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ALTERNATING SUMS OF THE POWERS OF FIBONACCI AND LUCAS NUMBERS

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Abstract. We shall consider alternating Melham's sums for Fibonacci and Lucas numbers of the form $\sum_{k=1}^{n}(-1)^k F_{2k+\delta}^{2m+\varepsilon}$ and $\sum_{k=1}^{n}(-1)^k L_{2k+\delta}^{2m+\varepsilon}$, where $\varepsilon,\delta \in \{0,1\}$.

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1. Introduction

The Fibonacci F_n and Lucas numbers L_n are defined by the following recursions: for $n > 0$,

$$F_{n+1} = F_n + F_{n-1} \text{ and } L_{n+1} = L_n + L_{n-1},$$

where $F_0 = 0$, $F_1 = 1$ and $L_0 = 2$, $L_1 = 1$, respectively.

If the roots of the characteristic equation $x^2 - x - 1 = 0$ are α and β, then the Binet formulas are

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } L_n = \alpha^n + \beta^n.$$

Wiemann and Cooper [6] raised certain conjectures for the Melham sum:

$$\sum_{k=1}^{n} F_{2k}^{2m+1}.$$

Ozeki [2] considered Melham's sum and he gave an explicit expansion for it as a polynomial in F_{2n+1}.

More generally, Prodinger [3] derived a formula for the sum:

$$\sum_{k=0}^{n} F_{2k+\delta}^{2m+\varepsilon},$$

where $\varepsilon,\delta \in \{0,1\}$. He also evaulated the corresponding sums for the Lucas numbers.

In this paper, we consider the alternating analogs of Melham's sums. We derive explicit formulas for the sums:

$$\sum_{k=1}^{n}(-1)^k F_{2k+\delta}^{2m+\varepsilon} \text{ and } \sum_{k=1}^{n}(-1)^k L_{2k+\delta}^{2m+\varepsilon},$$

where $\varepsilon, \delta \in \{0,1\}$.

2. Alternating Melham's Sums for Fibonacci numbers

In this section we will start with some lemmas and then we shall derive our results about the alternating Melham's sum.

Lemma 1. *For positive integers n,m and t such that $m \geq t$,*

$$i)\ (-1)^{t+1} F_{(2m-2t+1)n} + F_{(2m-2t+1)(n+1)} = \sum_{j=0}^{2m-2t} (-1)^{j(t-2)} F_{(2m-2t+1)n+j+\left(1-3(-1)^t\right)/2},$$

and

$$ii)\ F_{2(m-t)(n+1)} - (-1)^t F_{2(m-t)n} = \begin{cases} F_{2(m-t)n+m-t} L_{m-t} & \text{if } m \text{ is odd,} \\ L_{2(m-t)n+m-t} F_{m-t} & \text{if } m \text{ is even.} \end{cases}$$

Proof. $i)$ We can write

$$\begin{aligned}
& (-1)^{t+1} F_{(2m-2t+1)n} + F_{(2m-2t+1)(n+1)} \\
= \ & (-1)^{t+1}\left(\frac{\alpha^{(2m-2t+1)n} - \beta^{(2m-2t+1)n}}{\alpha-\beta}\right) \\
& + \left(\frac{\alpha^{(2m-2t+1)(n+1)} - \beta^{(2m-2t+1)(n+1)}}{\alpha-\beta}\right) \\
= \ & \frac{\alpha^{(2m-2t+1)n}\left((-1)^{t+1} + \alpha^{2m-2t+1}\right)}{\alpha-\beta} \\
& - \frac{\beta^{(2m-2t+1)n}\left((-1)^{t+1} + \beta^{2m-2t+1}\right)}{\alpha-\beta} \\
= \ & \begin{cases} \sum_{j=0}^{2m-2t} (-1)^j \left(F_{(2m-2t+1)n+j} + F_{(2m-2t+1)n+j+1}\right) & \text{if } t \text{ is odd,} \\ -\sum_{j=0}^{2m-2t} F_{(2m-2t+1)n+j} - F_{(2m-2t+1)n+j+1} & \text{if } t \text{ is even,} \end{cases}
\end{aligned}$$

$$= \begin{cases} \sum_{j=0}^{2m-2t} (-1)^j F_{(2m-2t+1)n+j+2} & \text{if } t \text{ is odd,} \\ \sum_{j=0}^{2m-2t} F_{(2m-2t+1)n+j-1} & \text{if } t \text{ is even,} \end{cases}$$

$$= \sum_{j=0}^{2m-2t} (-1)^{j(t-2)} F_{(2m-2t+1)n+j+(1-3(-1)^t)/2}.$$

ii) The proof is similar to the ones for the Binet formulas of $\{F_n\}$ and $\{L_n\}$. □

From [4], we have the following result for the Gibonacci sequence $\{G_n\}$, defined by for $n > 0$

$$G_{n+1} = G_n + G_{n-1},$$

with arbitrary initial values G_0 and G_1.

Lemma 2. *Let $a, p \neq 0$, q be arbitrary integers. Then for $n > 0$,*

$$\sum_{i=a}^{n} G_{pi+q} = \frac{G_{p(n+1)+q} + (-1)^{p+1} G_{pn+q} + (-1)^p G_{p(a-1)+q} - G_{pa+q}}{L_p - 1 - (-1)^p},$$

and

$$\sum_{i=a}^{n} (-1)^i G_{pi+q}$$
$$= \frac{(-1)^n G_{p(n+1)+q} + (-1)^{p+n} G_{pn+q} + (-1)^a G_{pa+q} + (-1)^{a+p} G_{p(a-1)+q}}{1 + (-1)^p + L_p}.$$

As a consequence of Lemma 2, for further use we state the following result:

Corollary 1. *For any integer r and positive even integer t,*

$$\sum_{k=1}^{n} (-1)^{kr} L_{kt} = \frac{(-1)^{nr} F_{t(n+1)} + (-1)^{nr-r} F_{tn} - F_t}{F_t}$$

and

$$\sum_{k=1}^{n} (-1)^{kr} F_{k(t+1)} = \frac{(-1)^{(n+1)r} F_{(t+1)n} + (-1)^{nr} F_{(t+1)(n+1)} - F_{t+1}}{L_{t+1}}.$$

Proof. Clearly

$$\sum_{k=1}^{n} (-1)^{kr} L_{kt} = (-1)^r L_t + L_{2t} + \ldots + (-1)^{nr} L_{nt}.$$

For the first claim, we consider two cases: the first case is when n is an odd integer. Here

$$\sum_{k=1}^{n}(-1)^{kr}L_{kt}=(-1)^{r}\sum_{j=1}^{(n+1)/2}L_{(2j-1)t}+\sum_{j=1}^{(n-1)/2}L_{2jt}. \tag{2.1}$$

If we take $a=1$, $p=2t$, $q=-t$ and $n\to\frac{n+1}{2}$ in Lemma 2, then we get

$$\sum_{j=1}^{(n+1)/2}L_{(2j-1)t}=\frac{L_{t(n+2)}-L_{tn}+L_{-t}-L_{t}}{L_{2t}-2}.$$

The following identities are well known [1,5]:

$$L_{c+t}-L_{c-t}=5F_{c}F_{t} \tag{2.2}$$

for even t, and

$$L_{2c}-(-1)^{c}2=5F_{c}^{2} \text{ and } L_{-c}=(-1)^{c}L_{c} \tag{2.3}$$

for any integer c. Thus we have

$$\sum_{j=1}^{(n+1)/2}L_{(2j-1)t}=\frac{5F_{t(n+1)}F_{t}}{5F_{t}^{2}}=\frac{F_{t(n+1)}}{F_{t}}. \tag{2.4}$$

Meanwhile, if we take $a=1, p=2t, q=0$ and $n\to\frac{n-1}{2}$ in Lemma 2, then we get

$$\sum_{j=1}^{(n-1)/2}L_{2jt}=\frac{L_{t(n+1)}-L_{t(n-1)}+L_{0}-L_{2t}}{L_{2t}-2}.$$

Since t is even, by (2.2) and (2.3), we rewrite the last equation as

$$\sum_{j=1}^{(n-1)/2}L_{2jt}=\frac{5F_{tn}F_{t}}{5F_{t}^{2}}-1=\frac{F_{tn}}{F_{t}}-1. \tag{2.5}$$

If we sustitute (2.4) and (2.5) in (2.1), then we obtain

$$\begin{aligned}\sum_{k=1}^{n}(-1)^{kr}L_{kt} &= (-1)^{r}\left(\frac{F_{t(n+1)}}{F_{t}}\right)+\left(\frac{F_{tn}}{F_{t}}-1\right)\\ &= \frac{(-1)^{r}F_{t(n+1)}+F_{tn}-F_{t}}{F_{t}}.\end{aligned} \tag{2.6}$$

For the second case, let n be an even integer, thus

$$\sum_{k=1}^{n}(-1)^{kr}L_{kt}=(-1)^{r}\sum_{j=1}^{n/2}L_{(2j-1)t}+\sum_{j=1}^{n/2}L_{2jt}. \tag{2.7}$$

By taking $a=1,\ p=2t,\ q=-t$ and $n\to\frac{n}{2}$ and $a=1,\ p=2t,\ q=0$ and $n\to\frac{n}{2}$ in Lemma 2, respectively, we obtain the following result by (2.2) and (2.3), for even t,

$$\sum_{j=1}^{n/2} L_{(2j-1)t} = \frac{F_{tn}}{F_t}, \tag{2.8}$$

$$\sum_{j=1}^{n/2} L_{2jt} = \frac{F_{t(n+1)}}{F_t}-1. \tag{2.9}$$

If we put (2.8) and (2.9) in (2.7), we get

$$\begin{aligned}\sum_{k=1}^{n}(-1)^{kr}L_{kt} &= (-1)^r\left(\frac{F_{tn}}{F_t}\right)+\left(\frac{F_{t(n+1)}}{F_t}-1\right)\\ &= \frac{(-1)^r F_{tn}+F_{t(n+1)}-F_t}{F_t}.\end{aligned} \tag{2.10}$$

Combining (2.6) and (2.10), we get the final result:

$$\sum_{k=1}^{n}(-1)^{kr}L_{kt}=\frac{(-1)^{nr}F_{t(n+1)}+(-1)^{nr-r}F_{tn}-F_t}{F_t},$$

as claimed.

Finally by taking $a=1,\ p=t+1,\ q=0$ in Lemma 2, the second claim is obtained similary to the first claim. □

Theorem 1. *i) For positive odd m,*

$$\begin{aligned}\sum_{k=1}^{n}(-1)^k F_k^{2m} &= \frac{1}{5^m}\sum_{i=0}^{m-1}(-1)^{i(n+1)+n}\binom{2m}{i}\frac{F_{(m-i)(2n+1)}}{F_{m-i}}\\ &\quad -\frac{1}{5^m}\binom{2m-1}{m}-\frac{1}{5^m}\binom{2m}{m}n,\end{aligned}$$

and, for positive even m,

$$\begin{aligned}&\sum_{k=1}^{n}(-1)^k F_k^{2m}\\ &= \begin{cases}\dfrac{1}{5^m}\left(\displaystyle\sum_{i=0}^{m-1}(-1)^i\binom{2m}{i}\frac{L_{(m-i)(2n+1)}}{L_{m-i}}+\binom{2m-1}{m}\right) & \text{if } n \text{ is even},\\ -\dfrac{1}{5^m}\left(\displaystyle\sum_{i=0}^{m-1}\binom{2m}{i}\frac{L_{(m-i)(2n+1)}}{L_{m-i}}+\binom{2m-1}{m-1}\right) & \text{if } n \text{ is odd},\end{cases}\end{aligned}$$

and

$$ii)\ \sum_{k=1}^{n}(-1)^k F_k^{2m+1} = \frac{1}{5^m}\sum_{i=0}^{m}\binom{2m+1}{i}\frac{(-1)^{i(n+1)+n}}{L_{2m-2i+1}} \times \sum_{j=0}^{2(m-i)}(-1)^{j(i-2)} F_{n(2m-2i+1)+j+\frac{1-3(-1)^i}{2}} - \frac{1}{5^m}\sum_{i=0}^{m}(-1)^i\binom{2m+1}{i}\frac{F_{2m-2i+1}}{L_{2m-2i+1}}.$$

Proof. $i)$ For odd m, consider

$$\begin{aligned}
&\sum_{k=1}^{n}(-1)^k F_k^{2m}\\
=\ & \sum_{k=1}^{n}(-1)^k\left(\frac{\alpha^k-\beta^k}{\alpha-\beta}\right)^{2m}\\
=\ & \sum_{k=1}^{n}(-1)^k\frac{1}{(\alpha-\beta)^{2m}}\sum_{i=0}^{2m}(-1)^i\binom{2m}{i}\alpha^{ki}\beta^{k(2m-i)}\\
=\ & \sum_{k=1}^{n}\frac{(-1)^k}{5^m}\left(\sum_{i=0}^{m}(-1)^i\binom{2m}{i}\left(\alpha^{k(2m-i)}\beta^{ki}+\alpha^{ki}\beta^{k(2m-i)}\right)\right.\\
&\left.-(-1)^m\binom{2m}{m}(\alpha\beta)^{km}\right)\\
=\ & \frac{1}{5^m}\left(\sum_{i=0}^{m-1}(-1)^i\binom{2m}{i}\sum_{k=1}^{n}(-1)^{k(i+1)}L_{2k(m-i)}\right.\\
&\left.+\binom{2m}{m}\sum_{k=1}^{n}(-1)^{m(k+1)+k}\right).
\end{aligned}$$

By taking $i+1=r$ and $2(m-i)=t$ in Corollary 1, we write

$$\begin{aligned}
&\sum_{k=1}^{n}(-1)^k F_k^{2m}\\
=\ & \frac{1}{5^m}\sum_{i=0}^{m-1}(-1)^i\binom{2m}{i}\\
&\times\left(\frac{(-1)^{n(i+1)}F_{2(m-i)(n+1)}+(-1)^{(n+1)(i+1)}F_{2(m-i)n}-F_{2(m-i)}}{F_{2(m-i)}}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{5^m}\binom{2m}{m}\sum_{k=1}^{n}(-1)^{m(k+1)+k} \\
= \ & \frac{1}{5^m}\sum_{i=0}^{m-1}(-1)^{i(n+1)+n}\binom{2m}{i}\left(\frac{F_{2(m-i)(n+1)}-(-1)^i F_{2(m-i)n}}{F_{2(m-i)}}\right) \\
&-\frac{1}{5^m}\sum_{i=0}^{m-1}(-1)^i\binom{2m}{i}+\frac{1}{5^m}\binom{2m}{m}\sum_{k=1}^{n}(-1)^{m(k+1)+k}.
\end{aligned}
$$

Using (ii) in Lemma 1, we have the claimed result. For even m, the desired result is also obtained.

$ii)$ Consider

$$
\begin{aligned}
&\sum_{k=1}^{n}(-1)^k F_k^{2m+1} \\
= \ & \sum_{k=1}^{n}(-1)^k\left(\frac{\alpha^k-\beta^k}{\alpha-\beta}\right)^{2m+1} \\
= \ & \sum_{k=1}^{n}\frac{(-1)^k}{(\alpha-\beta)^{2m+1}}\sum_{i=0}^{2m+1}\binom{2m+1}{i}(-1)^{i+1}\alpha^{ki}\beta^{k(2m+1-i)} \\
= \ & \sum_{k=1}^{n}\frac{(-1)^k}{(\alpha-\beta)^{2m+1}}\sum_{i=0}^{m}\binom{2m+1}{i}(-1)^i\left(\alpha^{k(2m+1-i)}\beta^{ki}-\alpha^{ki}\beta^{k(2m+1-i)}\right) \\
= \ & \frac{1}{5^m}\sum_{i=0}^{m}(-1)^i\binom{2m+1}{i}\sum_{k=1}^{n}(-1)^{k(i+1)}F_{k(2m-2i+1)}.
\end{aligned}
$$

By taking $r=i+1$ and $t=2(m-i)$ in Corollary 1, we write

$$
\begin{aligned}
&\sum_{k=1}^{n}(-1)^k F_k^{2m+1} \\
= \ & \frac{1}{5^m}\sum_{i=0}^{m}(-1)^i\binom{2m+1}{i}\times \\
&\frac{(-1)^{(n+1)(i+1)}F_{(2m-2i+1)n}+(-1)^{n(i+1)}F_{(2m-2i+1)(n+1)}-F_{2m-2i+1}}{L_{2m-2i+1}} \\
= \ & \frac{1}{5^m}\sum_{i=0}^{m}(-1)^{i+n(i+1)}\binom{2m+1}{i}\frac{(-1)^{i+1}F_{(2m-2i+1)n}+F_{(2m-2i+1)(n+1)}}{L_{2m-2i+1}}
\end{aligned}
$$

$$-\frac{1}{5^m}\sum_{i=0}^{m}(-1)^i\binom{2m+1}{i}\frac{F_{2m-2i+1}}{L_{2m-2i+1}}.$$

By Lemma 1, we obtain the claimed result. □

For further use, we state a consequence of Lemma 2:

Corollary 2. *For even positive integer t and $n>0$,*

$$\sum_{k=1}^{n}(-1)^k L_{2tk}=\frac{(-1)^n L_{t(2n+1)}}{L_t}-1.$$

Proof. Substituting $a=1,\ p=2t$ and $q=0$ in Lemma 2, we get

$$\begin{aligned}\sum_{k=1}^{n}(-1)^k L_{2tk} &= \frac{(-1)^n L_{2t(n+1)}+(-1)^n L_{2tn}-L_{2t}-2}{2+L_{2t}}\\ &= \frac{(-1)^n\left(L_{2t(n+1)}+L_{2tn}\right)-(L_{2t}+2)}{2+L_{2t}}.\end{aligned}$$

For even t, from [1,5], we have that

$$L_{c+t}+L_{c-t}=L_cL_t \tag{2.11}$$

and for any c,

$$L_{2c}+(-1)^c 2=L_c^2. \tag{2.12}$$

Thus we obtain

$$\sum_{k=1}^{n}(-1)^k L_{2tk}=(-1)^n\frac{L_{(2n+1)t}}{L_t}-1,$$

as claimed. □

Theorem 2. *For $m>0$,*

$$i)\qquad \sum_{k=1}^{n}(-1)^k F_{2k}^{2m}$$

$$=\begin{cases}\frac{1}{5^m}\sum_{i=0}^{m-1}(-1)^i\binom{2m}{i}\frac{L_{2(m-i)(2n+1)}}{L_{2(m-i)}}+\frac{(-1)^m}{5^m}\binom{2m-1}{m-1} & \text{if } n \text{ is even,}\\ \frac{1}{5^m}\sum_{i=0}^{m-1}(-1)^{i+1}\binom{2m}{i}\frac{L_{2(m-i)(2n+1)}}{L_{2(m-i)}}-\frac{(-1)^m}{5^m}\binom{2m-1}{m-1} & \text{if } n \text{ is odd,}\end{cases}$$

and

$$ii)\qquad \sum_{k=1}^{n}(-1)^k F_{2k}^{2m+1}$$

$$= \frac{1}{5^{m+1}} \sum_{i=0}^{m} (-1)^i \binom{2m+1}{i} \frac{(-1)^n L_{(2m-2i+1)(2n+1)} - L_{2m-2i+1}}{F_{2m-2i+1}}.$$

Proof. i) We write

$$\begin{aligned}
\sum_{k=1}^{n} (-1)^k F_{2k}^{2m} &= \sum_{k=1}^{n} (-1)^k \left(\frac{\alpha^{2k} - \beta^{2k}}{\alpha - \beta} \right)^{2m} \\
&= \sum_{k=1}^{n} (-1)^k \frac{1}{(\alpha-\beta)^{2m}} \sum_{i=0}^{2m} (-1)^i \binom{2m}{i} \alpha^{2ki} \beta^{2k(2m-i)} \\
&= \sum_{k=1}^{n} \frac{(-1)^k}{5^m} \left(\sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \left(\alpha^{2k(2m-i)} \beta^{2ki} + \alpha^{2ki} \beta^{2k(2m-i)} \right) \right. \\
&\quad \left. + (-1)^m \binom{2m}{m} (\alpha\beta)^{2km} \right) \\
&= \sum_{k=1}^{n} \frac{(-1)^k}{5^m} \left(\sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} L_{4k(m-i)} + (-1)^m \binom{2m}{m} \right) \\
&= \frac{1}{5^m} \left(\sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \sum_{k=1}^{n} (-1)^k L_{4k(m-i)} + (-1)^m \binom{2m}{m} \sum_{k=1}^{n} (-1)^k \right).
\end{aligned}$$

If we take $t = 2(m-i)$ in Corollary 2, we write

$$\begin{aligned}
&\sum_{k=1}^{n} (-1)^k F_{2k}^{2m} \\
&= \frac{1}{5^m} \sum_{i=0}^{m-1} \binom{2m}{i} (-1)^i \left(\frac{(-1)^n L_{2(m-i)(2n+1)}}{L_{2(m-i)}} - 1 \right) \\
&\quad + \frac{1}{5^m} \binom{2m}{m} (-1)^m \sum_{k=1}^{n} (-1)^k \\
&= \frac{1}{5^m} \sum_{i=0}^{m-1} (-1)^{n+i} \binom{2m}{i} \frac{L_{2(m-i)(2n+1)}}{L_{2(m-i)}} - \frac{1}{5^m} \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \\
&\quad + \frac{1}{5^m} \binom{2m}{m} (-1)^m \sum_{k=1}^{n} (-1)^k \\
&= \frac{1}{5^m} \sum_{i=0}^{m-1} (-1)^{n+i} \binom{2m}{i} \frac{L_{2(m-i)(2n+1)}}{L_{2(m-i)}}
\end{aligned}$$

$$+\frac{1}{5^m}(-1)^m\left(\binom{2m-1}{m-1}+\binom{2m}{m}\right)\left(\frac{(-1)^n-1}{2}\right).$$

According to the choice of n as an odd or as an even number, we have the conclusion.

$ii)$ The proof is obtained similary to the proof of (i). □

For further use, we state the following result:

Corollary 3. *For positive even integer t and $n>0$,*

$$\sum_{k=1}^{n}(-1)^k L_{(2k+1)t}=\frac{(-1)^n L_{2t(n+1)}-L_{2t}}{L_t}.$$

Proof. Substituting $a=1,\ p=2t$ and $q=t$ in Lemma 2, we get

$$\begin{aligned}\sum_{k=1}^{n}(-1)^k L_{(2k+1)t} &= \frac{(-1)^n L_{(2n+3)t}+(-1)^n L_{(2n+1)t}-L_{3t}-L_t}{2+L_{2t}}\\ &= \frac{(-1)^n\left(L_{(2n+3)t}+L_{(2n+1)t}\right)-(L_{3t}+L_t)}{2+L_{2t}}.\end{aligned}$$

By (2.11) and (2.12), we rewrite the last equation as for even t,

$$\begin{aligned}\sum_{k=1}^{n}(-1)^k L_{(2k+1)t} &= (-1)^n\frac{L_tL_{2t(n+1)}}{L_t^2}-\frac{L_{3t}+L_t}{L_t^2}\\ &= (-1)^n\frac{L_{2t(n+1)}}{L_t}-\frac{L_{2t}}{L_t}.\end{aligned}$$

Thus we have the conclusion. □

Theorem 3. *For $m>0$,*

$$i)\sum_{k=1}^{n}(-1)^k F_{2k+1}^{2m}$$

$$=\begin{cases}\dfrac{-1}{5^m}\displaystyle\sum_{i=0}^{m-1}\binom{2m}{i}\frac{L_{4(m-i)(n+1)}+L_{4(m-i)}}{L_{2(m-i)}}-\frac{1}{5^m}\binom{2m}{m} & \text{if } n \text{ is odd}\\[2ex] \dfrac{1}{5^m}\displaystyle\sum_{i=0}^{m-1}\binom{2m}{i}\frac{L_{4(m-i)(n+1)}-L_{4(m-i)}}{L_{2(m-i)}} & \text{if } n \text{ is even,}\end{cases}$$

and

$$ii)\sum_{k=1}^{n}(-1)^k F_{2k+1}^{2m+1}=\frac{1}{5^{m+1}}\sum_{i=0}^{m}\binom{2m+1}{i}\frac{(-1)^n L_{(4m-4i+2)(n+1)}-L_{4m-4i+2}}{F_{2m-2i+1}}.$$

Proof. $i)$ Consider

$$
\begin{aligned}
&\sum_{k=1}^{n}(-1)^k F_{2k+1}^{2m}\\
=\ &\sum_{k=1}^{n}(-1)^k\left(\frac{\alpha^{2k+1}-\beta^{2k+1}}{\alpha-\beta}\right)^{2m}\\
=\ &\sum_{k=1}^{n}\frac{(-1)^k}{(\alpha-\beta)^{2m}}\sum_{i=0}^{2m}(-1)^i\binom{2m}{i}\alpha^{(2k+1)i}\beta^{(2k+1)(2m-i)}\\
=\ &\sum_{k=1}^{n}\frac{(-1)^k}{5^m}\left(\sum_{i=0}^{m-1}(-1)^i\binom{2m}{i}\right.\\
&\left(\alpha^{(2k+1)(2m-i)}\beta^{(2k+1)i}+\alpha^{(2k+1)i}\beta^{(2k+1)(2m-i)}\right)\\
&\left.+\binom{2m}{m}(\alpha\beta)^{2km}\right)\\
=\ &\sum_{k=1}^{n}\frac{(-1)^k}{5^m}\left(\sum_{i=0}^{m-1}\binom{2m}{i}L_{2(2k+1)(m-i)}+\binom{2m}{m}\right)\\
=\ &\frac{1}{5^m}\left(\sum_{i=0}^{m-1}\binom{2m}{i}\sum_{k=1}^{n}(-1)^k L_{2(2k+1)(m-i)}+\binom{2m}{m}\sum_{k=1}^{n}(-1)^k\right)
\end{aligned}
$$

If we take $t=2(m-i)$ in Corollary 3, we write

$$
\begin{aligned}
&\sum_{k=1}^{n}(-1)^k F_{2k+1}^{2m}\\
=\ &\frac{1}{5^m}\sum_{i=0}^{m-1}\binom{2m}{i}\frac{(-1)^n L_{4(m-i)(n+1)}-L_{4(m-i)}}{L_{2(m-i)}}+\frac{1}{5^m}\binom{2m}{m}\sum_{k=1}^{n}(-1)^k\\
=\ &\frac{(-1)^n}{5^m}\sum_{i=0}^{m-1}\binom{2m}{i}\frac{L_{4(m-i)(n+1)}}{L_{2(m-i)}}\\
&-\frac{1}{5^m}\sum_{i=0}^{m-1}\binom{2m}{i}\frac{L_{4(m-i)}}{L_{2(m-i)}}+\frac{1}{5^m}\binom{2m}{m}\left(\tfrac{(-1)^n-1}{2}\right).
\end{aligned}
$$

According to the chice of n as an odd or as an even number, we have the conclusion.

$ii)$ The proof of (ii) is obtained similary to the proof of (i). □

3. Alternating Melham's sum for Lucas numbers

Theorem 4. *i) For positive odd m,*

$$\sum_{k=1}^{n}(-1)^k L_k^{2m} = \sum_{i=0}^{m-1}(-1)^{(i+1)n}\binom{2m}{i}\frac{F_{(m-i)(2n+1)}}{F_{m-i}} - 2^{2m-1} + \binom{2m-1}{m} + \binom{2m}{m}n,$$

and, for even m,

$$\sum_{k=1}^{n}(-1)^k L_k^{2m} = \begin{cases} \sum\limits_{i=0}^{m-1}\binom{2m}{i}\frac{L_{(m-i)(2n+1)}}{L_{m-i}} - 2^{2m-1} + \binom{2m-1}{m} & \text{if } n \text{ is even,} \\ \sum\limits_{i=0}^{m-1}(-1)^{i+1}\binom{2m}{i}\frac{L_{(m-i)(2n+1)}}{L_{m-i}} - 2^{2m-1} + \binom{2m-1}{m} - \binom{2m}{m} & \text{if } n \text{ is odd,} \end{cases}$$

and

$$ii)\ \sum_{k=1}^{n}(-1)^k L_k^{2m+1} = \sum_{i=0}^{m}\binom{2m+1}{i}\frac{(-1)^{(i+1)n+1}}{L_{2m-2i+1}} \times \sum_{j=0}^{2(m-i)}(-1)^{ji+1} L_{n(2m-2i+1)+j+\frac{1-3(-1)^i}{2}} + 2\sum_{i=0}^{m}(-1)^i\binom{2m+1}{i}\frac{1}{L_{2m-2i+1}} - 2^{2m}.$$

Proof. $i)$ By the Binet formula of $\{L_n\}$, we have

$$\sum_{k=1}^{n}(-1)^k L_k^{2m} = \sum_{k=1}^{n}(-1)^k\left(\alpha^k+\beta^k\right)^{2m}$$

$$= \sum_{k=1}^{n}(-1)^k\left(\sum_{i=0}^{m-1}\binom{2m}{i}\left(\alpha^{k(2m-i)}\beta^{ki}+\alpha^{ki}\beta^{k(2m-i)}\right)+\binom{2m}{m}(\alpha\beta)^{km}\right).$$

Since $\alpha\beta = -1$, we get

$$\sum_{k=1}^{n}(-1)^k L_k^{2m} = \sum_{i=0}^{m-1}\binom{2m}{i}\sum_{k=1}^{n}(-1)^{k(i+1)} L_{2k(m-i)} + \binom{2m}{m}\sum_{k=1}^{n}(-1)^{k(m+1)}.$$

From Corollary 1, we write

$$\begin{aligned}
&\sum_{k=1}^{n}(-1)^k L_k^{2m} \\
= \; & \sum_{i=0}^{m-1}\binom{2m}{i} \\
&\left(\frac{(-1)^{(i+1)n} F_{2(m-i)(n+1)} + (-1)^{(n+1)(i+1)} F_{2(m-i)n} - F_{2(m-i)}}{F_{2(m-i)}}\right) \\
&+\binom{2m}{m}\sum_{k=1}^{n}(-1)^{k(m+1)} \\
= \; & \sum_{i=0}^{m-1}(-1)^{(i+1)n}\binom{2m}{i}\frac{F_{2(m-i)(n+1)} + (-1)^{i+1} F_{2(m-i)n}}{F_{2(m-i)}} \\
&-\sum_{i=0}^{m-1}\binom{2m}{i} + \binom{2m}{m}\sum_{k=1}^{n}(-1)^{k(m+1)} \\
= \; & \begin{cases}
\sum_{i=0}^{m-1}(-1)^{(i+1)n}\binom{2m}{i}\frac{F_{(m-i)(2n+1)}}{F_{m-i}} - 2^{2m-1} + \binom{2m-1}{m} + \binom{2m}{m}n & \text{if } m \text{ is odd,} \\
\sum_{i=0}^{m-1}(-1)^{(i+1)n}\binom{2m}{i}\frac{L_{(m-i)(2n+1)}}{L_{m-i}} - 2^{2m-1} + \binom{2m-1}{m} + \binom{2m}{m}\left(\frac{(-1)^n-1}{2}\right) & \text{if } m \text{ is even.}
\end{cases}
\end{aligned}$$

According to the choice of n, the claim is obtained.

ii) The proof is similar to the proof of i). □

Theorem 5. i) *For $m > 0$ and even $n > 0$,*

$$\sum_{k=1}^{n}(-1)^k L_{2k}^{2m} = \sum_{i=0}^{m-1}\binom{2m}{i}\frac{L_{2(m-i)(2n+1)}}{L_{2(m-i)}} - 2^{2m-1} + \binom{2m-1}{m},$$

and, for odd $n > 0$,

$$\sum_{k=1}^{n} (-1)^k L_{2k}^{2m} = -\sum_{i=0}^{m-1} \binom{2m}{i} \frac{L_{2(m-i)(2n+1)}}{L_{2(m-i)}} - 2^{2m-1} - \binom{2m-1}{m}.$$

$ii)$ For positive integers m and n,

$$\sum_{k=1}^{n} (-1)^k L_{2k}^{2m+1} = (-1)^n \sum_{i=0}^{m} \binom{2m+1}{i} \frac{F_{(2m-2i+1)(2n+1)}}{F_{2m-2i+1}} - 2^{2m}.$$

Proof. $i)$ We write

$$\begin{aligned}
& \sum_{k=1}^{n} (-1)^k L_{2k}^{2m} \\
= \ & \sum_{k=1}^{n} (-1)^k \left(\alpha^{2k} + \beta^{2k}\right)^{2m} \\
= \ & \sum_{k=1}^{n} (-1)^k \\
& \left(\sum_{i=0}^{m-1} \binom{2m}{i} \left(\alpha^{2k(2m-i)} \beta^{2ki} + \alpha^{2ki} \beta^{2k(2m-i)}\right) + \binom{2m}{m} (\alpha\beta)^{2km} \right) \\
= \ & \sum_{i=0}^{m-1} \binom{2m}{i} \sum_{k=1}^{n} (-1)^k L_{4k(m-i)} + \binom{2m}{m} \sum_{k=1}^{n} (-1)^k .
\end{aligned}$$

If we take $t = 2(m-i)$ in Corollary 2, we can write

$$\begin{aligned}
& \sum_{k=1}^{n} (-1)^k L_{2k}^{2m} \\
= \ & \sum_{i=0}^{m-1} \binom{2m}{i} \left(\frac{(-1)^n L_{2(m-i)(2n+1)}}{L_{2(m-i)}} - 1 \right) + \binom{2m}{m} \sum_{k=1}^{n} (-1)^k \\
= \ & (-1)^n \sum_{i=0}^{m-1} \binom{2m}{i} \frac{L_{2(m-i)(2n+1)}}{L_{2(m-i)}} - \sum_{i=0}^{m-1} \binom{2m}{i} + \binom{2m}{m} \sum_{k=1}^{n} (-1)^k \\
= \ & (-1)^n \sum_{i=0}^{m-1} \binom{2m}{i} \frac{L_{2(m-i)(2n+1)}}{L_{2(m-i)}} - 2^{2m-1} + \binom{2m-1}{m-1} + \binom{2m}{m} \left(\tfrac{(-1)^n - 1}{2} \right).
\end{aligned}$$

According to the choice of n, the proof is complete by the fact that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

$ii)$ The proof of (ii) is similar to the proof of (i). □

Theorem 6. *For $m > 0$,*

$$i) \sum_{k=1}^{n} (-1)^k L_{2k+1}^{2m}$$

$$= \begin{cases} \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \frac{L_{4(m-i)(n+1)} - L_{4(m-i)}}{L_{2(m-i)}} & \text{if } n \text{ is even,} \\ (-1)^{m+1} \binom{2m}{m} - \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \frac{L_{4(m-i)(n+1)} + L_{4(m-i)}}{L_{2(m-i)}} & \text{if } n \text{ is odd,} \end{cases}$$

and

$$ii) \sum_{k=1}^{n} (-1)^k L_{2k+1}^{2m+1} =$$

$$\sum_{i=0}^{m} (-1)^i \binom{2m+1}{i} \frac{(-1)^n F_{2(2m-2i+1)(n+1)} - F_{2(2m-2i+1)}}{F_{2m-2i+1}}.$$

Proof. $i)$ Using the Binet formula of $\{L_n\}$, we have

$$\begin{aligned} &\sum_{k=1}^{n} (-1)^k L_{2k+1}^{2m} \\ = &\sum_{k=1}^{n} (-1)^k \left(\alpha^{2k+1} + \beta^{2k+1}\right)^{2m} \\ = &\sum_{k=1}^{n} (-1)^k \left(\sum_{i=0}^{m-1} \binom{2m}{i} \left(\alpha^{(2k+1)(2m-i)} \beta^{(2k+1)i} + \alpha^{(2k+1)i} \beta^{(2k+1)(2m-i)}\right) \right. \\ &\left. + \binom{2m}{m} (\alpha\beta)^{(2k+1)m} \right) \\ = &\sum_{k=1}^{n} (-1)^k \left(\sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \left(\alpha^{(2k+1)(2m-2i)} + \beta^{(2k+1)(2m-2i)}\right)\right. \end{aligned}$$

$$
\begin{aligned}
&\quad + \binom{2m}{m} (-1)^m \Big) \\
&= \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \sum_{k=1}^{n} (-1)^k L_{2(2k+1)(m-i)} + \binom{2m}{m} (-1)^m \sum_{k=1}^{n} (-1)^k .
\end{aligned}
$$

If we take $t = 2(m-i)$ in Corollary 3, we get

$$
\begin{aligned}
& \sum_{k=1}^{n} (-1)^k L_{2k+1}^{2m} \\
= & \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \left(\frac{(-1)^n L_{4(m-i)(n+1)} - L_{4(m-i)}}{L_{2(m-i)}} \right) \\
& + \binom{2m}{m} (-1)^m \sum_{k=1}^{n} (-1)^k \\
= & \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \frac{(-1)^n L_{4(m-i)(n+1)}}{L_{2(m-i)}} - \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \frac{L_{4(m-i)}}{L_{2(m-i)}} \\
& + \binom{2m}{m} (-1)^m \sum_{k=1}^{n} (-1)^k \\
= & (-1)^n \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \frac{L_{4(m-i)(n+1)}}{L_{2(m-i)}} - \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \frac{L_{4(m-i)}}{L_{2(m-i)}} \\
& + \binom{2m}{m} (-1)^m \left(\tfrac{(-1)^n - 1}{2} \right).
\end{aligned}
$$

According to the choice of n, the claimed result is clear.

ii) The proof is similar to the proof of i). □

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