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NEW PRECONDITIONERS FOR NONSYMMETRIC AND INDEFINITE SADDLE POINT LINEAR SYSTEMS

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Abstract. In this paper, we extend the preconditioners of Sturler and Liesen [*SIAM J. Sci. Comput.*, 26(2005): 1598-1619]. The spectral characteristics of the preconditioners are investigated and the results show that all eigenvalues of the preconditioned matrices are strongly clustered. Finally, numerical experiments are also reported for illustrating the efficiency of the presented preconditioners.

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1. INTRODUCTION

Consider the following general nonsingular saddle point linear system

$$A\xi \equiv \begin{pmatrix} G & B^T \\ C & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ q \end{pmatrix} = f, \tag{1.1}$$

where $G \in R^{n\times n}$, $B, C \in R^{n\times m}$ have full rank, $x, b \in R^n$ and $y, q \in R^m$, and the vectors x, y are unknown. Here we assume that A is nonsingular, that will be used in the following analysis. Under these assumptions, system (1.1) has a unique solution. This system is very important and appears in many different applications of scientific computing, such as constrained optimization [1, 16], the finite element or finite difference methods of solving the Navier-Stokes equation [4–7], fluid dynamics, constrained least problems and generalized least squares problems [10–12], and the discretized time-harmonic Maxwell equations in mixed form [9].

Results for the general systems have been obtained before, for example, Murphy, Golub and Wathen [3] proposed the block-diagonal preconditioner

$$\begin{pmatrix} G^{-1} & 0 \\ 0 & (CG^{-1}B^T)^{-1} \end{pmatrix}.$$

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Then the preconditioned matrices are diagonalizable and have at most three distinct eigenvalues. Hence, a Krylov subspace method will converge in at most three steps. Because such a general system is typically large and sparse, the preconditioner is expensive. Recently, Sturler and Liesen [14] considered the following preconditioner

$$\begin{pmatrix} D^{-1} & 0 \\ 0 & (CD^{-1}B^T)^{-1} \end{pmatrix},$$

where $G = D - E$ and D can be efficiently inverted. For such a preconditioner, one should assume that A and $CD^{-1}B^T$ are invertible. But it may be expensive to compute $CD^{-1}B^T$, and the matrix $CD^{-1}B^T$ may be singular, so we introduce the following preconditioner

$$\begin{pmatrix} D^{-1} & 0 \\ 0 & (F + CD^{-1}B^T)^{-1} \end{pmatrix},$$

where D and $F + CD^{-1}B^T$ are invertible. We can get a suitable F such that $F + CD^{-1}B^T$ is invertible and we can reduce the computing cost of the Schur complement. If we choose suitable D and F, we find that all of the block diagonal preconditioners can be included. When G is singular or ill-conditioned, such preconditioners can still be applied.

This paper is organized as follows. In Section 2, we will establish new preconditioners and study their spectral analysis for the saddle point systems. The related systems are given in Section 3. In Section 4, numerical experiments are given. Finally, the conclusions are presented in Section 5.

2. Block diagonal preconditioner

We split G as $G = D - E$, where D is easy to solve. $CD^{-1}B^T$ is the exact Schur complement of the matrix

$$\begin{pmatrix} D & B^T \\ C & 0 \end{pmatrix}.$$

In this paper, we introduce the following preconditioner which is a generalization of the preconditioners of [14]:

$$P(D) = \begin{pmatrix} D^{-1} & 0 \\ 0 & (F + CD^{-1}B^T)^{-1} \end{pmatrix}.$$

Then we get the left preconditioned matrix:

$$B(F) = P(D)A = \begin{pmatrix} I - D^{-1}E & D^{-1}B^T \\ (F + CD^{-1}B^T)^{-1}C & 0 \end{pmatrix} = \begin{pmatrix} I - S & N \\ M & 0 \end{pmatrix}, \tag{2.1}$$

where,

$$
\begin{aligned}
MN &= (F + CD^{-1}B^T)^{-1}CD^{-1}B^T \\
&= (F + CD^{-1}B^T)^{-1}(F + CD^{-1}B^T - F) \\
&= I + (-(F + CD^{-1}B^T)^{-1}F) \\
&= I + Q.
\end{aligned}
$$

When $F = 0$, $MN = I$ (it is the case in [14]), and the preconditioner reduces to the preconditioner in [14]. If $B = C$, $E = C^T W^{-1} B$ and $F = W - CD^{-1}B^T$, it reduces to the block diagonal preconditioner in [13] and [8].

In the following analysis, we discuss the eigendecomposition of the matrix

$$
B_0 = \begin{pmatrix} I & N \\ M & 0 \end{pmatrix}, \tag{2.2}
$$

and we assume that Q is diagonalizable and $5 + 4\delta_j \neq 0$, where $Qv_j = \delta_j v_j$.

Theorem 1. *Let B_0 be of the form (2.2). Then B_0 has the following eigenvalues and eigenvectors:*

(a) $n - m$ eigenpairs of the form $(1, [u_j^T, 0]^T)$, where $u_1, \cdots, u_{n-m}$ form a basis of Null(M), the nullspace of M.

(b) $2m$ eigenpairs of the form $(\lambda^{\pm}, [u_j, (\lambda^{\pm})^{-1}(Mu_j)^T]^T)$, where $\lambda^{\pm} = (1 \pm \sqrt{5 + 4\delta_j})/2$, and $Qv_j = \delta_j v_j$. Then the eigenvector matrix Y of B_0 is given by

$$
Y = \begin{pmatrix} U_1 & U_2 & U_2 \\ 0 & (\lambda^+)^{-1}MU_2 & (\lambda^-)^{-1}MU_2 \end{pmatrix}, \tag{2.3}
$$

where $U_1 = [u_1, \cdots, u_{n-m}]$ and $U_2 = [u_{n-m+1}, \cdots, u_n]$.

Proof. We consider the equation

$$
B_0[u^T, v^T]^T = \lambda [u^T, v^T]^T,
$$

which is equivalent to the following two equations

$$
u + Nv = \lambda u, \tag{2.4}
$$

$$
Mu = \lambda v. \tag{2.5}
$$

Since B_0 is nonsingular, we can assume $\lambda \neq 0$. So from (2.5), we have

$$
v = \lambda^{-1} M u.
$$

First, we assume $\lambda = 1$. Substituting this into (2.4) and (2.5), we have

$$
Nv = 0 \quad \text{and} \quad Mu = v. \tag{2.6}
$$

Since B^T has full column rank by assumption, this implies that $v = 0$ and that B_0 has only eigenpairs of the form $(1, [u_j^T, 0]^T)$, where $u \in null(M)$.

Since C has full row rank, so does M, and B_0 has precisely $n-m$ distinct eigenpairs of this type. Next, we assume $\lambda \neq 1$. From (2.4), we can get

$$u = (\lambda-1)^{-1} N v.$$

Substituting this into (2.5) yields

$$Q v_j = (\lambda^2 - \lambda - 1) v_j. \tag{2.7}$$

Hence, v_j must be eigenvectors of Q, we assume that $Q v_j = \delta_j v_j$. Then we solve (2.7) for λ to yield

$$\lambda^{\pm} = \left(1 \pm \sqrt{5+4\delta_j}\right)/2.$$

So, the remaining $2m$ eigenpairs are $(\lambda^{\pm}, [u_j, (\lambda^{\pm})^{-1}(M u_j)^T]^T)$. □

We are now ready to consider the perturbation bounds on the eigenvalues of $B(F)$. Throughout this paper $\|\cdot\|$ indicates the 2-norm, and we let

$$Y = \begin{pmatrix} U_1 & U_2 & U_2 \\ 0 & (\lambda^+)^{-1} M U_2 & (\lambda^-)^{-1} M U_2 \end{pmatrix} = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix},$$

$$\gamma^+ = \operatorname{diag}((\lambda_j^- - 1)/(\lambda_j^- - \lambda_j^+)),$$
$$\gamma^- = \operatorname{diag}((\lambda_j^+ - 1)/(\lambda_j^- - \lambda_j^+)).$$

Theorem 2. *Consider matrices $B(F)$ of the form (2.1). Let Y be the eigenvector matrix of B_0, as given by (2.2). Then for each eigenvalue λ_B of $B(F)$ there exists an eigenvalue λ of B_0 such that*

$$\begin{aligned} |\lambda_B - \lambda| &\le \left\| Y^{-1} \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} Y \right\| \\ &\le 2\max(1, \|\gamma^+\|, \|\gamma^-\|) \left\| Y_{11}^{-1} S Y_{11} \right\|. \end{aligned} \tag{2.8}$$

Proof. Since B_0 is diagonalizable, this follows from a classic result in perturbation theory [15]. We expand the right-hand side of (2.8) (see also [3, 14])

$$\begin{aligned} |\lambda_B - \lambda| &\le \left\| \begin{pmatrix} \hat{I} Y_{11}^{-1} S Y_{11} & \hat{I} Y_{11}^{-1} S Y_{12} \\ -[0,\gamma^-] Y_{11}^{-1} S Y_{11} & -[0,\gamma^-] Y_{11}^{-1} S Y_{12} \end{pmatrix} \right\| \\ &\le \sqrt{2}\max(1, \|\gamma^+\|, \|\gamma^-\|) \left\| \begin{bmatrix} Y_{11}^{-1} S Y_{11} \\ -[0, I] Y_{11}^{-1} S Y_{11} \end{bmatrix} \right\| \\ &\le 2\max(1, \|\gamma^+\|, \|\gamma^-\|) \left\| Y_{11}^{-1} S Y_{11} \right\|. \end{aligned}$$

□

We can find that, if $\delta_j \approx -5/4$, $\gamma^{\pm}$ can be very large. So we should take a suitable Q such that the value of δ_j are well separated from $-5/4$. In section 4, we will present some choices.

Now we will derive a bound on $\|Y_{11}^{-1}SY_{11}\|$, following the approach in [3, 14]. Recall that $Y_{11} = [\ U_1 \quad U_2\ ]$, where $U_1^T U_1 = I$ and $U_2 = NV$ with unit columns. Let $U_2 = V_2\Theta$, where $V_2^T V_2 = I$. Furthermore, denote $\omega_1 = \|U_1^T V_2\|$.

Theorem 3. *Define Y_{11}, S, U_1, U_2, V_2, Θ and ω_1 as above, and let $\kappa(.)$ denote the 2-norm condition number. Then*

$$\|Y_{11}^{-1}SY_{11}\| \le \kappa(\Theta)\left(\frac{1+\omega_1}{1-\omega_1}\right)^{1/2}\|S\|.$$

The proof is similar to that of Lemma 2.3 in [14].

3. Fixed-point method and its related system

We can derive the following splitting from (2.1):

$$B(F)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} I-S & N \\ M & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \left(B_0 - \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}\right)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{Bmatrix} f \\ g \end{Bmatrix}. \tag{3.1}$$

Note that

$$B_0^{-1} = \begin{pmatrix} I_n - N(MN)^{-1}M & N(MN)^{-1} \\ (MN)^{-1}M & -(MN)^{-1} \end{pmatrix}. \tag{3.2}$$

We left-multiply (3.1) by B_0^{-1} to yield the fixed-point iteration

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} (I-N(MN)^{-1}M)S & 0 \\ ((MN)^{-1}M)S & 0 \end{pmatrix}\begin{pmatrix} x_k \\ y_k \end{pmatrix} + \begin{pmatrix} \widehat{f} \\ \widehat{g} \end{pmatrix}. \tag{3.3}$$

So we can get the related system for the fixed-point iteration

$$(I-(I-N(MN)^{-1}M)S)x = \widehat{f}. \tag{3.4}$$

Theorem 4. *The spectral radius of the fixed point iteration matrix in (3.3) and the eigenvalues λ_R of the related system matrix in (3.4) satisfy*

$$\left.\begin{array}{r} \rho(I-(I-N(MN)^{-1}M)S) \\ |1-\lambda_R| \end{array}\right\} \le \frac{\|S\|}{(1-\omega_1^2)^{1/2}}, \tag{3.5}$$

where ω_1 is the largest singular value of $U_1^T U_2$.

Proof. Let $\widehat{N} = N(MN)^{-1}, \widehat{M} = M$. We have

$$(I-\widehat{N}\widehat{M})U_1 = U_1, \quad (I-\widehat{N}\widehat{M})U_2 = 0.$$

Thus

$$(I-\widehat{N}\widehat{M})S = (U_1,U_2)\begin{pmatrix} I_{n-m} & 0 \\ 0 & 0 \end{pmatrix}(U_1,U_2)^{-1}S,$$

Then, similary to the proof of Theorem 4.4 in [3], we have

$$\left.\begin{array}{r}\rho(I-(I-N(MN)^{-1}M)S)\\ |1-\lambda_R|\end{array}\right\} \leq \frac{\|S\|}{(1-\omega_1^2)^{1/2}}.$$

□

Generally, this leads to better clustering and tighter bounds for the related system than for the block-diagonally preconditioned system. Because of these advantages, the related system will generally have faster convergence than the block-diagonally preconditioned system, as we will see it in section 4.

4. NUMERICAL EXAMPLES

All the numerical experiments were performed with MATLAB 7.0. In all of our runs we used a zero initial guess. The stopping criterion is $||r^{(k)}||_2/||r^{(0)}||_2 \leq 10^{-6}$, where $r^{(k)}$ is the residual vector after the k-th iteration. The right-hand side vectors b and q are taken such that the exact solutions x and y are both vectors with all of their components equal to 1. The initial guess is chosen to be the zero vector. We will use preconditioned GMRES(10) to solve the saddle point-type systems. Our numerical experiments are similar to ones in [2]. The matrices considered are taken from [2] with a slightly altered notations. We construct the saddle point-type matrix A from a transformed matrix $\hat{A}$ of the following form

$$\hat{A} = \begin{pmatrix} F_1 & 0 & B_u^T \\ 0 & F_2 & B_v^T \\ B_u & B_v & 0 \end{pmatrix},$$

where $G = \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix}$ is positive real. The matrix $\hat{A}$ arises from the discretization by the maker and cell finite dirergence scheme of a leaky two dimensional liddriven cavity problem in a square domain $(0 \leq x \leq 1; 0 \leq y \leq 1)$. Then the matrix (B_u, B_v) is replaced by a random matrix $\hat{B}$ with the same sparsity as (B_u, B_v), replaced by $B_1 = \hat{B}(1:m, 1:m) - \frac{3}{2}I_m$, such that B_1 is nonsingular. Denote $B_2 = \hat{B}(1:m, m+1:n)$, then we have $B = (B_1, B_2)$ with $B_1 \in R^{m,m}$ and $B_2 \in R^{m,n-m}$. Obviously, the resulting saddle point-type matrix

$$A = \begin{pmatrix} G & B^T \\ B & 0 \end{pmatrix}$$

satisfies that $rank(B^T) = rank(B) = m$.

From the matrix A we construct the following saddle point-type matrices:

$$A_1 = \begin{pmatrix} G_1 & B^T \\ B & 0 \end{pmatrix},$$

where G_1 is constructed from G by replacing its first $\frac{m}{4}$ rows and columns with zero entries. Note that G_1 is semipositive real and its nullity is $\frac{m}{4}$.

In our numerical experiments the matrix W in the augmentation block preconditioners is taken as $W = I_m$ and the mesh $h = 1/16$. During the implementation of our augmentation block preconditioners, we assume

$$P1 = \begin{pmatrix} D^{-1} & 0 \\ 0 & (CD^{-1}B^T)^{-1} \end{pmatrix},$$

where $F = 0, D = G_1 + 10B^T C$, and

$$P2 = \begin{pmatrix} D^{-1} & 0 \\ 0 & I \end{pmatrix},$$

where $F = I - CD^{-1}B^T, D = G_1 + B^T C$,

$$P3 = \begin{pmatrix} D^{-1} & 0 \\ 0 & (0.01CD^{-1}B^T)^{-1} \end{pmatrix},$$

where $F = 1.1CD^{-1}B^T, D = G_1 + 10B^T C$, respectively.

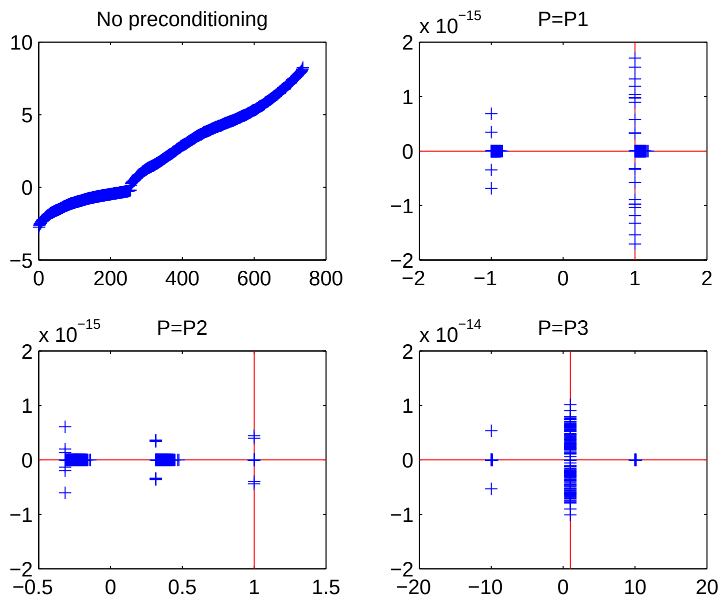

FIGURE 1. Spectrum of A and $P^{-1}A$ when $h = \frac{1}{16}(n + m = 736)$.

On Figure 1, we can observe that that the eigenvalues of $P^{-1}A$ are strongly clustered, which is expected by Theorem 2.1 and Theorem 2.2. In our example, we compared the three different preconditioners. On Figure 2, we see that the choice

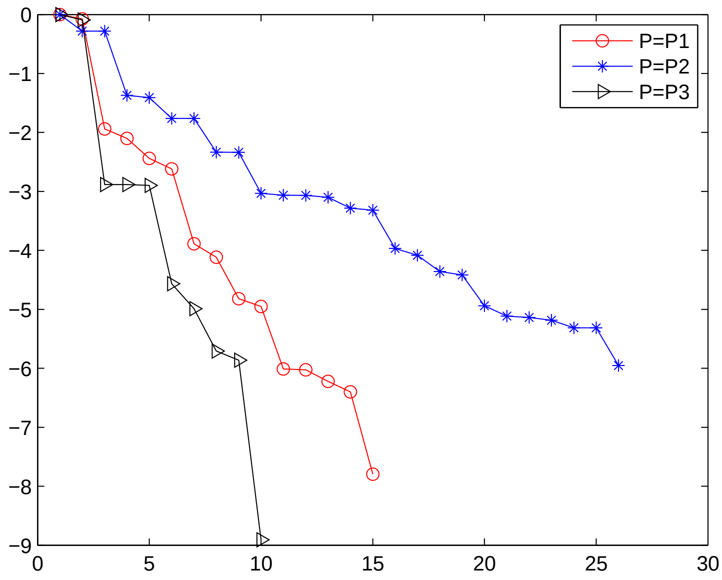

FIGURE 2. Convergence curve and total numbers of inner GMRES iterations when $h = \frac{1}{16}(n+m = 736)$.

of a suitable F improves the results.

5. CONCLUSION

In this paper, we extended the block positive definite preconditioner introduced in [3], and we proved some of its properties. Then, we gave numerical examples in order to illustrate the efficiency of our method.

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