Miskolc Mathematical Notes
Vol. 20 (2019), No. 2, pp. 1021–1037
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2019.2739

OPTIMAL CONTROL PROBLEMS FOR SOME CLASSES OF FUNCTIONAL-DIFFERENTIAL EQUATIONS ON THE SEMI-AXIS

O. KICHMARENKO AND O. STANZHYTSKYI

Received 07 November, 2018

Abstract. In this paper we study functional-differential equations on the semi-axis, which are non-linear with respect to the phase variables and linear with respect to the control. Sufficient conditions for existence of optimal control in terms of the right-hand side and the quality criterion are obtained. Relation between the solutions of the problems on infinite and finite intervals is studied and results that about these connections are proven.

2010 *Mathematics Subject Classification:* 34K35; 49K35; 49J99; 93C23

Keywords: control, finite interval, infinite interval, functional, convexity, weak convergent, compactness

1. INTRODUCTION

Let $h>0$ be a constant, describing the delay. By $|\cdot|$ we denote a vector norm in R^d, and by $||\cdot||$ the norm of $d\times m$–matrices, which agrees with the vector norm. We introduce the necessary functional spaces which we use in this paper. Let $C = C([-h,0];R^d)$ be the Banach space of continuous functions from $[-h,0]$ into R^d with the uniform norm $||\varphi||_C = \max_{\theta\in[-h,0]} |\varphi(\theta)|$, and let $L_p = L_p([-h,0];R^m)$, $p>1$ be the Banach space of p-integrable m-dimensional vector-valued functions with the norm $||\varphi||_{L_p} = \left(\int_{-h}^{0} |\varphi(s)|^p ds\right)^{1/p}$.

Let x be continuous function on $[0,\infty)$ and let $\varphi\in C$. If $x(0)=\varphi(0)$, then the function

$$x(t,\varphi) = \begin{cases} \varphi(t), & t\in[-h,0] \\ x(t), & t\geq 0 \end{cases}$$

is continuous for $t\geq 0$. In the standard way (see [11]) for each $t\geq 0$ we can introduce an element $x_t(\varphi)\in C$ by the expression $x_t(\varphi) = x(t+\theta,\varphi)$, $\theta\in[-h,0]$. Further, instead of $x_t(\varphi)$ we write x_t.

Let $t\in[0,\infty)$, and D be a domain in $[-h,\infty)\times C$ with boundary ∂D.

In this paper, we study optimal control problems for systems of functional-differential equations ($\dot{x} = dx(t)/dt$)

$$\dot{x}(t)= f_1(t,x_t)+\int\limits_{-h}^{0} f_2(t,x_t,y)\,u(t,y)dy,\ t\in[0,\tau],\ x(t)=\varphi_0(t),\ t\in[-h,0], \quad (1.1)$$

with one of the next cost criterion

$$J[u] = \int_0^{\tau} \left(e^{-\gamma t} A(t,x_t) + B(t,u(t,\cdot))\right) dt \ \to inf, \quad (1.2)$$

$$J[u] = \int_0^{\tau} \left(e^{-\gamma t} A(t,x_t) + \int_{-h}^{0} |u(t,y)|^2 dy\right) \to inf. \quad (1.3)$$

These problems are considered on the infinite horizon $t \geq 0$. Here $\varphi_0 \in C$ is a fixed element such that $(0,\varphi_0) \in D$, $x(t)$ is the phase vector in R^d, and x_t is the corresponding phase vector in C, τ is the moment when (t,x_t) reaches the boundary ∂D for the first time or $\tau = \infty$ otherwise. Also, $f_1 : D \to R^d$, $f_2 : D\times[-h,0] \to M^{d\times m}$ – $d\times m$-dimensional matrix, such that for each $(t,\varphi)\in D$ $f_2(t,\varphi,\cdot)$ belongs to the space $L_q([-h,0];M^{d\times m})$ with the norm

$$||f_2(t,\varphi)||_{L_q} = \left(\int_{-h}^{0} ||f_2(t,\varphi,y)||^q dy\right)^{1/q}, \quad \frac{1}{p}+\frac{1}{q}=1$$

$A : D \to R^+$, $B : [0,\infty)\times L_p \to R^+$ are given mappings.

The control parameter $u \in L_p([0,\infty)\times[-h,0])$ is m-dimensional vector function such that for almost all $(t,y), u(t,y) \in W$, $0 \in W$, where W is a convex and closed set in R^m.

For each control function, we define corresponding solution (trajectory) of (1.1). A continuous function $x(t)$ is a solution of (1.1) on the interval $[-h,T]$, if it satisfies the following conditions: $x(t) = \varphi_0(t), t \in [-h,0]$; $(t,\ x_t) \in D$ for $t \in [0,T]$; for $t \in [0,T]$ $x(t)$ satisfies the integral equation

$$x(t) = \varphi_0(0) + \int_0^t [f_1(s,x_s) + \int_{-h}^{0} f_2(s,x_s,y)\,u(s,y)\,dy]ds.$$

The control function $u(t,\cdot)$ is considered admissible for the problems (1.1)-(1.2) and (1.1), (1.3) if $u(t,y) \in L_p([0,\infty)\times[-h,0]$; $u(t,y)\in W$ for almost all $t\geq 0$, $y \in [-h,0]$; the solution $x(t)$ corresponding to the control $u(t,\cdot)$ exists on the interval $[-h,\tau], \tau > 0$; $|J[u]| < \infty$.

Let $V(\varphi_0)$ denote the Bellman function for the problem on the infinite horizon and let $V_T(\varphi_0)$ be the Bellman function for the corresponding problem on some finite interval $[0,T]$.

In [13] it was shown that system (1.1) includes as particular cases the usual optimal control problem for functional-differential equations

$$\dot{x}(t) = f(t, x_t) + g(t, x_t)u(t), \quad u \in L_p\left([0,\infty); R^m\right), \tag{1.4}$$

for equations with maximum and for system of ordinary differential equations.

The choice of the control $u(t,\cdot) \in L_p([0,\infty); [-h,0])$ for each t as an element of the function space is justified (determined) by two reasons:

1) the given problem to be similar to the general functional-operator form of an optimal control problem where $u(t) \in W$ and W is a topological space (see, for example, [2]).

2) the given class of problems includes some problems with applications to economics (see [4, 6]).

A great number of publications is devoted to the study functional differential systems [1, 5, 8, 11, 14] and also a lot of publications is devoted to the study of optimal control problems of type (1.4). In the monograph [14], optimal control problems for functional-differential equations are studied and method for dynamic programming and maximum principle are developed. However, in most of the cases, these methods give only necessary conditions for optimality. In the cases when they give sufficient conditions, the verification of those conditions is quite complicate and requires involving new objects, which were not presented in the initial problem. That is why, it is desirable to have the sufficient conditions for existence of optimal control in terms of the right-hand side of the system and the cost criterion. In this direction, we can mention the work [3], where under condition of compactness for the set of values of the admissible controls is obtained analogue of Filippov theorem. In the case when the set of the control values is unbounded, it is obtained analogue of the Cessari theorem. Note that if the condition for compactness of the control set is removed, then this leads to condition of the growth, which connect the right-hand side of the system and the function of quality criterion. In [9, 10], the authors study the problem for optimal control of the system

$$\dot{x}(t) = \tau x(t) + f_0\left(x(t), \int_{-T}^{0} a(y)\, x(t+y)\, dy\right) - u(t).$$

In [10] for some cost criteria Hamilton-Jacobi-Bellman equations are obtained and in the terms of their solutions sufficient conditions for optimality are obtained. In [7], similar questions are considered for problem with phase space restrictions. In [9], under the condition that the function $\tau x + f_0(x, y)$ is non-decreasing in both variables for quality criterion

$$J[u] = \int_0^{\infty} e^{-\rho t} u^{1-\sigma}(t)\, x(t)\, dt, \quad \sigma \in (0;1)$$

sufficient conditions for optimality are obtained. In [13], problem of type (1.1)-(1.2) is considered in more general settings, but only on finite interval $[0, T]$.

The goal of this work is to generalize the results obtained in [13] to the infinite horizon $[0,\infty)$ and to clarify the relation between problems on finite and infinite intervals. It turns out that by means of optimal control for finite interval, it is possible to construct easily minimizers for the problem on infinite horizon.

This paper is organized in the following way. In Section 2 we give a rigorous statement of the considered problem and the main result. Section 3 is devoted to the proof of the main result of this paper. In Subsection 3.1 a theorem for existence of optimal control for the problems (1.1), (1.2) and (1.1), (1.3) is proven. In Subsection 3.2 a theorem about the connection between the solution of the problem on infinite horizon $[0,\infty)$ and the solution of same problem on finite intervals is proven. In Subsection 3.3 existence of optimal control in the case when the domain D is unbounded is proven.

2. Statement of the problems and main results

We give rigorous statement of the problem and statement of the main result of this work. In this paper, we assume that the following conditions are satisfied. Let D be a domain in $[-h,\infty)\times C$, and ∂D be its boundary (see, for example [12] p. 18). We introduce the notations $D_t=\{\varphi\in C,(t,\varphi)\in D\}$, $D_c=\bigcup_{t\geq 0}D_t$, where D_c is bounded in C.

Assumption 1. The admissible controls are m-dimensional vector functions $u(t,y)\in L_p([0,\infty)\times[-h,0];R^m)$, such that for almost all $t\geq 0$ and $y\in[-h,0]$ we have $u(t,y)\in W$, where W is a convex closed set in R^m and $0\in W$ and there exists $J[u]$.

The set of admissible controls is denoted by $\mathcal{U}$.

Assumption 2. The mappings $f_1(t,\varphi):D\to R^d$ and $f_2(t,\varphi,y):D\times[-h,0]\to M^{d\times m}$ are defined and measurable with respect to all arguments in the domain D and $D_1=\{(t,\varphi)\in D, y\in[-h,0]\}$, respectively. Moreover, these functions satisfy in D and D_1, with respect to φ the condition for linear growth and the Lipchitz condition, i.e., there exists constant $K>0$, such that

$$|f_1(t,\varphi)|+\|f_2(t,\varphi,y)\|\leq K(1+\|\varphi\|_C), \tag{2.1}$$

for $(t,\varphi)\in D,\ y\in[-h,0]$,

$$|f_1(t,\varphi_1)-f_1(t,\varphi_2)|+\|f_2(t,\varphi_1,y)-f_2(t,\varphi_2,y)\|\leq K\|\varphi_1-\varphi_2\|_C, \tag{2.2}$$

for $(t,\varphi_1),(t,\varphi_2)\in D$.

Assumption 3. 1) The mapping $A:D\to R$, $A(t,\varphi)\geq 0$ for $(t,\varphi)\in D$ is defined and continuous in D and for $(t,\varphi)\in D$ there is a constant $K_A>0$, such that $A(t,\varphi)\leq K_A(1+\|\varphi\|_C)$;

2) The mapping $B : [0,\infty) \times L_p \to R$ is measurable with respect to all its arguments and there are constants $a > 0,\ a_1 > 0$, such that $a_1 \|z\|_{L_p}^p \geq B(t,z) \geq a\|z\|_{L_p}^p$ if $t \geq 0$;

3) For each $t \geq 0,\ B(t,z)$ is strongly differentiable with respect to z and for $t \geq 0$ and $z \in L_p$ the Frechet derivative $\frac{\partial B}{\partial z}$ satisfies the estimate

$$\left\| \frac{\partial B}{\partial z} \right\|_{\mathcal{L}(L_p;R^1)} \leq a_2 \|z\|_{L_p}^{p-1}$$

for some constant $a_2 > 0$, independently of t and z. Here $\|\cdot\|_{\mathcal{L}(L_p;R^1)}$ is the uniform operator norm in the space of linear continuous functionals over L_p.

The main results of this work are given by the following theorems.

Theorem 1. *Suppose that Assumptions 1-3 are satisfied. Then there exists a solution (x^*,u^*) of the problems (1.1), (1.2) and (1.1), (1.3).*

Let $T > 0$ be fixed. By (x_T^*, u_T^*) we denote the solution of the problems (1.1), (1.2) or (1.1), (1.3) on $[0,T]$.

For the problem on infinite horizon, we define

$$u_{T,\infty}(t,\cdot) = \begin{cases} u_T^*(t,\cdot), & t \in [0,T] \\ 0, & t > T \end{cases} \tag{2.3}$$

and $x^{T,\infty}(t)$ is the corresponding trajectory.

It is obvious that the given control is admissible for the original problem. Again, $(u^*(t,\cdot), x^*(t))$ is an optimal pair for the problem (1.1)-(1.2), τ – the time at which the solution x_t^* reaches the boundary ∂D.

Theorem 2. *Suppose that Assumptions 1-3 are satisfied, then we have:*

1)

$$V_T(\varphi_0) \to V(\varphi_0), \quad T \to \infty;$$

2) *there is a sequence $T_n \to \infty,\ n \to \infty$, such that the sequence $\{u_{T_n,\infty}\}$ is minimizer for the problem (1.1), (1.2) i.e.*

$$J\left[u_{T_n,\infty}\right] \to V, \quad n \to \infty, \tag{2.4}$$

3) *there is a sequence $T_n \to \infty,\ n \to \infty$, such that*

$$u_{T_n,\infty} \xrightarrow{w} u^*, \quad n \to \infty \tag{2.5}$$

weekly in $L_p([0,\infty) \times [-h,0]; R^m)$

4) *pointwise on $[0,\tau^*]$, uniformly on each finite interval*

$$x^{T_n,\infty}(t) \to x^*(t), \quad n \to \infty.$$

If the problem (1.1)-(1.2) has unique solution, then the convergence in (2.4), (2.5) occurs for all $T \to \infty$.

Proposition 1. *In the conditions of Theorem 2 for the functional (1.3) all statements of Theorem 2 are valid, and the weak convergence of optimal controls (2.5) is replaced with strong convergence in* $L_2([0,\infty)\times[-h,0];R^m)$.

The next theorem is about the case when the domain D_c in the statement of the problem is unbounded. As it is shown in [13], the solution of the original problem cannot go to infinity in finite time. However, it can increase without bound in such a way, that the integrals in (1.2) and (2.2) become divergent for all admissible controls. Now we give a theorem, which guarantees existence of optimal control in this case. So, we assume that it is possible that D is unbounded domain in $[-h,\infty)\times C$ but the set of control values W is bounded in R^m. Without loss of generality, we can assume that W is a ball with radius r.

Theorem 3. *If the conditions of Theorem 1 are satisfied and* $\gamma<(hr+1)K$*, then the problems (1.1), (1.2) and (1.1), (1.3) have solutions.*

3. Proofs of the theorems

3.1. *Proof of Theorem 1*

Proof. First, we note that in the conditions of Theorem 1 imply that the conditions of Theorem 2.1 in the work [13] are satisfied. Therefore, the solution of the problem (1.1) exists, it is unique, and it can be extended to the boundary of the domain D. Further, we note that the set $\mathcal{U}$ of admissible controls is nonempty, since $0\in\mathcal{U}$. Moreover, if $x(t,0)$ is a solution of the system (1.1), which exists for such control, and $x_t(0)$ is the corresponding element in C. Then by the condition 1) of Assumption 3 and the boundedness of D_c we have $J[0]<\infty$ here r is the radius of the ball, which contains D_c.

Since the quality functionals is non-negative quantity, then there is non-negative lower bound m of the values of $J[u]$ and therefore there is a sequence of admissible controls $\{u_n(t)\}$. Let $u^{(n)}(t,y)$ be minimizing sequence, such that $J\left[u^{(n)}\right]\to m$, $n\to\infty$ monotonically. Also, let $x^{(n)}(t)$ be a sequence of solutions of the equation (1.1), for which there exists controls $u^{(n)}$, and let $[-h,\tau_n]$ be the maximal intervals of their existence.

Note that $(\tau_n,x^{(n)}_{\tau_n})\in\partial D$. It is easy to check

$$m+1\ge a\int_0^{\tau_n}\int_{-h}^0\left|u^{(n)}(t,y)\right|^p dy\,dt\ge a\int_0^{\infty}\int_{-h}^0\left|u^{(n)}(t,y)\right|^p dy\,dt, \tag{3.1}$$

for n large enough. Hence, the sequence $u^{(n)}(t,y)$ is weakly compact in $L_p([0,\infty)\times[-h,0])$. This means that it contains a weakly convergent subsequence.

Without loss of generality, we can assume that $u^{(n)}(t,y)$ itself is weakly convergent to $u^*\in L_p([0,\infty)\times[-h,0])$. By the Mazur Lemma ([15], ch. V), some convex combination $b_k(t,y) = \sum_{i=1}^{n(k)} \alpha_i(k) u^{(i)}(t,y)$ of elements $u^{(i)}(t,y)$ converges strongly to u^* in $L_p([0,\infty)\times[-h,0])$. Therefore, there exists a subsequence $b_{k_j}(t,y)$ of the sequence $b_k(t,y)$, such that it converges to $u^*(t,y)$ almost everywhere in $[0,\infty)\times[-h,0]$. Since W is convex and closed, then $b_{k_j}(t,y)\in W$, and for this reason $u^*(t,y)\in W$, therefore, the control $u^*(t,y)$ is admissible.

Now we consider the sequence of solutions $x^{(n)}(t)$ of the system (1.1), which correspond to the controls $u^{(n)}(t,y)$. When for $t\in[0,\tau_n]$ we have

$$x^{(n)}(t) = \varphi_0(0) + \int_0^t \left[f_1\left(s, x_s^{(n)}\right) + \int_{-h}^0 f_2(s, x_s, y) u^{(n)}(s,y)\,dy \right] ds. \tag{3.2}$$

Using the functions $x^{(n)}(t)$ we construct the functions $z^{(n)}(t)$, which are determined on the semi-axis in the following way:

$$z^{(n)}(t) = \begin{cases} x^{(n)}(t), & t\in[0,\tau_n] \\ x^{(n)}(\tau_n), & t>\tau_n. \end{cases},$$

Since D_C is bounded, then there is $C>0$, such that

$$\left| z^{(n)}(t) \right| \le C, \quad t\ge 0. \tag{3.3}$$

We choose an arbitrary $T>0$ and fixed it. We are going to show that the family of functions $\left\{z^{(n)}(t)\right\}$ is compact on $[0,T]$. To do that, by (3.3), it is enough to prove that they are equicontinuous. For $t_1,t_2\in[0,\tau_n]$ from (3.2) and by (2.1) we have the estimate

$$\left| x^{(n)}(t_2) - x^{(n)}(t_1) \right| \le K(1+C)(t_2-t_1) + K(1+C)h^{1/q}(t_2-t_1)^{1/q}\left(\frac{m+1}{a}\right)^{1/p}.$$

Therefore, for $t_1\le t_2\le \tau_n$ and for some positive C_1, C_2 we get

$$\left| z^{(n)}(t_2) - z^{(n)}(t_1) \right| \le C_1(t_2-t_1) + C_2(t_2-t_1)^{1/q}. \tag{3.4}$$

It is easy to check if $t_1<\tau_n<t_2$, then estimate (3.4) holds. From here the equicontinuity of the family of the functions $z^{(n)}(t)$ on $[0,T]$, and therefore their compactness follows. In this way, there exists a subsequence $z_k^{(n)}(t)$ of the sequence $z^{(n)}(t)$ such that $z_k^{(n)}(t)$ converges uniformly to $z^*(t)$ on $[0,T]$. The function $z^*(t)$ is defined and continuous on $[0,T]$, and hence z_t^* exists as an element of the space C for all $t\in[0,T]$. Therefore, on each interval $[0,T]$ from the sequence $\left\{z^{(n)}(t)\right\}$ it is possible to take uniformly convergent subsequence. We show, that there exists subsequence of the sequence $\left\{z^{(n)}(t)\right\}$, which converges point-wise on $[0,\infty)$ to some continuous

function. To do that, we use the diagonal method. For $T=1$ there exists a subsequence $\left\{z_1^{(n)}(t)\right\}$ of the sequence $\left\{z^{(n)}(t)\right\}$ such that $z_1^{(n)}(t) \rightrightarrows z_1^*(t)$ for $n \to \infty$ on $[0,1]$. For $T=2$ there exists a subsequence $\left\{z_2^{(n)}(t)\right\}$ of the sequence $\left\{z_1^{(n)}(t)\right\}$ such that $z_2^{(n)}(t) \rightrightarrows z_2^*(t)$ for $t \in [0,2]$. We observe, that $z_2^*(t) = z_1^*(t)$ for $t \in [0,1]$. Continue this process, we obtain, for each natural number N existence of a subsequence $\left\{z_N^{(n)}(t)\right\}$ of the sequence $\left\{z_{N-1}^{(n)}(t)\right\}$ such that $z_N^{(n)}(t) \rightrightarrows z_N^*(t)$ on $[0,N]$ and $z_N^*(t) = z_{N-1}^*(t)$ for $t \in [0, N-1]$. Applying the diagonal method, from these sequences we choose the subsequences $\left\{z_n^{(n)}(t)\right\}$, such that, they obviously converge point-wise for $t \in [0,\infty)$ to the continuous function $z^*(t)$, determined in the following way: $z^*(t) = z_N^*(t)$ on $[0,N]$, where N is a natural number. For convenience, in our next considerations, we again denote the sequence $\left\{z_n^{(n)}(t)\right\}$ by $\{z^n(t)\}$, and the corresponding sequence of controls as $\left\{u^{(n)}(t)\right\}$. Since $z^*(t)$ is defined and continuous in $[0,\infty)$, then z_t^* exists as an element of the space C for all $t \ge 0$. Also, we note that $z_N^*(t)$ converges uniformly to $z^*(t)$ on each interval $[0,\tau]$.

Let τ^* be the moment when (t, z_t^*) reaches the boundary ∂D for the first time, then

$$\tau^* = \begin{cases} inf\ \{t \ge 0 : (t, z_t^*) \in \partial D \\ \infty,\ \left(t, z_t^*\right) \in D,\ t \ge 0 \end{cases}$$

Notice that $z_{\tau_n}^{(n)} = z^{(n)}(\tau_n + \theta) = x^{(n)}(\tau_n + \theta) = x_{\tau_n}^{(n)}$, and hence τ_n is the moment when $(t, z_t^{(n)})$ reaches ∂D for the first time.

We are going to show that $\tau^* \le \lim_{n\to\infty} inf\, \tau_n$. We consider two cases:

1) Let $\tau^* < \infty$. Then $\tau^* > \lim_{n\to\infty} inf\, \tau_n = \tau$.

We choose an arbitrary $T \in [0,\infty)$ such that $T \ge \tau^*$. Obviously, there is a subsequence $\left\{\tau_{n_k}\right\}$ of the sequence $\{\tau_n\}$, such that $\tau_{n_k} \to \tau$ for $n_k \to \infty$. So, for n_k large enough, we have $\tau_{n_k} < \tau^*$ and

$$(\tau_{n_k}, z_{\tau_{n_k}}^*) \in D,\ \left(\tau, z_\tau^*\right) \in D, \tag{3.5}$$

but $(\tau_{n_k}, z_{\tau_{n_k}}^{(n_k)}) \in \partial D$.

On the other hand, taking into account the uniform convergence on $[-h,T]$ of the sequences $z^{(n)}(t)$ to $z^*(t)$ and the uniform continuity of $z^*(t)$ on $[-h,T]$ we have $z_{\tau_{n_k}}^{(n_k)} \to z_\tau^*$, in C. Indeed,

$$\sup_{-h\le\theta\le 0} \left| z^{(n_k)}(\tau_{n_k}+\theta) - z^*(\tau+\theta) \right| \le \sup_{-h\le\theta\le 0} \left| z^{(n_k)}(\tau_{n_k}+\theta) - z^*(\tau_{n_k}+\theta) \right|$$

$$+ \sup_{-h\le\theta\le 0} \left| z^*(\tau_{n_k}+\theta) - z^*(\tau+\theta) \right| \to 0,\ n_k \to \infty,$$

Since the set ∂D is closed, then $(\tau, z^*_\tau) \in \partial D$. This contradicts (3.5).

2) Let $\tau^* = \infty$, and $\lim_{n\to\infty} inf \tau_n < \infty$. Choose $T_2 > 0$, $T_2 > \lim_{n\to\infty} inf \tau_n$. Now applying analogous reasoning on $[0, T_2]$, we obtain, that this case can be reduced to the previous one. Hence $\tau^* \le \lim_{n\to\infty} inf \tau_n = \tau$.

Set $x^*(t) = z^*(t)$ for $t \in [0, \tau^*]$ in the case of finite τ^* and $t \in [0, \infty)$ for $\tau^* = \infty$. We will show that $x^*(t)$ is a solution of the equation (1.1), to which corresponds the control $u^*(t, y)$ for all considered t.

We consider three cases.

1. If $\tau < \infty$, then the proof is analogous to the proof of the corresponding fact in [13], Theorem 2.2.

2. Let $\tau = \infty$, but $\tau^* < \infty$. In this case, either there exists subsequence $\{\tau_{n_k}\}$ of the sequence $\{\tau_n\}$ such that, $\tau_{n_k} \to \infty$, $n_k \to \infty$ or $\tau_n < \infty$ only for finite numbers. Then for large enough n_k, we have $z^{(n_k)}(t) = x^{(n_k)}(t)$ for $t \in [0, \tau^*]$ and $x^{(n_k)}(t) \rightrightarrows x^*(t)$ for $n_k \to \infty$ in $[0, \tau^*]$. From here the proof is similar to the proof of [13], Theorem 2.2.

3. Let $\tau^* = \infty$. We choose an arbitrary $T > 0$ and consider the interval $[0, T]$. Analogically to the previous case, there exists subsequence $\{\tau_{n_k}\}$ such that $\tau_{n_k} \to \infty$ for $n_k \to \infty$. Then on $[0, T]$ for large enough n_k, we have $z^{(n_k)}(t) = x^{(n_k)}(t)$ on $[0, T]$, and therefore $x^{(n_k)}(t) \rightrightarrows x^*(t)$ in $[0, T]$ for $n \to \infty$. After that, the proof is similar to the previous.

It remains to show that the control $u^*(t, y)$ is optimal. Again, we consider two cases.

1. Let $\tau^* < \tau$. In this case either there exists a subsequence $\{\tau_{n_k}\}$ of the sequence $\{\tau_n\}$ such that $\tau_{n_k} \to \tau$ for $n_k \to \infty$, or there exist only not more than finite number of finite $\{\tau_n\}$ (in the case $\tau = \infty$). Then for n_k large enough, again we have $z^{(n_k)}(t) = x^{(n_k)}(t)$ for $t \in [0, \tau^*]$ and $x^{(n_k)}(t) \rightrightarrows x^*(t)$ for $n_k \to \infty$ on $[0, \tau^*]$.

Also, obviously, we have that for all $t \in [0, \tau^*]$.

$$\left\| x_t^{(n_k)} - x_t^* \right\|_C \to 0, \quad n_k \to \infty, \tag{3.6}$$

Then

$$\int_0^{\tau_{n_k}} e^{-\gamma t} A\left(t, x_t^{(n_k)}\right) dt + \int_0^{\tau_{n_k}} B\left(t, u^{(n_k)}(t, \cdot)\right) dt \\ \ge \int_0^{\tau^*} e^{-\gamma t} A\left(t, x_t^{(n_k)}\right) dt + \int_0^{\tau^*} B\left(t, u^{(n_k)}(t, \cdot)\right) dt. \tag{3.7}$$

The integrand of the first summand in (3.7) for each t tends to $A(t, x_t^*)$ by (3.6) and condition 1) from Assumption 3. From the fact that D_c is bounded and the condition of linear growth $A(t, \varphi)$(condition 1) of Assumption 3), it follows for some constant $K_1 > 0$ we have the inequality $A\left(t, x_t^{(n_k)}\right) \le K_A(1 + K_1)$. Now using Lebesgue dominated convergence theorem, we get that

$$\int_0^{\tau^*} e^{-\gamma t} A\left(t, x_t^{(n_k)}\right) dt \to \int_0^{\tau^*} e^{-\gamma t} A\left(t, x_t^*\right) dt, n_k \to \infty. \tag{3.8}$$

Since $B(t,u)$ is convex with respect to u, then

$$B\left(t, \vartheta\left(t,\cdot\right)\right) \geq B\left(t, u^*\left(t,\cdot\right)\right) + \left\langle B_u'\left(t, u^*\left(t,\cdot\right)\right), \vartheta\left(t,\cdot\right) - u^*\left(t,\cdot\right)\right\rangle, \tag{3.9}$$

for each admissible control $\vartheta(t,y) \in \mathcal{U}$. Here $\left\langle L_u', \vartheta - u^*\right\rangle$ is the action of the linear continuous functional L_u' on the element $\vartheta(t,\cdot) - u^*(t,\cdot) \in L_p$. So, using condition 3) from Assumption 3 we have

$$\int_0^{\infty}\int_{-h}^{0}\left|\frac{\partial B}{\partial u}(t, u^*(t,y))\right|^q dy dt \leq a_3 \int_0^{\infty}\int_{-h}^{0}\left|u^*(t,y)\right|^p dy dt < \infty.$$

Therefore, by Riesz theorem the expression

$$\int_0^{\infty}\langle B_u'(t, u^*(t,\cdot), \vartheta(t,\cdot)) - u^*(t,\cdot)\rangle dt$$

defines a linear continuous functional on $L_p([0,\infty)\times[-h,0])$. Now, let $\vartheta(t,\cdot) = u^{n_k}(t,\cdot)$ in (3.9) and using the weak convergence of $u^{n_k}(t,y)$ to $u^*(t,y)$, for the second summand in (3.7) we get the inequality

$$\lim_{n_k\to\infty} inf \int_0^{\tau^*} B\left(t, u^{(n_k)}(t,\cdot)\right) dt \geq \int_0^{\tau^*} B\left(t, u^*(t,\cdot)\right) dt\ . \tag{3.10}$$

From (3.7), (3.8) and (3.10) we have

$$m = \lim_{n\to\infty}\left(\int_0^{\tau_{n_k}} e^{-\gamma t} A\left(t, x_t^{(n_k)}\right) dt + \int_0^{\tau_{n_k}} B\left(t, u^{(n_k)}(t,\cdot)\right) dt\right)$$
$$\geq \int_0^{\tau^*} e^{-\gamma t} A\left(t, x_t^*\right) dt + \int_0^{\tau^*} B\left(t, u^*(t,\cdot)\right) dt,$$

therefore, in this case the control $u^*(t,y)$ is optimal.

2. Let $\tau^* = \tau$. We choose an arbitrary $t_1 = \tau^*$ and consider the interval $[0,t_1]$. On this interval $x^{(n_k)}(t) \rightrightarrows x^*(t)$, when $n_k \to \infty$, and hence $x_t^{(n_k)} \to x_t^*$ in C, $n_k \to \infty$.

By the theorem for characterization of the lower bound, the set $\{\, n \in N \,|\, \tau_n < t_1 \,\}$ is finite, and the open interval (t_1, τ^*) can contain infinite number of points τ_n (if they are finite). We consider this sequence. Then by analogy as in the previous case, we obtain

$$m \geq \int_0^{t_1} e^{-\gamma t} A\left(t, x_t^*\right) dt + \int_0^{t_1} B\left(t, u^*(t,\cdot)\right) dt$$

From here by taking limit for $t_1 \to \tau^*$ we obtain that $J[u^*] = m$.

For the functional (1.3) the proof is similar. In this case, the condition 3) from Assumption 3 is satisfied automatically, since $\frac{\partial B}{\partial z} = 2u(t,\cdot)$. This proves the theorem. □

3.2. *Proof of Theorem 2*

Proof. First, we consider the problem (1.1)-(1.2). We choose an arbitrary $T > 0$ and fix it. As before, $V(\varphi_0)$ denotes the Bellman function for the given problem and $V_T(\varphi_0)$ denotes the Bellman function for the corresponding problem on $[0,T]$. From Theorem 1 and the corresponding theorem from [13], it follows that these problems have solutions $(x^*(t), u^*(t,\cdot))$ and $(x^{*,T}(t), u_T^*(t,\cdot))$ respectively. Note that the set of admissible controls on $[0,\infty)$ is a subset of admissible controls on $[0,T]$. From the admissible controls $u(t,\cdot)$ on $[0,T]$, which are not admissible on $[0,\infty)$ we construct the following controls

$$u_{T,\infty}(t,\cdot) = \begin{cases} u(t,\cdot), & t \in [0,T] \\ 0, & t > T. \end{cases} \tag{3.11}$$

Let $\mathcal{U}_T$ denote the union of the set of admissible controls on $[0,\infty)$ with the set of controls of type (3.11). Then on $[0,\infty)$ this set of admissible controls coincides with the set $\mathcal{U}$, and on the interval $[0,T]$ it coincides with the set of all admissible controls for the problems of type (1.1)-(1.2) on $[0,T]$. Indeed, from an arbitrary admissible control $u(t,\cdot)$ on $[0,T]$, which is not admissible on $[0,\infty)$ by the rule (3.11) we construct an admissible control on $[0,\infty)$. On the other side, $L_p([0,\infty)) \subset L_p([0,T])$.

Let $D_T = D \cap [-h,T]$, and we denote by τ_T^* the moment when $x_t^{*,T}$ reaches the boundary of domain D_T. Note that in the case $\tau_T^* < T$ the control $u_{T,\infty}^*(t,\cdot)$ will be optimal for the problem (1.1), (1.2) on $[0,\infty)$. Then we conclude that $V = V_T$.

Consider now the case $\tau^*{}_T = T$. Denote by $x(t) = x(t, u_{T,\infty}^*(t,\cdot))$ the solution of the initial problem (1.1), which corresponding to the control $u_{T,\infty}^*(t,\cdot)$. We note that if $t \in [0,T]$ $u^*(t,\cdot) = u_T^*(t,\cdot)$, then by uniqueness of the solution of the initial value problem (1.1) $x(t) = x^{*,T}(t)$ for $t \in [0,T]$. Thus, from the definition of the Bellman function and using condition 2) from Assumption 3, we have

$$\begin{aligned} V \le J\left(u_{T,\infty}^*\right) &= \int_0^T (e^{-\gamma t} A(t, x_t^{*,T}) + B\left(t, u_T^*(t,\cdot)\right) dt + \int_T^{\tau_T} e^{-\gamma t} A(t, x_t)\, dt \\ &= V_T + \int_T^{\tau_T} e^{-\gamma t} A(t, x_t)\, dt. \end{aligned} \tag{3.12}$$

Here τ_T denotes the moment when x_t reaches the boundary of the domain D. The second term in (3.12) goes to zero for $T \to \infty$ by the boundedness of D_c and by the Lebesgue dominated convergence theorem.

We recall, that τ^* is the moment when the optimal trajectory x_t^* of the problem (1.1)-(1.2) on $[0,\infty)$, reaches the boundary of D. Also, note that if $\tau^* \le T$, then the pair $(x^*(t), u^*(t,\cdot))$ will be optimal for the problem on the interval $[0,T]$ and in this case again we have $V = V_T$. Let $\tau^* > T$, then

$$V = J[u^*] \geq V_T + \int_T^{\tau^*} (e^{-\gamma t} A(t, x_t) + B(t, u^*(t,\cdot)))dt.$$

But for $T \to \infty$

$$\int_T^{\tau^*} (e^{-\gamma t} A(t, x_t^*) + B(t, u^*(t,\cdot)))dt \leq \int_T^{\infty} e^{-\gamma t} K_A (1 + K_1) dt + \int_T^{\infty} a_1 \|u^*(t,\cdot)\|_{L_p}^p dt$$

$$= \int_T^{\infty} e^{-\gamma t} K_A (1 + K_1) dt + \int_T^{\infty} a_1 \int_{-h}^{0} |u^*(t,y)|^p dy dt \to 0. \tag{3.13}$$

Then, on one side, from (3.12) we have $V - V_T \leq \int_T^{\tau_T} e^{-\gamma t} A(t, x_t) dt$, and on the other side, we obtain $V - V_T \geq \int_T^{\tau_T} (e^{-\gamma t} A(t, x_t^*) + B(t, u^*(t,\cdot)))dt$, from here, taking into account (3.13), we get the statement 1) of the Theorem 2, namely (2).

Further, for convenience, without loss of generality, we can assume that $T = n \in N$ is a natural number. Let $u_n^*(t,\cdot)$ be an optimal control on $[0,n]$, and let $u_{n,\infty}^*(t,\cdot)$ be an admissible control of the problem (1.1), (1.2) on $[0,\infty)$, which is determined by the formula (2.3).

Again, if $\tau_n^* < n$ for some n, then $u_{n,\infty}^*$ is optimal for the problem on infinite horizon. Here τ_n^* is the moment when $x_t^{n,\infty}$ reaches the boundary of the domain D.

Now let $\tau_n^* = n$ for all n. Since $V_n = J[u_n^*] \to V$, then there exists a constant L, such that $V_n \leq L$. But from the conditions 2) of Assumption 3 and from (2.3), we have:

$$L \geq V_n = I\left[u_n^*\right] \geq a \int_0^n \int_{-h}^0 |u_n^*(t,y)|^p dy dt = a \int_0^\infty \int_{-h}^0 |u_{n,\infty}^*(t,y)|^p dy dt,$$

and therefore, the sequence of admissible controls $\{u_{n,\infty}^*\}$ is weekly compact in $L_p([0,\infty)\times[-h,0])$. Therefore, there exists a weekly convergent subsequence, which without loss of generality, we again denote by $\{u_{n,\infty}^*\}$. Moreover, we have

$$u_{n,\infty}^* \xrightarrow{\omega} u^*, \quad n \to \infty, \quad L_p([0,\infty)\times[-h,0]). \tag{3.14}$$

Analogically to Theorem 1, by using the Mazur lemma, we get that $u^*(t,y) \in W$ for almost all (t,y).

We denote by $x^{n,\infty}(t)$ the solution of our original problem (1.1), corresponding to the control $u_{n,\infty}^*(t,\cdot)$. Let τ_n be the moment when the solution $x_t^{n,\infty}$ reaches the boundary of the domain D. It is obvious, that $\tau_n > n$. Then, we have

$$J\left[u_{n,\infty}\right] = V_n + \int_n^{\tau_n} (e^{-\gamma t} A(t, x_t^{n,\infty}) + B(t, u_{n,\infty}^*(t,\cdot)))dt.$$

From the construction of the sequence $u_{n,\infty}$ and condition 2) from Assumption 3, we have $B\left(t, u^*_{n,\infty}(t,\cdot)\right) = 0$ for $t \geq n$. By condition 1) of Assumption 3, we have

$$J\left[u_{n,\infty}\right] = V_n + \int_n^{\tau_n} e^{-\gamma t} A\left(t, x_t^{n,\infty}\right) dt, \tag{3.15}$$

but $\int_n^{\tau_n} e^{-\gamma t} A\left(t, x_t^{n,\infty}\right) dt \leq \int_n^{\tau_n} e^{-\gamma t} K_A\left(1 + \left\|x_t^{n,\infty}\right\|_c\right) dt$. Since the set D_c is bounded, then, as it was shown before, the estimation $\left\|x_t^{n,\infty}\right\| \leq K_1$ for $t \leq \tau_n$ holds. Then, we get

$$\int_n^{\tau_n} e^{-\gamma t} K_A\left(1 + K_1\right) dt \leq \int_n^{\infty} e^{-\gamma t} K_A\left(1 + K_1\right) dt \to 0, \;\; n \to \infty. \tag{3.16}$$

From (3.15) and (3.16) we have that

$$J\left[u_{n,\infty}\right] \to V, \;\; n \to \infty. \tag{3.17}$$

Therefore, $\{u^*_{n,\infty}\}$ is a minimizing sequence for the problem (1.1), (1.2). This proves the statement 2) of the Theorem 2. We denote by $x^*(t)$ the solution of the initial problem (1.1), with corresponding control $u^*(t,\cdot)$ from (3.14). It follows, from theorem 2.1 in [13], that such solution exists and it is unique. The statement that the pair $(u^*(t,\cdot), x^*(t))$ is optimal for the problem (1.1), (1.2) can be proven in the similar way as in the corresponding proof of Theorem 1. The statement 3) of this theorem, now becomes obvious. The proof of the statement 4) can be carried out in the similar way as the proof of the corresponding fact of the Theorem 1. If the problem (1.1), (1.2) has unique solution, then the convergence in (2.4) and (2.5) holds for all $T \to \infty$. Obviously, the later follows from the fact that from each subsequence $\{u^*_{n_k,\infty}\}$ of the sequence $\{u^*_{n,\infty}\}$ in (3.14) we can choose a sequence that weekly converges to the optimal control $u^*(t,\cdot)$ and this control is unique. This proves the theorem. □

3.3. *Proof of Proposition 1*

Proof. We consider the optimal control problem (1.1), (1.3). Obviously, the proof only requires to establish the fact that the sequence $u^*_{T_n,\infty}$ converges strongly to u^*. In the similar way as in the Theorem 1, we have

$$V \geq \int_0^{\tau^*} A\left(t, x_t^*\right) e^{-\gamma t} dt + \lim_{n\to\infty} \int_0^{\tau_n} \int_{-h}^0 \left|u^*_{n,\infty}(t,y)\right|^2 dy dt, \tag{3.18}$$

where the last limit in (3.18) exists, and therefore coincides with its lower bound. From the construction of $u^*_{n,\infty}$ it follows that

$$\int_0^{\tau_n} \int_{-h}^0 \left|u^*_{n,\infty}(t,y)\right|^2 dy dt = \int_0^{\infty} \int_{-h}^0 \left|u^*_{n,\infty}(t,y)\right|^2 dy dt.$$

Thus, from (3.18) we have

$$V \geq \int_0^{\tau^*} e^{-\gamma t} A\left(t, x_t^*\right) dt + \int_0^{\tau^*} \int_{-h}^{0} \left|u^*(t,y)\right|^2 dy dt = V$$

In this way, we get

$$\lim_{n\to\infty} \int_0^{\infty} \int_{-h}^{0} \left|u_{n,\infty}^*(t,y)\right|^2 dt = \int_0^{\infty} \int_{-h}^{0} \left|u^*(t,y)\right|^2 dy dt.$$

From this, taking into account (3.14) the strong convergence of $u_{n,\infty}^*$ to u^* in $L_2([0,\infty)\times(-h,0])$ follows, which proves Proposition 1. □

3.4. *Proof of Theorem 3*

Proof. Assuming the conditions of this theorem, we consider the problem (1.1), (1.3). First, we show that the set of admissible controls $\mathcal{U}$ is non-empty. To do that, as in Theorem 1, we show that $0 \in \mathcal{U}$.

Indeed, let $x(t,0)$ be a solution of the system (1.1), corresponding to the control $u = 0$, let x_t be the corresponding element from C, τ be the moment of its exit on the boundary of D.

For $t \in [0,\tau)$ we write the integral representation of $x(t,0)$

$$x(t,0) = \varphi_0(0) + \int_0^t f_1(s, x_s) ds. \tag{3.19}$$

From here, taking into account (2.1), we obtain:

$$|x(t)| \leq |\varphi_0(0)| + \int_0^t K(1+\|x_s\|_c) ds \leq |\varphi_0(0)| + K \int_0^t (1 + \max_{s_1\in[-h,s]} |x(s_1)|) ds. \tag{3.20}$$

From (3.20) we have, that

$$\max_{s\in[0,t]} |x(s)| \leq |\varphi_0(0)| + K \int_0^t (1+\|x_s\|) ds,$$

and

$$\max_{s\in[-h,t]} |x(s)| \leq 2 \max_{s\in[-h,0]} |\varphi_0(s)| + Kt + K \int_0^t \max_{s_1\in[-h,s]} |x(s_1)| ds,$$

and therefore, also

$$\max_{t\in[0,\tau]} \|x_t(0)\|_c \leq \left(2 \max_{s\in[-h,0]} |\varphi(t)| + Kt\right) e^{Kt}. \tag{3.21}$$

From here, using the condition 1) of Theorem 3, we have

$$J(0) = \int_0^{\tau} e^{-\gamma t} A(t, x_t(0)) dt \leq \int_0^{\tau} e^{-\gamma t} K_A \left(1 + 2 \max_{t\in[-h,0]} \varphi_0(t) + Kt\right) e^{Kt} dt < \infty,$$

by virtue of (2.5).

Since the functional of quality is non-negative, then again there exists a non-negative lower bound m for the value of $J[u]$ and let $\left\{u^{(n)}(t,\cdot)\right\}$ be minimizing sequence such that

$$J\left(u^{(n)}\right) \to m, \quad n \to \infty. \tag{3.22}$$

Let $\left\{x^{(n)}(t)\right\}$ be a sequence of solutions of the equation (1.1), with corresponding controls $u^{(n)}$, and let $[-h,\tau_n)$ be the maximal intervals of their existence, $\left(\tau_n, x^n_{\tau_n}\right) \in \partial D$. Analogously to Theorem 1, we again obtain that the sequence $\left\{u^{(n)}(t,\cdot)\right\}$ is weekly compact in $L_p([0,\infty)\times(-h,0])$. Moreover, the estimation (3.1) holds. Without loss of generality, we can assume that the sequence $\left\{u^{(n)}(t,\cdot)\right\}$ converges weekly to $u^*(t,\cdot) \in L_p([0,\infty)\times[-h,0])$. As in the proof of Theorem 1, we have that $u^*(t,\cdot) \in \mathcal{U}$ – the set of admissible controls for the problem (1.1)-(1.2).

Analogously to (3.2) for $x^{(n)}(t)$ we have the integral representation

$$x^{(n)}(t) = \varphi_0(0) + \int_0^t \left[f_1(s,x_s) + \int_{-h}^0 f_2(s,x_s,y)u^{(n)}(s,y)dy\right]ds,$$

from which, taking into account (2.1) and the boundedness of the set W, similarly to (3.21), we have

$$\max_{s\in[-h,]} \left|x^{(n)}(s)\right| \le b_1 + b_2 t + (hRK + K)\int_0^t \max_{s_1\in[-h,s]} \left|x^{(n)}(s_1)\right| ds$$

for some positive constant b_1, b_2. Then, using the Gronwall's inequality, we obtain $\max_{s\in[-h,t]} \left|x^{(n)}(s)\right| \le (b_1 + b_2 t)\, e^{(hRK+K)t}$, from which we get

$$\max_{s\in[0,t]} \left\|x^{(n)}_s\right\| \le (b_1 + b_2 t)\, e^{(hRk+K)t}. \tag{3.23}$$

So, we conclude that $x^{(n)}_t$ cannot reach the infinite boundary ∂D for finite time. In other words, $x^{(n)}_t$ can go to infinity only when $t \to \infty$. The later allows us, to construct on an arbitrary interval $[0,T]$ the sequence of functions

$$z^{(n)}(t) = \begin{cases} x^{(n)}(t), t \in [0,\tau_n] \\ x^{(n)}(\tau_n), t \in [\tau_n, T], \end{cases}$$

if $\tau_n \le T$, and if $\tau_n > T$, then $z^{(n)}(t) = x^{(n)}(t)$.

Similarly to Theorem 1, we can show that the sequence $\left\{z^{(n)}(t)\right\}$ contains a subsequence, which converges point-wise on $[0,\infty)$ to some continuous function $z^*(t)$, and this convergence is uniform on each finite interval $[0,T]$. Again, we can assume that the sequence $\left\{z^{(n)}(t)\right\}$ itself has this property. We denote by τ^* – the moment when z_t reaches ∂D. Let $x^*(t) = z^*(t)$ then for $t \in [0,\tau^*]$, we can show (as in

Theorem 1), that $x^*(t)$ is a solution of the equation (1.1), to which corresponds the control $u^*(t,\cdot)$. The proof that the pair $(u^*(t,\cdot),x^*(t))$ is optimal can be done in the similar way as the proof of Theorem 1, where the passage of the limit under the integral of type (3.8) is possible by Lebesgue dominated convergence theorem. From the estimation (3.23) and the condition for linear growth of $A(t,\varphi)$ we have

$$e^{-\gamma t}A\left(t,x_t^{(n_k)}\right)\le K_A e^{-\gamma t}\left(1+\|x_t\|_c\right)\le K_A e^{-\gamma t}\left(b_1+b_2 t\right)e^{(hRK+K)t}.$$

If the condition (2.5) of the Theorem is taken into account, then the quantity $K_A e^{-\gamma t}\left(b_1+b_2 t\right)e^{(hRK+K)t}$ is now integrable upper bound. This proves the theorem. □

REFERENCES

[1] R. P. Agarwal, S. R. Grace, I. Kiguradze, and D. O'Regan, "Oscillation of functional differential equations." *Math. Comput. Modelling*, vol. 41, no. 4-5, pp. 417–461, 2005, doi: 10.1016/j.mcm.2004.06.018.

[2] V. M. Alekseev, V. M. Tikhomirov, and S. V. Fomin, *Optimal control. Transl. from the Russian by V. M. Volosov.* New York etc.: Consultants Bureau, 1987. doi: 10.1007/978-1-4615-7551-1.

[3] T. S. Angell, "Existence theorems for optimal control problems involving functional differential equations." *J. Optim. Theory Appl.*, vol. 7, pp. 149–169, 1971, doi: 10.1007/BF00932473.

[4] P. K. Asea and P. J. Zak, "Time-to-build and cycles." *J. Econ. Dyn. Control*, vol. 23, no. 8, pp. 1155–1174, 1999, doi: 10.3386/t0211.

[5] A. Ashyralyev, D. Agirseven, and B. Ceylan, "Bounded solutions of delay nonlinear evolutionary equations." *J. Comput. Appl. Math.*, vol. 318, pp. 69–78, 2017, doi: 10.1016/j.cam.2016.11.046.

[6] M. Bambi, "Endogenous growth and time-to-build: the AK case." *J. Econ. Dyn. Control*, vol. 32, no. 4, pp. 1015–1040, 2008, doi: 10.1.1.582.8187.

[7] G. Carlier and R. Tahraoui, "On some optimal control problems governed by a state equation with memory." *ESAIM, Control Optim. Calc. Var.*, vol. 14, no. 4, pp. 725–743, 2008, doi: 10.1051/cocv:2008005.

[8] S. Dashkovskiy, M. Kosmykov, A. Mironchenko, and L. Naujok, "Stability of interconnected impulsive systems with and without time delays, using Lyapunov methods." *Nonlinear Anal., Hybrid Syst.*, vol. 6, no. 3, pp. 899–915, 2012, doi: 10.1016/j.nahs.2012.02.001.

[9] S. Federico, B. Goldys, and F. Gozzi, "HJB equations for the optimal control of differential equations with delays and state constraints. I: Regularity of viscosity solutions." *SIAM J. Control Optim.*, vol. 48, no. 8, pp. 4910–4937, 2010, doi: 10.1137/09076742X.

[10] C. Guo-Ping, H. Jin-Zhi, and X. Y. Simon, "An optimal control method for linear systems with time delay." *J. Computers and Structures*, vol. 81, no. 15, pp. 1539 – 1546, 2003, doi: 10.1016/S0045-7949(03)00146-9.

[11] J. K. Hale, *Theory of functional differential equations. 2nd ed.* Springer, New York, NY, 1977, vol. 3, doi: 10.1007/978-1-4615-9968-5.

[12] T. Kato, *Perturbation theory for linear operators. Reprint of the corr. print. of the 2nd ed. 1980.*, reprint of the corr. print. of the 2nd ed. 1980 ed. Berlin: Springer-Verlag, 1995. doi: 10.1007/978-3-642-66282-9.

[13] O. Kichmarenko and O. Stanzhytskyi, "Sufficient conditions for the existence of optimal controls for some classes of functional-differential equations." *Nonlinear Dyn. Syst. Theory*, vol. 18, no. 2, pp. 196–211, 2018.

[14] V.B.Kolmanovskii and L.E.Shaikhet, *Control of systems with after-effect.*, ser. Translations of Mathematical Monographs (Book 157). New York: American Mathematical Society, Providence, 1996.

[15] K. Yosida, “Functional analysis. 6th ed.” Berlin-Heidelberg-New York, p. 501, 1980, doi: 10.1007/978-3-642-61859-8.

Authors’ addresses

O. Kichmarenko
Odesa National I.I.Mechnikov University, Optimal Control and Economical Cybernatics Department, 2, Dvoryanskaya Str., Odesa, 65082, Ukraine
E-mail address: olga.kichmarenko@gmail.com

O. Stanzhytskyi
Taras Shevchenko National University of Kyiv, Department of General Mathematics, 64/13, Volodymyrska Str., Kyiv, 01601, Ukraine
E-mail address: ostanzh@gmail.com