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FRACTIONAL DIFFERENTIAL EQUATIONS OF VARIABLE ORDER: EXISTENCE RESULTS, NUMERICAL METHOD AND ASYMPTOTIC STABILITY CONDITIONS

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1. Introduction

The concept of variable and distributed order fractional derivative firstly appeared in [9] since many physical processes exhibited memory effects that may vary with time or space variables. Some new variable order fractional derivatives and applications were suggested, for example, Hamilton's principle [2], variable–order mechanics [3], constitutive relation for vis-coelasiticity [12], fractional diffusion equations [8, 13, 14]. Although the variable-order fractional derivative provides more freedom degrees and new ways to understand the complicated dynamics, the main difficulty is to consider the qualitative theories. Hence, it is a challenging work to define a variable-order function not only can be efficient in explanation of physical phenomena but also for convenience of mathematical analysis.

In this paper, we propose a kind of short memory fractional differential equations and try to address this problem which is our main purpose. We investigate the following fractional differential equation and give existence results

$$\begin{cases} {}_{t_{l_k}}^{C}D_t^{\alpha_{k+1}}x &= f(x,t), t\in\left[t_{l_k},t_{l_{k+1}}\right] \\ x\left(t_{l_0}\right) &= \eta, \qquad 0<\alpha_{k+1}\le 1, k=0,1,\ldots,m-1, l_0=0, t_{l_m}=T, 1\le m, \end{cases} \tag{1.1}$$

where t_{l_k} is the initial point, ${}_{t_{l_k}}^{C}D_t^{\alpha_{k+1}}x$ is the Caputo derivative of the function $x(t)$, $f:\mathbb{R}\times[t_0,T]\to\mathbb{R}$ and the fractional order α_{k+1} is a piecewise constant defined over each $\left[t_{l_k},t_{l_{k+1}}\right]$.

The paper in the organized in following sections. Section 2 compares the classical fractional differential equations with Eq. (1.1). Then it gives existence results. Section 3 applies predictor-corrector method to obtain numerical solutions. Section 4 derives the exact solution of linear equations. Section 5 investigates the linear fractional variable–order system's asymptotic stability. Finally, conclusion is made in Section 6 and some possible applications are discussed.

2. EXISTENCE RESULTS

We need to point out Eq. (1.1) is totally different from classical fractional differential equations with initial conditions

$$\begin{cases} {}_{t_0}^{C}D_t^{\alpha}x(t) & = f(x,t) \\ x(t_0) & = \eta, \qquad 0<\alpha\leq 1. \end{cases} \tag{2.1}$$

Eq. (1.1) has "moving" initial points. We call it as a short memory fractional differential equation since the solution $x(t)$ only depends on the information from $x(t_{l_k})$ for $t\in[t_{l_k},t_{l_{k+1}}]$. There is no need to start from t_0 in fractional modelling and this provides more freedom degrees in real-world applications. Besides, this feature is much easier for mathematical analysis of variable-order problems. In the rest of the paper, we give existence results and numerical solutions of Eq. (1.1).

Now, let's revisit some results in the fractional calculus and introduce the following definitions in [11].

Definition 1. For $\alpha>0$, the Riemann-Liouville integral of α order for function y on $[t_0,+\infty)$ is defined as

$$ {}_{t_0}I_t^{\alpha}y=\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}y(s)\mathrm{d}s,\ t>t_0. \tag{2.2}$$

Definition 2. For $0<\alpha<1$ and $y(t)\in AC^1[t_0,+\infty)$, the Caputo derivative of α order is defined by

$$ {}_{t_0}^{C}D_t^{\alpha}y:=\frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t}(t-s)^{-\alpha}y'(s)\mathrm{d}s,\ t>t_0. \tag{2.3}$$

For $\alpha=1$, then ${}_{t_0}^{C}D_t^{\alpha}y(t)=y'(t)$.

Assume that $B(b,a)=\{(x,t):|x-x^*|\leq b,|t-t^*|\leq a\}$. Let the function $f:B(b,a)\to\mathbb{R}$ be bounded by M^*, and f is Lipschitz continuous with respect to x with the constant L^*.

Lemma 1 ([1,7])**.** *$x(t)$ is a solution of the fractional differential equation*

$$\begin{cases} {}_{t^*}^{C}D_t^{\alpha}x & = f(x,t),\ (x,t)\in B(b,a), \qquad 0<\alpha\leq 1, \\ x(t^*) & = x^*. \end{cases} \tag{2.4}$$

if and only if $x(t)$ is a solution of the following equivalent integral equation

$$x(t)=x(t^*)+{}_{t^*}I_t^{\alpha}f(x,t). \tag{2.5}$$

Lemma 2 ([1,7])**.** *The system (2.4) has a unique solution over the interval $[t^*,\ t^*+h^*]$ if f satisfies the Lipschitz condition*

$$|f(x,t)-f(y,t)|\leq L^*|x-y|,(x,t),(y,t)\in B(b,a). \tag{2.6}$$

where

$$h^{*\alpha}=\min\left\{a^{\alpha},\frac{\Gamma(1+\alpha)}{L^*},\frac{\Gamma(1+\alpha)b}{M^*}\right\}. \tag{2.7}$$

and $M^* = \max_{(x,t)\in B(b,a)}(|f(x,t)|)$.

Considering the fractional variable order system (1.1), let $t^* = t_{l_k}$, $k = 0, \ldots, m-1$, $l_0 = 0$ and the initial condition becomes (t_{l_k}, x_{l_k}). For example $t^* = t_0$, we can determine h_0 and get the interval $[t_0, t_{l_1}]$ where $t_{l_1} = t_{l_0} + h_0$. With the new initial condition (t_{l_1}, x_{l_1}) and by use of the existence condition (2.7), we can determine h_1, $[t_{l_1}, t_{l_1} + h_1]$ and (t_{l_2}, x_{l_2}). More generally, we can obtain each h_k and $[t_{l_k}, t_{l_k} + h_k]$ successively in this way. Hence, we now arrive at existence results of Eq. (1.1).

Theorem 1. *$f(x,t)$ is globally Lipschitz continuous with respect to x*

$$|f(x_2^*,t) - f(x_1^*,t)| \le L^* |x_2^* - x_1^*|, (x_1^*,t), (x_2^*,t) \in B(b,a), k = 0,1,\ldots,m-1. \quad (2.8)$$

Eq. (1.1) has a unique solution on $\left[t_0, t_0 + \sum_{k=0}^{m-1} h_k\right]$, where h_k is defined

$$h_k^{\alpha_{k+1}} = \min\left\{a^{\alpha_{k+1}}, \frac{\Gamma(1+\alpha_{k+1})}{L^*}, \frac{\Gamma(1+\alpha_{k+1})b}{M^*}\right\}. \quad (2.9)$$

Theorem 2. *Eq. (1.1) has a unique solution for $t \in [t_0, t_0 + ml]$ where*

$$l = \min\{h_0, \ldots, h_{m-1}\}. \quad (2.10)$$

3. Numerical method

Although we can use Picard's method to obtain series solutions, the accuracy is not high enough to get the update initial conditions (t_{kl}, x_{kl}). Hence, in this section, we consider the numerical solutions. Let us first illustrate general steps for exact solutions of the linear equations. Then we consider the predictor-corrector method for the nonlinear case.

The predictor-corrector method developed in [6] is the most popular numerical method for chaotic analysis of fractional differential equations. Recently, several improved versions and other applications are considered [4,5]. Eq. (2.1) is equal to

$$x(t) = x(t_0) + {}_{t_0}I_t^{\alpha} f(x,t). \quad (3.1)$$

Diethelm proposed the rectangle and trapezoid formulae for the fractional integral [6] where the coefficients were derived as

$$b_{j,n+1} = \frac{1}{\alpha}\left((n+1-j)^{\alpha} - (n-j)^{\alpha}\right) \quad (3.2)$$

and

$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^{\alpha} & \text{if} \quad j = 0; \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1} & \text{if} \quad 1 \le j \le n; \\ 1 & \text{if} \quad j = n+1. \end{cases} \quad (3.3)$$

For the variable–order fractional differential equation,

$$\begin{cases} {}_{t_{kl}}^{\;C}D_t^{\alpha_{k+1}} x(t) &= f(x,t), t \in \left[t_{kl}, t_{(k+1)l}\right], \\ x(t_0) &= \eta, \qquad 0 < \alpha_{k+1} \le 1, k = 0,1,\ldots,m-1, \end{cases} \quad (3.4)$$

it has the same numerical formulae on the first interval $[t_0, t_l]$. From $t \in \left[t_{kl}, t_{(k+1)l}\right]$, $1 \le k \le m-1$, $m = 2, 3, \ldots$, and $\Delta t = \frac{l}{s}$ where s is a positive integer, we obtain the numerical formula

$$\begin{cases} x^p_{ks+i+1} & = x_{ks} + \frac{\Delta t^{\alpha_{k+1}}}{\Gamma(\alpha_{k+1})} \sum_{j=0}^{i} b_{j,i+1} f\left(x_{j+ks}, t_{j+ks}\right), i = 0, \ldots, s-1, \\ x_{ks+i+1} & = x_{ks} + \frac{\Delta t^{\alpha_{k+1}}}{\Gamma(\alpha_{k+1}+2)} \sum_{j=0}^{i} a_{j,i+1} f\left(x_{j+ks}, t_{j+ks}\right) + \frac{\Delta t^{\alpha_{k+1}}}{\Gamma(\alpha_{k+1}+2)} f\left(x^p_{ks+i+1}, t_{ks+i+1}\right). \end{cases} \tag{3.5}$$

Here x_n is the numerical solution, $x_n := x(t_n)$ and Δt is the step–length of the numerical formulae. The error estimation is $O\left(\Delta t^p\right)$ and $p = 1 + \alpha_{k+1}$.

Example 1. Consider the following fractional differential equation

$$\begin{cases} {}_{t_{kl}}^{C}D_t^{\alpha_{k+1}} x & = \sin(x), t \in \left[t_{kl}, t_{(k+1)l}\right], \\ x(t_0) & = 0.1, \qquad 0 < \alpha_{k+1} \le 1, k = 0, 1, \ldots, m-1. \end{cases} \tag{3.6}$$

We adopt the following parameters: $m = 3$, $L = 1$, $\alpha_1 = 0.7$, $\alpha_2 = 0.8$ and $\alpha_3 = 0.9$. According to Theorem 2, we can use solutions' interval as $[0,\ 3l]$ and $l = 0.8$.

By use of the numerical method, the numerical solutions are given in Figs. 1 and 2. With different time domains, the fractional order is varied in Fig. 1. And the constant order case is compared in Fig. 2 where we set the order to 0.8. From the solution's behavior, we can see that although the fractional order is the same on different domains, the solution is not differentiable at the ends t_{kl} due to the short memory effects.

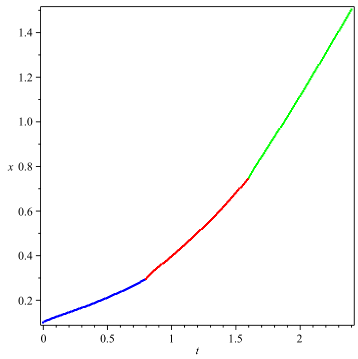

FIGURE 1. Numerical solutions of variable order system (3.6) (the blue: $\alpha_1 = 0.7$ and $t \in [0, 0.8]$; the red: $\alpha_2 = 0.8$ and $t \in [0.8, 1.6]$ the green: $\alpha_3 = 0.9$ and $t \in [1.6, 2.4]$.).

4. EXACT SOLUTIONS OF LINEAR EQUATIONS

In this subsection, we discuss two linear equations.

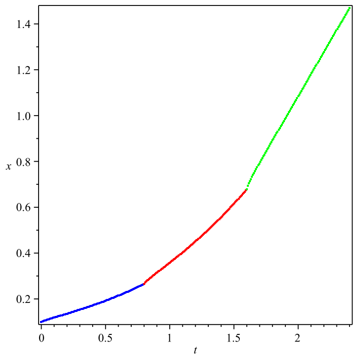

FIGURE 2. Numerical solutions of constant order system (3.6) on different time domains (the blue: $\alpha_1 = 0.8$ and $t \in [0, 0.8]$; the red: $\alpha_2 = 0.8$ and $t \in [0.8, 1.6]$; the green: $\alpha_3 = 0.8$ and $t \in [1.6, 2.4]$.).

Theorem 3. *The fractional differential equation*

$$\begin{cases} {}_{t_{kl}}^{C}D_t^{\alpha_{k+1}}x(t) & = \lambda x(t), t \in \left[t_{kl}, t_{(k+1)l}\right], \\ x(t_0) & = \eta, \qquad 0 < \alpha_{k+1} \le 1, k = 0, 1, \ldots \end{cases} \tag{4.1}$$

has a unique solution as

$$x(t) = \eta \left[\prod_{i=1}^{k} E_{\alpha_i}\left(\lambda, \left(t_{il} - t_{(i-1)l}\right)^{\alpha_i}\right)\right] E_{\alpha_{k+1}}\left(\lambda, (t - t_{kl})^{\alpha_{k+1}}\right), t \in \left[t_{kl}, t_{(k+1)l}\right]. \tag{4.2}$$

Proof. For $t \in [t_0, t_l]$, we derive that

$$x(t) = x(t_0) + \lambda_{t_0}I_t^{\alpha_1}x(t),$$
$$x(t) = \eta E_{\alpha_1}\left(\lambda, (t - t_0)^{\alpha_1}\right)$$

and

$$x(t_l) = \eta E_{\alpha_1}\left(\lambda, (t_l - t_0)^{\alpha_1}\right)$$

where $E_\alpha(\lambda, t)$ is the Mittag-Leffler function defined by

$$E_\alpha(\lambda, t) = \sum_{k=0}^{+\infty} \frac{\lambda^k t^{k\alpha}}{\Gamma(1 + k\alpha)}.$$

For $t \in [t_l, t_{2l}]$, we have

$$x(t) = x(t_l) + \lambda_{t_l}I_t^{\alpha_2}x(t),$$
$$x(t) = \eta E_{\alpha_1}\left(\lambda, (t_l - t_0)^{\alpha_1}\right) E_{\alpha_2}\left(\lambda, (t - t_l)^{\alpha_2}\right).$$

Finally, we get

$$x(t) = x(t_{kl}) + \lambda\, {}_{t_{kl}}I_t^{\alpha_{k+1}}x(t),$$

$$x(t) = \eta \left[\prod_{i=1}^{k} E_{\alpha_i} \left(\lambda, \left(t_{il} - t_{(i-1)l} \right)^{\alpha_i} \right) \right] E_{\alpha_{k+1}} \left(\lambda, (t - t_{kl})^{\alpha_{k+1}} \right), t \in \left[t_{kl}, t_{(k+1)l} \right].$$

which completes the proof. □

We can use the predictor corrector method to derive the numerical solutions in Figs. 3 and 4 where $\lambda = 0.8$ and $\lambda = -0.8$, respectively. Other parameters are set to $\eta = 1$, $l = 3$, $m = 3$, $\alpha_1 = 0.7$, $\alpha_2 = 0.8$ and $\alpha_3 = 0.9$.

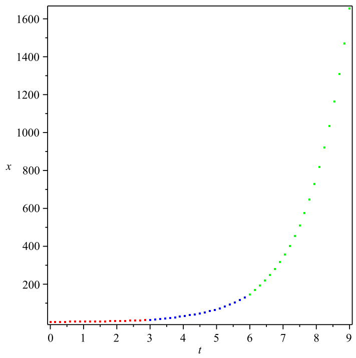

FIGURE 3. Mittag-Leffler function of variable order (5.1) (the red: $\alpha_1 = 0.7$ and $t \in [0,3]$; the blue: $\alpha_2 = 0.8$ and $t \in [3,6]$; the green: $\alpha_3 = 0.9$ and $t \in [6,9]$.).

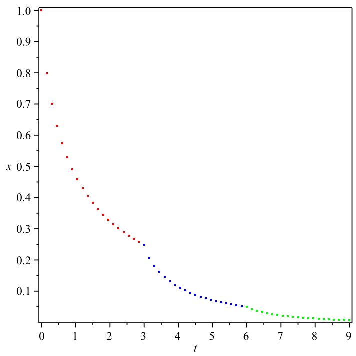

FIGURE 4. Mittag-Leffler function of variable order (5.1) on different time domains (the blue: $\alpha_1 = 0.7$ and $t \in [0,3]$; the red $\alpha_2 = 0.8$ and $t \in [3,6]$; the green: $\alpha_3 = 0.9$ and $t \in [6,9]$.).

5. Asymptotic stability

We can define a Mittag–Leffler function of variable order as

$$\varepsilon_{\alpha_{k+1}}(\lambda,t) := \left[\prod_{i=1}^{k} E_{\alpha_i}\left(\lambda, \left(t_{il}-t_{(i-1)l}\right)^{\alpha_i}\right)\right] E_{\alpha_{k+1}}\left(\lambda, (t-t_{kl})^{\alpha_{k+1}}\right), t \in \left[t_{kl}, t_{(k+1)l}\right] \tag{5.1}$$

where $0 < \alpha_{k+1} \leq 1$, for $t \in \left[t_{kl}, t_{(k+1)l}\right]$ and $k = 0, 1, \cdots, m-1$.

If m is a positive integer number, for $\lambda < 0$, $t \in [t_{ml}, \infty)$ and $t \to +\infty$, we can obtain

$$x(t) = \eta \varepsilon_{\alpha_{k+1}}(\lambda,t) \to 0. \tag{5.2}$$

Much more generally, according to Matigon's stability conditions [10], we know the following stability result of the standard fractional linear systems.

Lemma 3. *[10] Suppose λ is an eigenvalue of the coefficient matrix A. The fractional linear autonomous system*

$$\begin{cases} {}_{t_0}^{C}D_t^{\alpha} x &= Ax, \qquad 0 < \alpha \leq 1 \\ x(t_0) &= \eta, \end{cases}$$

where $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$ is asymptotically stable if and on if $|arg(\lambda)| > \frac{\alpha\pi}{2}$ is satisfied for all eigenvalues of matrix A.

We can extend Theorem 3 and the exact solution can be presented in form of a matrix Mittag–Leffler function of variable order $\varepsilon_{\alpha_{k+1}}(A,t)$. We now give the stability theorem.

Theorem 4. *The fractional linear system*

$$\begin{cases} {}_{t_{kl}}^{C}D_t^{\alpha_{k+1}} x(t) &= Ax(t), \qquad t \in \left[t_{kl}, t_{(k+1)l}\right], \\ x(t_0) &= \eta, \qquad 0 < \alpha_{k+1} \leq 1, k = 0, 1, \ldots \end{cases}$$

is asymptotically stable if there exists a positive integer N such that $|arg(\lambda)| > \frac{\alpha_{k+1}\pi}{2}$ for $k > N$.

6. Conclusions

Fractional derivative has non-locality or memory effects. This feature has made it be a powerful tool in various applied sciences and the fractional differential equation has become one of the popular directions. The concept of variable-order fractional derivative was proposed about fifteen years ago and it was efficient to reveal complicated fractional dynamics. However, less theories contributed except some numerical methods for numerical solutions. This paper provides a new concept of short memory which is very convenient to define a variable-order function. We then give existence conditions of such equations with variable orders. The predictor-corrector method is used to show the new concept both suitable for theoretical analysis and numerical calculation. We only give the existence results in this paper and we believe the following topics are important in future:

1) Numerical methods of high accuracy. We only illustrate the application of the predictor-corrector method. There are many numerical methods developed and available. They also can be used in this study. Besides, we notice that the computational time is saved a lot, particularly when the m becomes very large. The fractional differential equation itself is a short memory one and it saves much storage space in numerical calculations.

2) New applications of the short memory. Many applications now all considered the memory or non-locality of the whole interval. However, we may only need some of the information or data. That means we need a fractional approach between non–locality and locality. This study gives some a possible way for fractional modeling. We now can consider some other applications such as short memory Euler-Lagrange equations, fractional diffusion equations and signal processing.

7. Acknowledgements

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