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Sándor Radeleczki

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CLASSIFICATION SYSTEMS AND THE DECOMPOSITIONS OF A LATTICE INTO DIRECT PRODUCTS

Sándor Radeleczki
Institute of Mathematics, University of Miskolc
3515 Miskolc – Egyetemváros, Hungary
matradi@gold.uni-miskolc.hu

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Abstract. We introduce the notion of a classification system for an arbitrary complete lattice. We give a characterization of those pseudocomplemented lattices which have the property that any of their classification systems yields a decomposition of the lattice into a direct product. By applying these results we prove new stucture theorems for particular classes of complete pseudocomplemented lattices.

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1. Introduction

The notion of the classification system has its origin in an application of Concept Lattices to one of the main problems of Group Technology, namely, to classify some technical objects on the basis of their properties.

Given a set G of (technical) objects and a set M of (possible) properties, a binary relation $I \subseteqq G \times M$ is defined as follows:

$(g, m) \in I$ if and only if the object $g \in G$ has the property $m \in M$.

The triple (G, M, I) is called *formal context* in mathematical literature and (by using the basic construction of Formal Concept Analysis) a complete lattice $\mathcal{L}(G, M, I)$ is associated with it, which is called the *concept lattice of the context* (G, M, I). (For details see [3].) If the objects with the same properties are considered identical - and this is the case in our technical application - then the context (G, M, I) is called *row-reduced.* A *classification of the elements of* G means a partition $\pi = \{G_i, i \in I\}$ of the set G, where any block G_i of π is characterized by the common properties of its objects. The observation (see [7]) which led us to our investigations was the following:

"If (G, M, I) is a row-reduced context, then to any classification $\pi = \{G_i, i \in I\}$ of the elements of G corresponds a system $\{a_i, i \in I\}$ of nonzero elements of the concept lattice $\mathcal{L}(G, M, I)$ satisfying the conditions (1) and (2) of Definition 2.1. Conversely, any set of elements of $\mathcal{L}(G, M, I)$ which satisfies conditions (1) and (2) induces a classification of the elements of G."

In the next Section, we define and study classification systems in an arbitrary complete lattice presenting some of their basic properties, making abstraction from the original problem and the theory of Concept Lattices. In Section 3 we give a characterization of those complete pseudocomplemented lattices which have the property that any of their classification systems yields a decomposition of the given lattice into a direct product. In Section 4 we apply these results to describe direct product decompositions of certain pseudocomplemented lattices.

2. Basic notions

Let 0 and 1 stand for the least and the greatest element of a bounded lattice L, respectively. $(x]$ is our notation for the principal ideal generated by an $x \in L$. The supremum of a set $A \subseteqq L$ (if it exists) is denoted by $\vee A$. We set $\vee\varnothing = 0$, as usual.

Definition 2.1. Let L be a complete lattice. A set $S = \{a_i \mid i \in I\}$, $I \neq \varnothing$ of nonzero elements of L is called a *classification system* of L if the following conditions are satisfied:

(1) $\quad a_i \wedge a_j = 0$, for all $i \neq j$,

(2) $\quad x = \bigvee_{i\in I} (x \wedge a_i)$, for all $x \in L$.

If $S = \{1\}$, then we say that S is *trivial.* S is called *weakly independent* if $a_j \wedge (\bigvee_{i\in I\setminus\{j\}} a_j) = 0$ holds for all $j \in I$. We say that S is *strongly independent* if the relation $(\bigvee_{i\in J} a_i) \wedge (\bigvee_{i\in K} a_i) = (\bigvee_{i\in J\cap K} a_j)$ holds for all $K, J \subseteqq I$.

The following lemma contains some simple properties of the classification systems.

Lemma 2.2. *Let $S = \{a_i \mid i \in I\}$ be a classification system of a complete lattice L. Then the following statements hold:*

(i) *We have $\bigvee_{i\in I} a_i = 1$ and for any $b \in L$, $b \neq 0$ the nonzero elements of the set $\{b \wedge a_i \mid i \in I\}$ form a classification system of the sublattice $(b]$.*

(ii) *If $T = \{b_j \mid j \in J\}$ is a classification system of $(a_{i_0}]$ ($i_0 \in I$), then $S' = \{a_i \mid i \in I \setminus \{i_0\}\} \cup \{b_j \mid j \in J\}$ is also a classification system for L.*

(iii) *Let $K \subseteqq I$, $K \neq \varnothing$ arbitrary and $b = \bigvee_{i\in K} a_i$. If S is weakly independent then $S^* = \{a_i \mid i \in I \setminus K\} \cup \{b\}$ is also a classification system of L.*

Proof. (i) Substituting $x = 1$ in relation (2) we obtain $1 = \bigvee_{i\in I} a_i$. Let $V = \{b \wedge a_i \mid i \in K\}$ be the subset of nonzero elements of $\{b \wedge a_i \mid i \in I\}$ (here $K \subseteqq I$). Since by (2) we have $b = \bigvee_{i\in I} (b \wedge a_i)$, V cannot be empty. As we have $(b \wedge a_i) \wedge (b \wedge a_j) = b \wedge (a_i \wedge a_j) = 0$ for $i \neq j$, relation (1) is satisfied by V. Now take an $x \in (b]$, then

$x \wedge (b \wedge a_i) = (x \wedge b) \wedge a_i = x \wedge a_i$, $\forall i \in I$. Therefore, we can write:

$$x = \bigvee_{i \in I} (x \wedge a_i) = \bigvee_{i \in I} (x \wedge (b \wedge a_i)) = \bigvee_{i \in K} (x \wedge (b \wedge a_i)).$$

Thus V satisfies (2).

(ii) Relation (1) can be easily checked for S'. Since $\{b_j \mid j \in J\}$ is a classification system of $(a_{i_0}]$ and since $x \wedge a_{i_0} \in (a_{i_0}]$, we get

$$x \wedge a_{i_0} = \bigvee_{j \in J} ((x \wedge a_{i_0}) \wedge b_j) = \bigvee_{j \in J} (x \wedge (a_{i_0} \wedge b_j)) = \bigvee_{j \in J} (x \wedge b_j), for all x \in L.$$

Thus we can write:

$$x = \bigvee_{i \in I} (x \wedge a_i) = (\ x \wedge a_{i_0}) \vee (\bigvee_{i \in I \setminus \{i_0\}} (x \wedge a_i)) = (\bigvee_{j \in J} (x \wedge b_j)) \vee (\bigvee_{i \in I \setminus \{i_0\}} (x \wedge a_i)),$$

and this proves that relation (2) holds for S'.

(iii) We have $a_i \wedge b = a_i \wedge (\bigvee_{i \in K} a_i) \leqq a_i \wedge (\bigvee_{i \in I \setminus \{i\}} a_i)$. Since S is weakly independent $a_i \wedge (\bigvee_{i \in I \setminus \{i\}} a_i) = 0$. Thus we obtain $a_i \wedge b = 0$ for all $i \in I \setminus K$, and this ensures that relation (1) holds for S^*. Now for an $x \in L$ we have

$$x = \bigvee_{i \in I} (x \wedge a_i) \leqq \ (\bigvee_{i \in I \setminus K} (x \wedge a_i)) \vee (\bigvee_{i \in K} (x \wedge a_i)) \leqq \ (\bigvee_{i \in I \setminus K} (x \wedge a_i)) \vee (x \wedge b) \leqq x.$$

Hence we get $(\bigvee_{i \in I \setminus K} (x \wedge a_i)) \vee (x \wedge b) = x$, thus condition (2) is satisfied by S^*. □

Definition 2.3. An element $a \in L$ is called a *central element* of the lattice L if for all $x, y \in L$ the sublattice generated by $\{a, x, y\}$ is distributive and a is complemented.

A complement of an element $a \in L$ (if it exists) is denoted by $\overline{a}$. We note that the complement of a central element is unique. The central elements of L form a Boolean sublattice of L denoted by Cen$L$. We say that an ordered pair (a, b) of elements of a lattice L is a *modular pair* and we write $(a, b) \in M$ if, for all $x \in L$ $x \leqq b$ implies $x \vee (a \wedge b) = (x \vee a) \wedge b$ (see [6] or [8]). Clearly, we have $(a, b) \in M$ for any $a \in$Cen$L$ and $b \in L$. The following lemma from [6] will be useful in our proofs.

Lemma 2.4. *Let a be an element of a bounded lattice L. Then the following statements are equivalent:*

(i) $a \in$Cen$L$,

(ii) *There exists an element $a' \in L$ such that*

$x = (x \wedge a) \vee (x \wedge a') = (x \vee a) \wedge (x \vee a')$,

(iii) *There exists an element $a' \in L$ such that $a \wedge a' = 0$, $(a, a') \in M$, $(a', a) \in M$ and $x = (x \wedge a) \vee (x \wedge a')$ for every $x \in L$.*

Remark 2.5. It is easy to check that if the element a' in (ii) or (iii) exists, then $a' = \overline{a}$. Consequently, for any $a \in$Cen$L$, $\{a, \overline{a}\}$ is a classification system both for L and its dual L^d.

Definition 2.6. Let L be a complete lattice. A classification system $S = \{a_i \mid i \in I\}$ is called a *decomposition system* of L if $a_i \in \mathrm{Cen}L$, for all $i \in I$.

The following proposition clarifies the meaning of the former definition.

Proposition 2.7. *Let L be a complete lattice. Then the following assertions are true:*

(i) *If $\{a_i \mid i \in I\}$ is a (nontrivial) decomposition system of L, then $L \cong \prod_{i\in I}(a_i]$.*

(ii) *For any (nontrivial) direct decomposition $L = \prod_{i\in I} L_i$ there exist elements $a_i \in \mathrm{Cen}L$, $i \in I$ such that $L_i \cong (a_i]$ and such that $\{a_i \mid i \in I\}$ is a (nontrivial) decomposition system of L.*

Proof. Since the above statements are more or less known in the literature, we outline only the principal steps of the proof.

(i) The isomorphism is given by the map $h : L \longrightarrow \prod_{i\in I}(a_i]$, $h(x) = (x \wedge a_i)_{i\in I}$.

Indeed, it is easy to check that h is a homomorphism, since for any $a_i \in \mathrm{Cen}L$ and $x, y \in L$ we have $(x \vee y) \wedge a_i = (x \wedge a_i) \vee (y \wedge a_i)$ (and of course, $(x \wedge y) \wedge a_i = (x \wedge a_i) \wedge (y \wedge a_i)$). Since by the assumption of (i) $x = \bigvee_{i\in I}(x \wedge a_i)$ holds for all $x \in L$, $h(x_1) = h(x_2)$ implies $x_1 = x_2$, therefore h is injective. Finally, the surjectivity of h can be shown proving the equality

$$h\left(\bigvee_{i\in I} x_i\right) = (x_i)_{i\in I}, \quad \text{for all} \quad x_i \leqq a_i, i \in I.$$

(ii) Let 0_i and 1_i stand for the least and the greatest element of L_i, respectively. We define the elements $a_i \in \prod_{i\in I} L_i = L$ as follows: $(a_i)_j = 0_j$, for $j \in I$, $j \neq i$ and $(a_i)_i = 1_i$. Then obviously we have $L_i \cong (a_i]$ and $a_i \wedge a_j = 0$ for all $i \neq j$. It is also easy to check that any a_i is a central element of $\prod_{i\in I} L_i$, i.e. of L (see [4] or [6]). Clearly, for all $x = (x_i)_{i\in I} \in \prod_{i\in I} L_i$ we can write $x = \bigvee_{i\in I}(x \wedge a_i)$, thus $S = \{a_i \mid i \in I\}$ is a decomposition system of $L = \prod_{i\in I} L_i$. Evidently, whenever the set I contains more than one element, S is not trivial. □

Proposition 2.8. *Any decomposition system $S = \{c_i \mid i \in I\}$ of a complete lattice L is a strongly independent classification system of L. Moreover, we have $\bigvee_{i\in K} c_i \in \mathrm{Cen}L$, for any $K \subseteqq I$.*

Proof. In view of Proposition 2.7 we have $L \cong \prod_{i\in I}(c_i]$. For each $M \subseteqq I$ we define the elements $c^M \in \prod_{i\in I}(c_i]$ as follows: $(c^M)_i = c_i$ for all $i \in M$, otherwise $(c^M)_i = 0_i$. It is easy to check that for any $J, K \subseteqq I$ we have $c^J \wedge c^K = c^{J\cap K}$. Let $h : L \longrightarrow \prod_{i\in I}(c_i]$, $h(x) = (x \wedge c_i)_{i\in I}$ be the isomorphism as in the proof of Proposition 2.7.(i). Since we have $h\left(\bigvee_{i\in I} x_i\right) = (x_i)_{i\in I}$ for all $x_i \leqq a_i$ (as was indicated), h^{-1} maps any $x = (x_i)_{i\in I} \in \prod_{i\in I}(c_i]$ to $\bigvee_{i\in I} x_i$. Thus we get $h^{-1}(c^M) = \bigvee_{i\in M} c_i$, for all $M \subsetneqq I$. Since h^{-1} is an isomorphism, now we can write:

$$\left(\bigvee_{i\in J} c_i\right) \wedge \left(\bigvee_{i\in K} c_i\right) = h^{-1}(c^J) \wedge h^{-1}(c^K) = h^{-1}(c^J \wedge c^K) = h^{-1}(c^{J\cap K}) = \bigvee_{i\in J\cap K} c_i.$$

Thus $\{c_i \mid i \in I\}$ is a strongly independent system.

In order to prove the second assertion, let us define for any $K \subsetneqq I$ the direct products $A_K = \prod_{i\in K}(c_i]$ and $B_K = \prod_{i\in I\setminus K}(c_i]$, and denote by 0_{A_K},1_{A_K} and 0_{B_K},1_{B_K} the least and the greatest elements of the lattices A_K and B_K, respectively. Clearly, we have $L \cong \prod_{i\in I}(c_i] \cong A_K \times B_K$. By [4], $(1_{A_K}, 0_{B_K})$ and $(0_{A_K}, 1_{B_K})$ are central elements of the lattice $A_K \times B_K$ and correspond by isomorphism to the elements $c^K \in \prod_{i\in I}(c_i]$ and $c^{I\setminus K} \in \prod_{i\in I}(c_i]$, respectively. Since $h^{-1}(c^K) = \bigvee_{i\in K} c_i$, and since the central elements of a lattice are preserved by isomorphisms, we get $\bigvee_{i\in K} c_i \in$Cen$L$. □

3. Classification systems in particular lattices

A lattice L with 0 is said to be *0-modular* if $a, b \in L, a \leqq c$ and $b \wedge c = 0$ imply $(a \vee b) \wedge c = a$. By J.C. Varlet's result (see e.g. [8]), this definition is equivalent to the fact that there is no N_5 sublattice in L including the element 0.

Proposition 3.1. *If L is a 0-modular complete lattice, then any of its weakly independent classification systems of it is a decomposition system.*

Proof. Let $S = \{a_i \mid i \in I\}$ be a weakly independent classification system of L and take $b_i = \bigvee_{j\in I\setminus\{i\}} a_j$, $i \in I$. Since S is weakly independent, we have $a_i \wedge b_i = 0$. In view of Lemma 2.2(iii) the set $\{a_i, b_i\}$ is a classification system of L, thus we get $x = (x \wedge a_i) \vee (x \wedge b_i)$ for all $x \in L$. Since L is 0-modular, we have $(x \vee a_i) \wedge b_i = x$ for all $x \leqq a_i$. Thus we obtain $x \vee (a_i \wedge b_i) = x = (x \vee a_i) \wedge b_i$ and this relation means that $(a_i, b_i) \in M$. Similarly we can prove $(b_i, a_i) \in M$. Now, by applying Lemma 2.4 we get $a_i \in$Cen$L$, $i \in I$, therefore S is a decomposition system of L. □

A lattice L with 0 element is called a *pseudocomplemented lattice* if any element $x \in L$ has a *pseudocomplement* x^*, that is, for any $x \in L$ there exists an element $x^* \in L$ such that $y \wedge x = 0$ iff $y \leqq x^*$.

Definition 3.2. (R. Beazer, [2]) An element a of a pseudocomplemented lattice L is called a *semicentral element* if

$x = (x \wedge a) \vee (x \wedge a^*)$ holds for all $x \in L$. $(*)$

Remark 3.3. The above definition implies $a^* = \overline{a}$ and $a^{**} = a$ for any semicentral element $a \in L$. From here it follows that the pseudocomplement of a, i.e. a^*, is a semicentral element too. We note that in view of relation $(*)$ the set $\{a, a^*\}$ is a classification system of L. It is also clear that any $c \in \mathrm{Cen}L$ is a semicentral element.

Proposition 3.4. *Any classification system $S = \{a_i \mid i \in I\}$ of a complete pseudocomplemented lattice L is weakly independent and it consists of semicentral elements of L. Moreover, for any $K \subseteqq I$, $K \neq \emptyset$ $\bigvee_{i \in K} a_i$ is a semicentral element of L.*

Proof. As $a_i \wedge a_j = 0$ for $i \neq j$, we have $a_j \leqq a_i^*$ for all $j \neq i$. Thus we can write $\bigvee_{j \in I \setminus \{i\}} a_j \leqq a_i^*$. As a consequence, we get $a_i \wedge \left(\bigvee_{j \in I \setminus \{i\}} a_j \right) = 0$, $i \in I$, i.e. S is weakly independent. Furthermore, we can write $a_i^* = \bigvee_{j \in I} (a_i^* \wedge a_j) = (a_i^* \wedge a_i) \vee \left(\bigvee_{j \in I \setminus \{i\}} (a_i^* \wedge a_j) \right) \leqq \bigvee_{j \in I \setminus \{i\}} a_j$. Hence we obtain $a_i^* = \bigvee_{j \in I \setminus \{i\}} a_j$. Since S is a weakly independent classification system, by applying Lemma 2.2(iii) with $K = I \setminus \{i\}$ we get that $\{a_i, a_i^*\}$ is a classification system, too. Therefore $x = (x \vee a_i) \vee (x \wedge a_i^*)$ holds for all $x \in L$ (and $i \in I$). Thus each a_i is a semicentral element.

Finally, set $b = \bigvee_{i \in K} a_i$, for a $K \subseteqq I$, $K \neq \emptyset$. Then $S^* = \{a_i \mid i \in I \setminus K\} \cup \{b\}$ is also a classification system of L, by Lemma 2.2(iii) again. Thus, as we have already proved above, $b = \bigvee_{i \in K} a_i$ is a semicentral element. $\square$

Corollary 3.5. *If L is a complete 0-modular pseudocomplemented lattice, then any classification system of L is its decomposition system.*

Proof. We apply Proposition 3.4 and Proposition 3.1. $\square$

Example 3.6. Let Tol$L$ stand for the tolerance lattice of a lattice L. By [1], Tol$L$ is a pseudocomplemented and 0-modular complete lattice. Thus, according to Corollary 3.5, any classification system of Tol$L$ is its decomposition system.

The last two results naturally raise the question: "Under what conditions do the classification systems of a complete lattice coincide with its decomposition systems?" In the case of pseudocomplemented lattices the following is an answer.

Theorem 3.7. *Let L be a complete pseudocomplemented lattice. Then the following assertions are equivalent:*

(i) *Any semicentral element of L is its central element,*

(ii) *Any classification system of L is its decomposition system,*

(iii) *For any semicentral element $a \in L$ we have $(a, a^*) \in M$.*

(iv) *The complemented congruences of the algebra $(L, \wedge, ^*)$ and the factor congruences of the lattice L are the same.*

Proof. (i)$\Rightarrow$(ii) is obvious, since any classification system of a pseudocomplemented lattice consists of semicentral elements.

(ii)$\Rightarrow$(iii). Since for any semicentral element $a \in L$, the set $\{a, a^*\}$ is a classification system, by assumption of (ii) it follows that $a, a^* \in$Cen$L$. Hence $(a, a^*) \in M$.

(iii)$\Rightarrow$(i). Let $a \in L$ be a semicentral element. Then, according to Remark 3.3, a^* is a semicentral element, too, and $a^{**} = a$. Thus by assumption we have $(a, a^*) \in M$ and $(a^*, a) \in M$ and $x = (x \wedge a) \vee (x \wedge a^*)$, $\forall x \in L$. As $a \wedge a^* = 0$, by applying Lemma 2.4, we get that $a \in$Cen$L$.

(i)$\Leftrightarrow$(iv). Let $\theta_a = \{(x, y) \in L^2 \mid x \wedge a = y \wedge a\}$. In view of [2], θ_a is a complemented congruence of the algebra $(L, \wedge, ^*)$ if and only if a is a semicentral element of L. However $a \in$Cen$L$ if and only if θ_a is a factor congruence of the lattice L (see [4]), whence the required equivalence follows. $\square$

4. Classification systems in CJ-generated pseudocomplemented lattices

An element p of a complete lattice L is called *completely join-irreducible* if for any system of elements $x_i \in L$, $i \in I$, the equality $p = \bigvee\{x_i \mid i \in I\}$ implies $p = x_{i_0}$ for some $i_0 \in I$. If any element of L is a join of the completely join-irreducible elements, then L is called a *CJ-generated lattice.* The set of completely join-irreducible elements of L is denoted by $J(L)$. For an $a \in L$ let $J(a) = \{p \in J(L) \mid p \leq a\}$. It is clear that for any system of elements $a_i \in L$, $i \in I$ we have $J\left(\bigwedge\limits_{i \in I} a_i\right) = \bigcap\limits_{i \in I} J(a_i)$ and $\bigcup\limits_{i \in I} J(a_i) \subseteqq J\left(\bigvee\limits_{i \in I} a_i\right)$, moreover, if L is CJ-generated, then for any $x, y \in L$ $J(x) \subseteqq J(y) \Longleftrightarrow x \leqq y$.

For an arbitrary relation $\rho \subseteqq X \times X$ and for a set $B \subseteqq X$ we define $\rho(B) = \{x \in X \mid \exists b \in B$ such that $(b, x) \in \rho\}$. We say that the set B is *closed relative to the relation* ρ if $\rho(B) \subseteqq B$, i.e. for any $b \in B$ and $x \in X$ from $(b, x) \in \rho$ it follows $x \in B$. If ρ is an equivalence relation, then the equivalence class of an element $x \in X$ is denoted by $[x]_\rho$.

Let L be a CJ-generated (complete) pseudocomplemented lattice. On the set $J(L) \setminus \{0\}$ we define the relation R as follows: $pRq \Leftrightarrow p \wedge q \neq 0$ $(p, q \in J(L) \setminus \{0\})$. A partition $\pi = \{A_i \mid i \in I\}$ of $J(L) \setminus \{0\}$ is called *R-closed* if every block A_i is closed relative to the relation R.

Now we can formulate the following

Proposition 4.1.

(i) *If* $\{a_i \mid i \in I\}$ *is a classification system of* L*, then the sets* $J(a_i) \setminus \{0\}$, $i \in I$ *form an* R*-closed partition of* $J(L) \setminus \{0\}$.

(ii) *If* $\pi = \{A_i \mid i \in I\}$ *is an* R*-closed partition of the set* $J(L) \setminus \{0\}$ *then the elements* $a_i = \bigvee A_i$, $i \in I$ *form a classification system of* L *with* $J(a_i) \setminus \{0\} = A_i$.

Proof. (i) Since $a_i \neq 0$, we have $J(a_i) \setminus \{0\} \neq \emptyset$, and for $i \neq j$, $a_i \wedge a_j = 0$ gives $(J(a_i) \setminus \{0\}) \cap (J(a_j) \setminus \{0\}) = \emptyset$. We prove $\bigcup\limits_{i \in I} (J(a_i) \setminus \{0\}) = J(L) \setminus \{0\}$. The inclusion

$\bigcup_{i\in I}(J(a_i)\setminus\{0\}) \subseteqq J(L)\setminus\{0\}$ is obvious. To prove the converse inclusion take an arbitrary $p\in J(L)\setminus\{0\}$. Then $p=\bigvee_{i\in I}(p\wedge a_i)$, because $\{a_i \mid i\in I\}$ is a classification system of L. As p is completely join-irreducible, there exists an $i_0\in I$ such that $p=p\wedge a_{i_0}$, i.e. such that $p\in J(a_{i_0})\subseteqq \bigcup_{i\in I}(J(a_i)\setminus\{0\})$. Thus $\{J(a_i)\setminus\{0\} \mid i\in I\}$ is a partition of $J(L)\setminus\{0\}$.

Take now any $p\in J(a_i)\setminus\{0\}$ and assume that pRq holds for some $q\in J(L)\setminus\{0\}$. Since $\{J(a_i)\setminus\{0\} \mid i\in I\}$ is a partition, we have $q\in J(a_j)\setminus\{0\}$ for some $j\in I$. Then $p\leqq a_i$ and $q\leqq a_j$ implies $p\wedge q\leqq a_i\wedge a_j$. As $a_i\wedge a_j=0$ for $i\neq j$, the assumption $p\wedge q\neq 0$ gives $i=j$. Therefore $q\in J(a_i)\setminus\{0\}$ and this means that $J(a_i)\setminus\{0\}$ is R-closed.

(ii) Let $\pi=\{A_i \mid i\in I\}$ be an R-closed partition of $J(L)\setminus\{0\}$ and put $a_i=\bigvee A_i$, $i\in I$. Since $A_i\neq\emptyset$, we have $a_i\neq 0$. First, we show that $J(a_i)\setminus\{0\}=A_i$.

Let $p\in J(a_i)\setminus\{0\}$. We claim that $p\wedge q\neq 0$ for some $q\in A_i$. Indeed, $p\wedge q=0, \forall q\in A_i$ would imply that $a_i=\bigvee\{q \mid q\in A_i\}\leqq p^*$, thus we would get $p=p\wedge a_i\leqq p\wedge p^*=0$ - a contradiction. Thus we must have pRq for some $q\in A_i$. Since A_i is R-closed by assumption, we get $p\in A_i$ and this proves $J(a_i)\setminus\{0\}\subseteqq A_i$. The inclusion $A_i\subseteqq J(a_i)\setminus\{0\}$ is obvious.

If $i\neq j$ then $J(a_i\wedge a_j)=J(a_i)\cap J(a_j)=(A_i\cup\{0\})\cap(A_j\cup\{0\})=\{0\}$, and since L is CJ-generated, from here it follows $a_i\wedge a_j=0$.

Finally, observe that, in order to prove that $\{a_i \mid i\in I\}$ is a classification system of L, it is enough to show the inequality $x\leqq\bigvee_{i\in I}(x\wedge a_i)$ for any $x\in L$. For this purpose take any $p\in J(x)\setminus\{0\}$. Since $\{A_i \mid i\in I\}$ is a partition of $J(L)\setminus\{0\}$, there exists an $i_p\in I$ such that $p\in A_{i_p}$. Then we get $p\leqq x\wedge a_{i_p}$, therefore $p\leqq\bigvee_{i\in I}(x\wedge a_i)$. Thus we obtain $x=\bigvee\{p \mid p\in J(x)\}\leqq\bigvee_{i\in I}(x\wedge a_i)$. $\square$

Let $\overline{R}$ denote the transitive hull of the relation R. Since R is reflexive and symmetric by its definition, $\overline{R}$ is an equivalence on $J(L)\setminus\{0\}$. Clearly, the equivalence classes $[x]_{\overline{R}}$, $x\in J(L)\setminus\{0\}$ form an R-closed partition of $J(L)\setminus\{0\}$. Let $D=\{d \mid d=\bigvee[x]_{\overline{R}}, x\in J(L)\setminus\{0\}\}$. Evidently the elements of D can be indexed by the elements of the factor set $\mathcal{C}_R=(J(L)\setminus\{0\})/\overline{R}$, i.e. we can write: $D=\{d_i \mid i\in\mathcal{C}_R\}$. Now, by applying Proposition 4.1(ii) we obtain the following

Corollary 4.2. *For any CJ-generated complete pseudocomplemented lattice L, the set $D=\{d_i \mid i\in C_R\}$ is a classification system of L.*

We call a CJ-generated lattice L *connected*, if $\overline{R}$ is the total relation on the set $J(L)\setminus\{0\}$, i.e. if for any $p,q\in J(L)\setminus\{0\}$ there exists a sequence of elements $p_0,p_1,...,p_n\in J(L)\setminus\{0\}$, such that $p_0=p$, $p_n=q$ and $p_{i-1}\wedge p_i\neq 0$, for all $1\leqq i\leqq n$. Now we have the following

Corollary 4.3 *Let L be a CJ-generated complete pseudocomplemented lattice. Then*

(i) *L admits only the trivial classification system if and only if L is connected.*

(ii) *If L is connected, then it is directly irreducible.*

Proof. (i) If L is not a connected lattice, then the equivalence $\overline{R}$ has more than one class, thus applying Corollary 4.2 we get that $D = \{d_i \mid i \in \mathcal{C}_R\}$ is a nontrivial classification system of L.

Conversely, suppose that L is connected, and let $\{a_i \mid i \in I\}$ be a classification system of L. In view of Proposition 4.1(i) the sets $J(a_i) \setminus \{0\}$, $i \in I$ form an R-closed partition of $J(L) \setminus \{0\}$. Thus for every a_i and any $x \in J(a_i) \setminus \{0\}$ we have $[x]_{\overline{R}} \subseteqq J(a_i)$. As L is connected, we have $[x]_{\overline{R}} = J(L) \setminus \{0\}$ for all $x \in J(L) \setminus \{0\}$, whence we get that $J(a_i) = J(L)$, i.e. that $a_i = 1$, for all $i \in I$. Hence any classification system of L is trivial.

(ii) On the contrary, suppose that the lattice L is directly reducible. Then, by Proposition 2.7, L admits a nontrivial decomposition system S. Since S at the same time is a classification system, the above (i) gives that L is not connected, which is a contradiction. □

Now we are able to formulate the main result of this section, which is the following

Theorem 4.4. *Let L be a CJ-generated complete pseudocomplemented lattice. Then the following assertions are equivalent.*

(i) *Any classification system of L is a decomposition system,*

(ii) *L is a direct product of connected lattices,*

(iii) *Any semicentral element of L is central.*

Proof. The equivalence (i)⇔(iii) was proved by Theorem 3.7.

(i)⇒(ii). Let $D = \{d_i \mid i \in \mathcal{C}_R\}$ be the classification system induced by the equivalence $\overline{R}$. Since by assumption D is a decomposition system of L, Proposition 2.7 gives $L \cong \prod_{i \in \mathcal{C}_R} (d_i]$. Clearly, any $(d_i]$ as a principal ideal of L is a pseudocomplemented CJ-generated complete lattice and the set of its completely join-irreducible elements is the same as $J(d_i) \subseteqq J(L)$. Since any d_i is of the form $d_i = \bigvee [x]_{\overline{R}}$ for some $x \in J(L) \setminus \{0\}$, and since $J(d_i) \setminus \{0\} = [x]_{\overline{R}}$ in view of Proposition 4.1(ii), we have $p\overline{R}q$ for all $p, q \in J(d_i) \setminus \{0\}$. Thus any lattice $(d_i]$ is connected.

(ii)⇒(iii). By assumption of (ii) we have $L \cong \prod_{i \in I} L_i$, where all L_i are connected. Then, in view of Proposition 2.7, there are $c_i \in$Cen$L$ such that $L_i \cong (c_i]$, $i \in I$ and $S = \{c_i \mid i \in I\}$ is a decomposition system of L. Now, let a be a semicentral element of L. Since $\{a, a^*\}$ is a classification system of L, if $c_i \wedge a \neq 0$ and $c_i \wedge a^* \neq 0$ for some $i \in I$, then by applying Lemma 2.2(i) we get that $\{c_i \wedge a, c_i \wedge a^*\}$ is a nontrivial classification system of the lattice $(c_i]$. (The system is nontrivial, since $c_i \wedge a = c_i$ would imply $c_i \leqq a$, i.e. $c_i \wedge a^* = 0$, - a contradiction.) Since any $(c_i]$ is a connected lattice, it admits only the trivial classification system. Thus for each $i \in I$ we must have either $c_i \wedge a = 0$ or $c_i \wedge a^* = 0$, i.e. either $c_i \leqq a^*$ or $c_i \leqq a^{**} = a$. Let $K = \{i \in I \mid c_i \leqq a\}$. Then we have $c_i \leqq a^*$ for all $i \in I \setminus K$. Set $b = \bigvee_{i \in K} c_i$ and $c = \bigvee_{i \in I \setminus K} c_i$. Then Proposition 2.8 gives $b, c \in$Cen$L$, and since S is also a classification system, we have $b \vee c = \bigvee_{i \in I} c_i = 1$.

On the other hand, we have $b = \bigvee_{i \in K} c_i \leqq a$ and $c = \bigvee_{i \in I \setminus K} c_i \leqq a^*$, so we get $c \wedge a \leqq a^* \wedge a = 0$ and $c \vee a \geqq c \vee b = 1$, whence $a = \overline{c}$. As the complement of a central element is also central, we obtain that $a \in$Cen$L$. □

Corollary 4.5. *Any finite 0-modular pseudocomplemented lattice L is a direct product of connected directly irreducible lattices.*

Proof. In view of Corollary 3.5 any classification system of L is a decomposition system. Since any finite lattice is a CJ-generated complete lattice, we can apply Theorem 4.4 and this gives that L is a direct product of connected lattices, i.e. that $L \cong \prod_{i \in I} L_i$, where any L_i is connected and of course, finite and pseudocomplemented, too. Applying now Corollary 4.3(ii) we get that each L_i is directly irreducible. □

Finally, we present an application of our former results to the theory of Stone lattices.

A bounded pseudocomplemented lattice L is called a *Stone lattice* if $x^* \vee x^{**} = 1$ holds for all $x \in L$. We say that a lattice L with 0 is *handled*, if it contains an atom $\omega \in L$ such that $\omega \leqq x$ hold for all $x \in L \setminus \{0\}$. Clearly, any bounded handled lattice L is a pseudocomplemented lattice where $x^* = 0$ and $x^{**} = 1$, for all $x \in L \setminus \{0\}$. Thus any bounded handled lattice is a Stone lattice. It is also obvious that any CJ-generated handled lattice is connected. In [5], G. Grätzer and E.T. Schmidt gave the following fine characterization of finite distributive Stone lattices:

"A finite distributive lattice is a Stone lattice if and only if it is a direct product of finite distributive handled lattices."

Remark 4.6. It is easy to check that in a distributive Stone lattice L we have $x^* \in$Cen$L$ for all $x \in L$. Conversely, if $x^* \in$Cen$L$ holds for all $x \in L$ in a bounded lattice L, then L is a Stone lattice. (We note that a lattice with this property is not distributive in general.). In view of these observations, the next theorem can be considered to be a generalization of the above cited result of G. Grätzer and E.T. Schmidt.

Theorem 4.7. *Let L be a CJ-generated complete lattice. Then the following statements are equivalent:*

(i) *L is a pseudocomplemented atomic lattice, and for all $x \in L$ we have $x^* \in$*Cen$L$.

(ii) *L is a direct product of handled lattices.*

Proof. (i)⇒(ii). Since any semicentral element $a \in L$ satisfies $a = a^{**}$, the assumption of (i) implies that any semicentral element of L is central. Thus, by applying Theorem 4.4 we get that L is of the form $L \cong \prod_{i \in I} L_i$, where all L_i are connected CJ-generated lattices. Evidently, any lattice L_i, as a factor of L, is a complete pseudocomplemented atomic lattice.

Now we prove that any L_i $(i \in I)$ is a handled lattice. Denote by 0_i and 1_i the smallest and the largest element of L_i, respectively. Let α be an atom of L_i, we have to show that $\alpha \leqq x$ for all $x \in L_i$, $x \neq 0_i$. On the contrary, let us assume that $b \wedge \alpha = 0$, for some $b \in L_i$, $b \neq 0_i$. Then we have $b^* \neq 1_i$ and $b^* \geqq \alpha > 0_i$. The latter relation implies $b^{**} \neq 1_i$, and since $b^{**} \geqq b$, we have $b^{**} \neq 0_i$. Thus we get $b^*, b^{**} \notin \{0_i, 1_i\}$.

Set now an element $z = (z_k)_{k\in I} \in \prod_{k\in I} L_k$, with $z_i = b$. Since by assumption $z^*, z^{**} \in$Cen$L$, and since $z^{**} = \overline{(z^*)}$ the set $\{z^*, z^{**}\}$ is a decomposition system of L. Thus we have $x = (x \wedge z^*) \vee (x \wedge z^{**})$ for all $x \in \prod_{i\in I} L_i$. Clearly, we get $x_i = (x_i \wedge (z^*)_i) \vee (x_i \wedge (z^{**})_i)$ for all $x_i \in L_i$. Observe now, that for all $x = (x_k)_{k\in I} \in \prod_{k\in I} L_k$ we have $x^* = (x_k^*)_{k\in I}$. Hence $(z^*)_i = b^*$ and $(z^{**})_i = b^{**}$. Summarizing, we get that $\{b^*, b^{**}\}$ is a classification system of L_i. Since $b^* \neq 1_i$, this system is nontrivial. As L_i is a connected lattice, this result is in contradiction with Corollary 4.3(i).

(ii)$\Rightarrow$(i). Let $L = \prod_{i\in I} L_i$, with all L_i handled. As L is a bounded lattice, each L_i must be bounded, and consequently, it must be pseudocomplemented. Then it is not hard to check that L itself, as a direct product of pseudocomplemented atomic lattices L_i, is a pseudocomplemented atomic lattice, too. Take now an arbitrary $x = (x_i)_{i\in I} \in L$. Then we have $x^* = (x_i^*)_{i\in I}$. Since any L_i is a bounded handled lattice, we must have $x_i^* = 0_i$, for all $x_i \neq 0_i$, and $x_i^* = 1_i$ otherwise. Thus for all $i \in I$ we have either $(x^*)_i = 0_i$ or $(x^*)_i = 1_i$, and this fact proves that x^* is a central element of the product $\prod_{i\in I} L_i$. Hence $x^* \in$Cen$L$. $\square$

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