([5, Lemma 7.4])**.** *Let L be a lattice, $a, b \in L$ and $a \leq b$. If $a \ll b/0$ then $a \ll L$.*

Lemma 3 ([5, Lemma 7.5])**.** *In a lattice L let $c' \ll c/0$ and $d' \ll d/0$. Then $c' \vee d' \ll (c \vee d)/0$.*

Lemma 4 ([5, Lemma 7.6])**.** *If $a \ll L$, then $a \leq r(L)$.*

Lemma 5 ([5, Exercise 7.3])**.** *If L is a lattice and $a \in L$, then $r(a/0) \leq r(L)$.*

Lemma 6 ([5, Lemma 12.3])**.** *In any modular lattice $[(c \vee d) \wedge b] \leq [c \wedge (b \vee d)] \vee [d \wedge (b \vee c)]$ holds for every $b, c, d \in L$.*

Lemma 7. *Let L be a lattice, $a, b \in L$ and $a \leq b$. Then b lies above a if and only if $a\beta_* b$.*

Proof. $(\Longrightarrow)$ See [8, Theorem 3].

$(\Longleftarrow)$ Let $b \vee t = 1$ with $t \in 1/a$. Since $a\beta_* b$, $a \vee t = 1$ and since $a \leq t$, $t = 1$. Hence $b \ll 1/a$ and b lies above a. □

Lemma 8 ([8, Lemma 2])**.** *Let L be a lattice and $a, b, c \in L$. If $a \vee b = 1$ and $(a \wedge b) \vee c = 1$, then $a \vee (b \wedge c) = b \vee (a \wedge c) = 1$.*

2. $\oplus-$SUPPLEMENTED LATTICES

Definition 1. Let L be a lattice. L is called a $\oplus-$supplemented lattice, if every element of L has a supplement that is a direct summand of L.

Clearly we see that every $\oplus-$supplemented lattice is supplemented and every complemented lattice is $\oplus-$supplemented. We also clearly see that hollow and local lattices are $\oplus-$supplemented.

Proposition 1. *Let L be a lattice. Then L is $\oplus-$supplemented if and only if for every $b \in L$, there exists a direct summand c of L such that $b \vee c = 1$ and $b \wedge c \ll c/0$.*

Proof. Clear from definition. □

Proposition 2. *Let L be a lattice. If every element of L has a weak supplement that is a direct summand of L, then L is $\oplus-$supplemented.*

Proof. Let a be a weak supplement of b in L and a be a direct summand of L. Since a is a weak supplement of b in L, $a \wedge b \ll L$ and since a is a direct summand of L, $a \wedge b \ll a/0$. Hence a is a supplement of b in L and L is $\oplus-$supplemented. □

Lemma 9. *Let L be a lattice, and $a, b \in L$. If x is a supplement of $a \vee b$ in L and y is a supplement of $a \wedge (x \vee b)$ in $a/0$ then $x \vee y$ is a supplement of b in L.*

Proof. Since x is a supplement of $a \vee b$ in L and y is a supplement of $a \wedge (x \vee b)$ in $a/0$, then $1 = a \vee b \vee x$, $(a \vee b) \wedge x \ll x/0$, $a = [a \wedge (x \vee b)] \vee y$ and $(x \vee b) \wedge y = a \wedge (x \vee b) \wedge y \ll y/0$. Here $1 = a \vee b \vee x = [a \wedge (x \vee b)] \vee y \vee b \vee x = b \vee x \vee y$. By Lemma 6, $(x \vee y) \wedge b \leq [(y \vee b) \wedge x] \vee [(x \vee b) \wedge y] \leq [(a \vee b) \wedge x] \vee [(x \vee b) \wedge y] \ll (x \vee y)/0$. Hence $x \vee y$ is a supplement of b in L. □

Lemma 10. *Let L be a lattice and $a_1, a_2 \in L$ where $a_1/0$ and $a_2/0$ are $\oplus-$supplemented and $1 = a_1 \oplus a_2$. Then L is $\oplus-$supplemented.*

Proof. Let x be any element of L. Then $1 = a_1 \vee a_2 \vee x$ and $a_1 \vee a_2 \vee x$ has a supplement 0 in L. Since $a_2/0$ is $\oplus-$supplemented, $a_2 \wedge (a_1 \vee x)$ has a supplement y that is a direct summand in $a_2/0$. By Lemma 9, y is a supplement of $a_1 \vee x$ in L. Since $a_1/0$ is $\oplus-$supplemented, $a_1 \wedge (x \vee y)$ has a supplement z that is a direct summand in $a_1/0$. By Lemma 9, $y \vee z$ is a supplement of x in L. Since y is a direct summand of $a_2/0$ and z is a direct summand of $a_1/0$, by $1 = a_1 \oplus a_2$, $y \vee z = y \oplus z$ is a direct summand of L. Finally, L is $\oplus-$supplemented. □

Corollary 1. *Let L be a lattice, $a_1, a_2, ..., a_n \in L$ and $1 = a_1 \oplus a_2 \oplus ... \oplus a_n$. If $a_i/0$ is $\oplus-$supplemented for every i$= 1, 2, \ldots, n$, then L is $\oplus-$supplemented.*

Proof. Clear from Lemma 10. □

Lemma 11. *Let L be a lattice, $a \in L$ and $a = (a \wedge a_1) \oplus (a \wedge a_2)$ for every $a_1, a_2 \in L$ with $1 = a_1 \oplus a_2$. If L is $\oplus-$supplemented, then $1/a$ is also $\oplus-$supplemented.*

Proof. Let $x \in 1/a$. Since L is $\oplus-$supplemented, there exist $y, z \in L$ such that $1 = x \vee y$, $x \wedge y \ll y/0$ and $1 = y \oplus z$. Since y is a supplement of x in L and $a \leq x$, by Lemma 1, $a \vee y$ is a supplement of x in $1/a$. Since $1 = y \oplus z$, by hypothesis, $a = (a \wedge y) \oplus (a \wedge z)$. Then $(a \vee y) \wedge (a \vee z) = [(a \wedge y) \vee (a \wedge z) \vee y] \wedge [(a \wedge y) \vee (a \wedge z) \vee z] = [y \vee (a \wedge z)] \wedge [(a \wedge y) \vee z] = (a \wedge y) \vee [(y \vee (a \wedge z)) \wedge z] = (a \wedge y) \vee [(y \wedge z) \vee (a \wedge z)] = (a \wedge y) \vee (0 \vee (a \wedge z)) = (a \wedge y) \vee (a \wedge z) = a$. Hence $1/a$ is $\oplus-$supplemented. □

Corollary 2. *Let L be a distributive lattice. If L is $\oplus-$supplemented, then $1/a$ is $\oplus-$supplemented for every $a \in L$.*

Proof. Clear from Lemma 11. □

Lemma 12. *Let L be a supplemented lattice and $a/0$ is a quotient sublattice such that $a \wedge r(L) = 0$. Then every element of $a/0$ is a direct summand of $a/0$.*

Proof. Let $x \in a/0$. Since L is supplemented, there exists an element y of L with $1 = x \vee y$ and $x \wedge y \ll y/0$. Since $1 = x \vee y$ and $x \leq a$, $a = x \vee (a \wedge y)$. Since $x \wedge y \ll y/0$, by Lemma 4, $x \wedge y \leq r(L)$. Then $x \wedge (a \wedge y) = a \wedge x \wedge y \leq a \wedge r(L) = 0$ and $a \wedge x \wedge y = 0$. Hence $a = x \oplus (a \wedge y)$ in $a/0$ and x is a direct summand of $a/0$. □

Corollary 3. *Let L be a supplemented lattice and $a/0$ is a quotient sublattice such that $a \wedge r(L) = 0$. Then $a/0$ is complemented.*

Proof. Clear from Lemma 12. □

Proposition 3. *Let L be a $\oplus-$supplemented lattice. Then there exist $a_1, a_2 \in L$ such that $1 = a_1 \oplus a_2$, $r(a_1/0) \ll a_1/0$ and $r(a_2/0) = a_2$.*

Proof. Since L is $\oplus-$supplemented, there exist $a_1, a_2 \in L$ such that $1 = r(L) \vee a_1 = a_1 \oplus a_2$ and $r(L) \wedge a_1 \ll a_1/0$. Then by Lemma 5, $r(a_1/0) \leq r(L) \wedge a_1 \ll a_1/0$.

Assume x be a maximal ($\neq a_2$) element of $a_2/0$. Since $1/(a_1 \vee x) = (a_1 \oplus a_2)/(a_1 \vee x) = (a_1 \vee x \vee a_2)/(a_1 \vee x) \cong a_2/[a_2 \wedge (a_1 \vee x)] = a_2/[(a_2 \wedge a_1) \vee x] = a_2/x$, $a_1 \vee x$ is a maximal element ($\neq 1$) of L and since $1 = r(L) \vee a_1 \leq a_1 \vee x$, this is a contradiction. Hence $r(a_2/0) = a_2$. □

Definition 2. Let L be a lattice. L is called a completely $\oplus-$supplemented lattice, if every quotient sublattice $a/0$ such that a is a direct summand of L is $\oplus-$supplemented.

Theorem 1. *Let L be a $\oplus-$supplemented lattice with $(D3)$. Then L is completely $\oplus-$supplemented.*

Proof. Let u be a direct summand of L and $x \in u/0$. Since L is $\oplus-$supplemented, then there exists a direct summand y of L such that $1 = x \vee y$ and $x \wedge y \ll y/0$.

Because of $1 = x \vee y$, $u \vee y = 1$ and because of L has $(D3)$, $u \wedge y$ is a direct summand of L and hence $u \wedge y$ is a direct summand of $u/0$. Since $1 = x \vee y$ and $x \leq u$, $u = x \vee (u \wedge y)$. By $x \wedge u \wedge y = x \wedge y \ll y/0$, $x \wedge u \wedge y \ll L$. By $x \wedge u \wedge y \leq u \wedge y$ and $u \wedge y$ is a direct summand of L, $x \wedge u \wedge y \ll u \wedge y$. Thus $u/0$ is $\oplus-$supplemented. □

Definition 3. Let L be a supplemented lattice. L is called a strongly $\oplus-$supplemented lattice if every supplement element in L is a direct summand of L.

Clearly we see that every strongly $\oplus-$supplemented lattice is $\oplus-$supplemented and every complemented lattice is strongly $\oplus-$supplemented. Hollow and local lattices are strongly $\oplus-$supplemented.

Lemma 13. *Let a be a supplement of b in L and $x, y \in a/0$. Then y is a supplement of x in $a/0$ if and only if y is a supplement of $b \vee x$ in L.*

Proof. ($\Longrightarrow$) Let y be a supplement of x in $a/0$ and $b \vee x \vee z = 1$ with $z \leq y$. Because of $x, y \in a/0$ and $z \leq y$, $x \vee z \leq a$. Since a is a supplement of b in L, $a = x \vee z$. Since y is a supplement of x in $a/0$, $z = y$. Hence y is a supplement of $b \vee x$ in L.

($\Longleftarrow$)Let y be a supplement of $b \vee x$ in L. So, $b \vee x \vee y = 1$ and $(b \vee x) \wedge y \ll y/0$. Since $x \vee y \leq a$ and a is a supplement of b in L, $x \vee y = a$ and $x \wedge y \leq (b \vee x) \wedge y \ll y/0$. Hence y is a supplement of x in $a/0$. □

Lemma 14. *Let L be a strongly $\oplus-$supplemented lattice. Then for every direct summand a of L, the quotient sublattice $a/0$ is strongly $\oplus-$supplemented.*

Proof. Let $1 = a \oplus b$ with $b \in L$, $x, y \in a/0$ and y be supplement of x in $a/0$. By Lemma 13, y is a supplement of $b \vee x$ in L. Since L is strongly $\oplus-$supplemented, every supplement element is a direct summand of L and y is a direct summand of L. Here there exists $z \in L$ such that $1 = y \oplus z$. By modularity, $a = a \wedge 1 = a \wedge (y \oplus z) = y \oplus (a \wedge z)$. Thus y is a direct summand of $a/0$. □

Corollary 4. *Every strongly $\oplus-$supplemented lattice is completely $\oplus-$supplemented.*

Proof. Clear from Lemma 14. □

Lemma 15. *Let L be a distributive lattice and $a_1, a_2 \in L$ with $1 = a_1 \oplus a_2$. If $a_1/0$ and $a_2/0$ are strongly $\oplus-$supplemented, then L is also strongly $\oplus-$supplemented.*

Proof. Let a be a supplement of b in L. Since L is distributive, $a = a \wedge 1 = a \wedge (a_1 \oplus a_2) = (a \wedge a_1) \oplus (a \wedge a_2)$ holds. By Lemma 13, $a \wedge a_1$ is a supplement of $(a \wedge a_2) \vee b$ in L. We can also see that $a \wedge a_1$ is a supplement of $a_1 \wedge ((a \wedge a_2) \vee b)$ in $a_1/0$. Since $a_1/0$ is strongly $\oplus-$supplemented, $a \wedge a_1$ is a direct summand of $a_1/0$. Similarly we can see that $a \wedge a_2$ is a direct summand of $a_2/0$. Since $1 = a_1 \oplus a_2$ and $a = (a \wedge a_1) \oplus (a \wedge a_2)$, a is a direct summand of L. Hence L is strongly $\oplus-$supplemented. □

Corollary 5. *Let L be a distributive lattice, $a_1, a_2, ..., a_n \in L$ and $1 = a_1 \oplus a_2 \oplus ... \oplus a_n$. If $a_i/0$ is strongly $\oplus-$supplemented for every i$= 1, 2, \ldots, n$, then L is strongly $\oplus-$supplemented.*

Proof. Clear from Lemma 15. □

Lemma 16. *Let L be a supplemented lattice. The following statements are equivalent.*
(i) *L is strongly $\oplus-$supplemented.*
(ii) *Every supplement element of L lies above a direct summand in L.*
(iii) (a) *For every nonzero supplement element a in L, $a/0$ contains a nonzero direct summand of L.*
(b) *For every nonzero supplement element a in L, $a/0$ contains a maximal direct summand of L.*

Proof. $(i) \Longrightarrow (ii)$ Clear, since every element of L lies above itself.
$(ii) \Longrightarrow (iii)$ Let a be a nonzero supplement element in L. Assume a is a supplement of b in L. By hypothesis, there exists a direct summand x of L such that a lies above x in L. By Lemma 7, $a\beta_* x$ and since $a \vee b = 1$, $x \vee b = 1$. Since a is a supplement of b in L and $x \leq a$, $a = x$ and a is a nonzero direct summand of L.
$(iii) \Longrightarrow (i)$ Let a be a supplement of b in L and x be a maximal direct summand of L with $x \leq a$. Assume $1 = x \oplus y$ with $y \in L$. Then $a = a \wedge 1 = a \wedge (x \oplus y) = x \oplus (a \wedge y)$ and by Lemma 13, $a \wedge y$ is a supplement of $b \vee x$ in L. If $a \wedge y$ is not zero, then by hypothesis, $(a \wedge y)/0$ contains a nonzero direct summand c of L. Here $x \oplus c$ is a direct summand of L and $x \oplus c \leq a$. This contradicts the choice of x. Hence $a \wedge y = 0$ and $a = x$. Thus a is a direct summand of L and L is strongly $\oplus-$supplemented. □

Proposition 4. *Let L be a lattice. The following statements are equivalent.*
(i) *L has $(D1)$ property.*
(ii) *Every element of L lies above a direct summand in L.*
(iii) *L is amply supplemented and strongly $\oplus-$supplemented.*

Proof. $(i) \Longrightarrow (ii)$ Let $a \in L$. Since L has $(D1)$ property, there exist $a_1, a_2 \in L$ such that $1 = a_1 \oplus a_2$, $a_1 \leq a$ and $a_2 \wedge a \ll a_2/0$. Let $a \vee t = 1$ with $t \in 1/a_1$. Since $a_1 \leq a$ and $1 = a_1 \oplus a_2$, $a = a \wedge 1 = a \wedge (a_1 \oplus a_2) = a_1 \oplus (a \wedge a_2)$. Then $1 = a \vee t = a_1 \vee (a \wedge a_2) \vee t = (a \wedge a_2) \vee t$ and since $a \wedge a_2 \ll L$, $t = 1$. Hence $a \ll 1/a_1$ and a lies above a_1.
$(ii) \Longrightarrow (iii)$ Let $a \vee b = 1$ with $a, b \in L$. By hypothesis, $a \wedge b$ lies above a direct summand in L. Here there exist $x, y \in L$ such that $1 = x \oplus y$ and $a \wedge b$ lies above x. Since $1 = x \oplus y$ and $x \leq b$, $b = b \wedge 1 = b \wedge (x \vee y) = x \vee (b \wedge y)$. Then $1 = a \vee b = a \vee x \vee (b \wedge y) = a \vee (b \wedge y)$. By hypothesis, $b \wedge y$ lies above a direct summand in L. Here there exist $x_1, y_1 \in L$ such that $1 = x_1 \oplus y_1$ and $b \wedge y$ lies above x_1. By Lemma 7, $(b \wedge y)\beta_* x_1$ and since $1 = a \vee (b \wedge y)$, $1 = a \vee x_1$ holds.

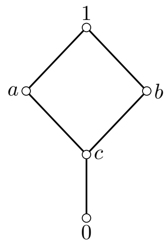
Let $(a \wedge x_1) \vee t = 1$ with $t \in L$. By $a \wedge x_1 \leq a \wedge b \wedge y$, $(a \wedge b \wedge y) \vee t = 1$ holds. Here $y = y \wedge 1 = y \wedge ((a \wedge b \wedge y) \vee t) = (a \wedge b \wedge y) \vee (y \wedge t)$ and $1 = x \vee y = x \vee (a \wedge b \wedge y) \vee (y \wedge t) = x \vee (a \wedge b) \vee (y \wedge t)$. Since $a \wedge b$ lies above x, by Lemma 7, $(a \wedge b)\, \beta_* x$. Then $1 = x \vee (a \wedge b) \vee (y \wedge t) = x \vee (y \wedge t)$ and since y is a supplement of x in L and $y \wedge t \leq y$, $y \wedge t = y$ and $y \leq t$. Hence $1 = (a \wedge b \wedge y) \vee t = t$ and $a \wedge x_1 \ll L$. Since x_1 a direct summand of L, $a \wedge x_1 \ll x_1/0$ and x_1 is a supplement of a in L. Moreover, $x_1 \leq b$. Hence L is amply supplemented. By Lemma 16, L is strongly ⊕−supplemented.

$(iii) \Longrightarrow (i)$ Let a be any element of L. By hypothesis, a has a supplement b in L. Here $1 = a \vee b$ and $a \wedge b \ll b/0$. Since L is amply supplemented, b has a supplement x in L with $x \leq a$. By hypothesis, x is a direct summand of L and there exists an element y of L such that $1 = x \oplus y$. Let $(a \wedge y) \vee t = 1$ with $t \in L$. Since $1 = x \vee y = a \vee y$, by Lemma 8, $a \vee (y \wedge t) = 1$. Since $1 = x \vee b$ and $x \leq a$, $a = a \wedge 1 = a \wedge (x \vee b) = x \vee (a \wedge b)$. Then $1 = a \vee (y \wedge t) = x \vee (a \wedge b) \vee (y \wedge t)$ and since $a \wedge b \ll L$, $1 = x \vee (y \wedge t)$. Since $1 = x \vee (y \wedge t)$ and y is a supplement of x in L, $y \wedge t = y$ and $y \leq t$. Then $1 = (a \wedge y) \vee t = t$ and $a \wedge y \ll L$. Since y is a direct summand of L, $a \wedge y \ll y/0$. Hence L has $(D1)$ property. □

Corollary 6. *Let L be a lattice with $(D1)$ property. Then L is ⊕−supplemented.*

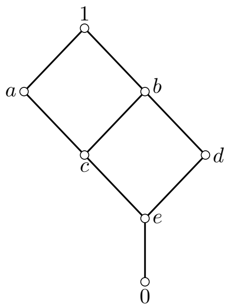
Proof. Clear from Proposition 4 and Corollary 4. □

Example 1. Consider the lattice $L = \{0, a, b, c, 1\}$ given by the following diagram.



Then L is supplemented but not ⊕−supplemented.

Example 2. Consider the lattice $L = \{0, a, b, c, d, e, 1\}$ given by the following diagram.



Then L is supplemented but not $\oplus-$supplemented.

Example 3. Consider the interval $[0,1]$ with natural topology. Let P be the set of all closed subsets of $[0,1]$. P is complete modular lattice by the inclusion (See [1, Example 2.10]). Here $\underset{i\in I}{\wedge} C_i = \underset{i\in I}{\cap} C_i$ and $\underset{i\in I}{\vee} C_i = \overline{\underset{i\in I}{\cup} C_i}$ for every $C_i \in P$ $(i \in I)$ $\left(\overline{\underset{i\in I}{\cup} C_i}\text{ is the closure of } \underset{i\in I}{\cup} C_i\right)$. Let $X \in P$ and $X \vee Y = [0,1]$ with $Y \in P$. Then $[0,1]-X \subset Y$ and since Y is closed $\overline{[0,1]-X} \subset Y$. Let $X^{'} = \overline{[0,1]-X}$. Then $X^{'} \in P$, $X \vee X^{'} = X \cup X^{'} = [0,1]$ and $X^{'} \subset Y$ for every $Y \in P$ with $X \vee Y = [0,1]$. Hence X has ample supplements in P (here $X^{'} = \overline{[0,1]-X}$ is the only supplement of X in P) and P is amply supplemented. Let $A = [0,a] \in P$ with $0 < a < 1$. Here $A^{'} = \overline{[0,1]-A} = [a,1]$ is the only supplement of A in P. Let $A^{'} \vee B = A^{'} \cup B = [0,1]$ with $B \in P$. Since $A^{'} \cup B = [0,1]$, $[0,a) = [0,1]-A^{'} \subset B$ and since B is closed, $[0,a] \subset B$. This case $a \in B$ and since $a \in A^{'}$, $A^{'} \wedge B = A^{'} \cap B \neq \varnothing$. Hence $A^{'}$ is not a direct summand of P and P is not $\oplus-$supplemented.

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