



LOGARITHMIC MEANS OF WALSH-FOURIER SERIES

USHANGI GOGINAVA

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Abstract. In this paper we discuss some convergence and divergence properties of subsequences of logarithmic means of Walsh-Fourier series. We give necessary and sufficient conditions for the convergence regarding logarithmic variation of numbers.

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1. WALSH FUNCTIONS

We shall denote the set of all non-negative integers by \mathbb{N} , the set of all integers by \mathbb{Z} and the set of dyadic rational numbers in the unit interval $\mathbb{I} := [0, 1)$ by \mathbb{Q} . In particular, each element of \mathbb{Q} has the form $\frac{p}{2^n}$ for some $p, n \in \mathbb{N}$, $0 \leq p \leq 2^n$.

Denote the dyadic expansion of $n \in \mathbb{N}$ and $x \in \mathbb{I}$ by

$$n = \sum_{j=0}^{\infty} \varepsilon_j(n) 2^j, \varepsilon_j(n) = 0, 1$$

and

$$x = \sum_{j=0}^{\infty} \frac{x_j}{2^{j+1}}, x_j = 0, 1.$$

In the case of $x \in \mathbb{Q}$ chose the expansion which terminates in zeros. Define the dyadic addition $+$ as

$$x + y = \sum_{k=0}^{\infty} |x_k - y_k| 2^{-(k+1)}.$$

The sets $I_n(x) := \{y \in \mathbb{I} : y_0 = x_0, \dots, y_{n-1} = x_{n-1}\}$ for $x \in \mathbb{I}$, $I_n := I_n(0)$ for $0 < n \in \mathbb{N}$ and $I_0(x) := \mathbb{I}$ are the dyadic intervals of \mathbb{I} . For $0 < n \in \mathbb{N}$ denote by $|n| := \max \{j \in \mathbb{N} : n_j \neq 0\}$, that is, $2^{|n|} \leq n < 2^{|n|+1}$.

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The Rademacher system is defined by

$$\rho_n(x) := (-1)^{x_n} \quad (x \in \mathbb{I}, n \in \mathbb{N}).$$

The Walsh-Paley system is defined as the sequence of the Walsh-Paley functions:

$$w_n(x) := \prod_{k=0}^{\infty} (\rho_k(x))^{n_k} = (-1)^{\sum_{k=0}^{|n|} n_k x_k} \quad (x \in \mathbb{I}, n \in \mathbb{N}).$$

The Walsh-Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x) \quad (n \in \mathbb{N}), D_0 = 0.$$

Recall that (see [20])

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in I_n(0) \\ 0, & \text{if } x \in \mathbb{I} \setminus I_n(0) \end{cases}. \quad (1.1)$$

As usual, denote by $L_1(\mathbb{I})$ the set of measurable functions defined on \mathbb{I} , for which

$$\|f\|_1 := \int_{\mathbb{I}} |f(t)| dt < \infty.$$

Let $f \in L_1(\mathbb{I})$. The partial sums of the Walsh-Fourier series are defined as follows:

$$S_M(x, f) := \sum_{i=0}^{M-1} \widehat{f}(i) w_i(x),$$

where the number

$$\widehat{f}(i) = \int_{\mathbb{I}} f(t) w_i(t) dt$$

is said to be the i th Walsh-Fourier coefficient of the function f . Set $E_n(x, f) = S_{2^n}(x, f)$. The maximal function is defined by

$$E^*(x, f) = \sup_{n \in \mathbb{N}} E_n(x, |f|).$$

The notation $a \lesssim b$ in the proofs stands for $a < c \cdot b$, where c is an absolute constant.

2. LOGARITHMIC MEANS

In the literature, there is the notion of Riesz's logarithmic means of a Fourier series. The n -th Riesz's logarithmic means of the Fourier series of an integrable function f is defined by

$$R_n(x, f) := \frac{1}{l_n} \sum_{k=1}^n \frac{S_k(x, f)}{k},$$

where $l_n := \sum_{k=1}^n (1/k)$.

Riesz's logarithmic means with respect to the trigonometric system was studied by a lot of authors. This means with respect to the Walsh and Vilenkin systems was discussed by Simon [21], Blahota, Gát [1], Gát [4], Gát, Goginava [8].

Let $\{q_k : k \geq 0\}$ be a sequence of nonnegative numbers. The n -th Nörlund means for the Fourier series of f is defined by

$$\frac{1}{Q_n} \sum_{k=0}^{n-1} q_{n-k} S_k(f),$$

where

$$Q_n := \sum_{k=1}^n q_k.$$

If $q_k = k$, then we get the Nörlund logarithmic means

$$t_n(x, f) := \frac{1}{l_n} \sum_{k=0}^{n-1} \frac{S_k(x, f)}{n-k}.$$

In this paper we call it logarithmic mean although, it is a kind of "reverse" Riesz's logarithmic mean.

It is easy to see that

$$t_n(x, f) = \int_{\mathbb{T}} f(t) F_n(x+t) dt,$$

where by $F_n(t)$ we denote n th logarithmic kernel, i. e.

$$F_n(t) := \frac{1}{l_n} \sum_{k=0}^{n-1} \frac{D_k(t)}{n-k}.$$

and Fejér kernel is defined by

$$K_n(t) := \frac{1}{n} \sum_{k=1}^n D_k(t).$$

3. L_1 - ESTIMATION FOR LOGARITHMIC KERNEL

For $n = \sum_{j=0}^{\infty} \varepsilon_j(n) 2^j$, $\varepsilon_j(n) = 0, 1$ we define

$$n(k) := \sum_{j=0}^k \varepsilon_j(n) 2^j.$$

It is easy to see that $n(|n|) = n$. In this paper for L_1 -norm of logarithmic means we prove the following two sides estimation.

Theorem 1. *Let $n \in \mathbb{N}$. Then*

$$\left\| \frac{1}{l_n} \sum_{j=1}^n \frac{D_{n-j}}{j} \right\|_1 \sim \frac{1}{|n|} \sum_{k=1}^{|n|} |\varepsilon_k(n) - \varepsilon_{k+1}(n)| l_{n(k-1)}.$$

Proof. We can write

$$\begin{aligned} & \sum_{j=1}^n \frac{D_{n-j}(t)}{j} \\ &= \sum_{j=1}^{n(|n|-1)} \frac{D_{n-j}(t)}{j} + \sum_{j=n(|n|-1)+1}^n \frac{D_{n-j}(t)}{j} \\ &= \sum_{j=1}^{n(|n|-1)} \frac{D_{\varepsilon_{|n|}(n)2^{|n|}+n(|n|-1)-j}(t)}{j} + \sum_{j=n(|n|-1)+1}^n \frac{D_{n-j}(t)}{j}. \end{aligned} \quad (3.1)$$

Since

$$D_{\varepsilon_{|n|}(n)2^{|n|}+n(|n|-1)-j}(t) = \varepsilon_{|n|}(n) D_{2^{|n|}}(t) + (w_{2^{|n|}}(t))^{\varepsilon_{|n|}(n)} D_{n(|n|-1)-j}(t),$$

from (3.1) we have

$$\begin{aligned} & \sum_{j=1}^n \frac{D_{n-j}(t)}{j} \\ &= \varepsilon_{|n|}(n) D_{2^{|n|}}(t) l_{n(|n|-1)} \\ &+ (w_{2^{|n|}}(t))^{\varepsilon_{|n|}(n)} \sum_{j=1}^{n(|n|-1)} \frac{D_{n(|n|-1)-j}(t)}{j} \\ &+ \varepsilon_{|n|}(n) \sum_{j=1}^{2^{|n|}-1} \frac{D_{2^{|n|}-j}(t)}{j + n(|n|-1)}. \end{aligned}$$

Iterating this equality we obtain

$$\begin{aligned}
 & \sum_{j=1}^n \frac{D_{n-j}(t)}{j} \\
 &= \left(\sum_{j=2}^{|n|} \varepsilon_j(n) D_{2^j}(t) l_{n(j-1)} \right) \prod_{k=j+1}^{|n|} (\rho_k(t))^{\varepsilon_k(n)} \\
 &+ \left(\sum_{j=2}^{|n|} \varepsilon_j(n) \sum_{k=1}^{2^j-1} \frac{D_{2^j-k}(t)}{k+n(j-1)} \right) \prod_{s=j+1}^{|n|} (\rho_s(t))^{\varepsilon_s(n)} \\
 &+ \left(\sum_{j=1}^{n(1)} \frac{D_{n(1)-j}(t)}{j} \right) \prod_{k=2}^{|n|} (\rho_k(t))^{\varepsilon_k(n)}.
 \end{aligned} \tag{3.2}$$

Since

$$\varepsilon_j(n) D_{2^j}(t) \prod_{k=0}^j (\rho_k(t))^{\varepsilon_k(n)} = \varepsilon_j(n) D_{2^j}(t) \rho_j(t)$$

we have

$$\begin{aligned}
 & \left(\sum_{j=2}^{|n|} \varepsilon_j(n) D_{2^j}(t) l_{n(j-1)} \right) \prod_{k=j+1}^{|n|} (\rho_k(t))^{\varepsilon_k(n)} \\
 &= w_n(t) \left(\sum_{j=2}^{|n|} \varepsilon_j(n) D_{2^j}(t) l_{n(j-1)} \right) \prod_{k=0}^j (\rho_k(t))^{\varepsilon_k(n)} \\
 &= w_n(t) \left(\sum_{j=2}^{|n|} \varepsilon_j(n) D_{2^j}(t) \rho_j(t) l_{n(j-1)} \right).
 \end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3) we conclude that

$$\begin{aligned}
& \sum_{j=1}^n \frac{D_{n-j}(t)}{j} \\
&= w_n(t) \left(\sum_{j=2}^{|n|} \varepsilon_j(n) D_{2^j}(t) \rho_j(t) l_{n(j-1)} \right) \\
&+ \left(\sum_{j=2}^{|n|} \varepsilon_j(n) \sum_{k=1}^{2^j-1} \frac{D_{2^j-k}(t)}{k+n(j-1)} \right) \prod_{s=j+1}^{|n|} (\rho_s(t))^{\varepsilon_s(n)} \\
&+ \left(\sum_{j=1}^{n(1)-1} \frac{D_{n(1)-j}(t)}{j} \right) \prod_{k=2}^{|n|} (\rho_k(t))^{\varepsilon_k(n)} \\
&=: H_n^{(1)}(t) + H_n^{(2)}(t) + H_n^{(3)}(t).
\end{aligned} \tag{3.4}$$

Since (see [9])

$$D_{2^j-k}(t) = D_{2^j}(t) - w_{2^j-1}(t) D_k(t), k = 1, 2, \dots, 2^j - 1$$

for $H_n^{(2)}(t)$ we can write

$$\begin{aligned}
& H_n^{(2)}(t) \\
&= \left(\sum_{j=2}^{|n|} \varepsilon_j(n) D_{2^j}(t) \sum_{k=1}^{2^j-1} \frac{1}{k+n(j-1)} \right) \prod_{s=j+1}^{|n|} (\rho_s(t))^{\varepsilon_s(n)} \\
&- \left(\sum_{j=2}^{|n|} \varepsilon_j(n) w_{2^j-1}(t) \sum_{k=1}^{2^j-1} \frac{D_k(t)}{k+n(j-1)} \right) \prod_{s=j+1}^{|n|} (\rho_s(t))^{\varepsilon_s(n)} \\
&=: H_n^{(21)}(t) + H_n^{(22)}(t).
\end{aligned} \tag{3.5}$$

Since

$$\begin{aligned}
H_n^{(21)}(t) &= \left(\sum_{j=2}^{|n|} \varepsilon_j(n) D_{2^j}(t) (l_{n(j)} - l_{n(j-1)}) \right) \\
&\times \prod_{s=j+1}^{|n|} (\rho_s(t))^{\varepsilon_s(n)}
\end{aligned}$$

from (1.1) we get

$$\left\| H_n^{(21)} \right\|_1 \leq \sum_{j=2}^{|n|} \varepsilon_j(n) (l_{n(j)} - l_{n(j-1)}) \leq c |n|. \tag{3.6}$$

Usin Abel's transformation we obtain

$$\begin{aligned} & \sum_{k=1}^{2^j-1} \frac{D_k(t)}{k+n(j-1)} \\ &= \sum_{k=1}^{2^j-2} \left(\frac{1}{k+n(j-1)} - \frac{1}{k+1+n(j-1)} \right) k K_k(t) \\ & \quad + \frac{2^j-1}{2^j-1+n(j-1)} K_{2^j-1}(t). \end{aligned}$$

Since (see [20]) $\sup_n \|K_n\|_1 < \infty$ for $H_n^{(22)}(t)$ we can write

$$\begin{aligned} & \left\| H_n^{(22)} \right\|_1 \\ & \lesssim \sum_{j=2}^{|n|} \varepsilon_j(n) \sum_{k=1}^{2^j-1} \left(\frac{1}{k+n(j-1)} - \frac{1}{k+1+n(j-1)} \right) k \\ & \quad + \sum_{j=2}^{|n|} \varepsilon_j(n) \frac{2^j-1}{2^j-1+n(j-1)} \\ & \lesssim \sum_{j=2}^{|n|} \varepsilon_j(n) \sum_{k=1}^{2^j-1} \frac{1}{(k+n(j-1))} \\ & \quad + \sum_{j=2}^{|n|} \varepsilon_j(n) \sum_{k=1}^{2^j-1} \frac{n(j-1)}{(k+n(j-1))^2} + \sum_{j=2}^{|n|} \varepsilon_j(n) \frac{2^j-1}{2^j-1+n(j-1)} \\ & \lesssim \sum_{j=2}^{|n|} \varepsilon_j(n) (l_{n(j)} - l_{n(j-1)}) + |n| \lesssim |n|. \end{aligned} \tag{3.7}$$

Combining (3.5)-(3.7) we conclude that

$$\left\| H_n^{(2)} \right\|_1 \leq c |n|. \tag{3.8}$$

It is easy to see that

$$\sup_n \left\| H_n^{(3)} \right\|_1 \leq c. \tag{3.9}$$

First, we find upper estimation for $\left\| H_n^{(1)} \right\|_1$. We can write

$$H_n^{(1)}(t) = w_n(t) \left(\sum_{j=2}^{|n|} \varepsilon_j(n) l_{n(j-1)} (D_{2^{j+1}}(t) - D_{2^j}(t)) \right)$$

$$\begin{aligned}
&= w_n(t) \left(\sum_{j=2}^{|n|-1} (\varepsilon_j(n) l_{n(j-1)} - \varepsilon_{j+1}(n) l_{n(j)}) D_{2^{j+1}}(t) \right) \\
&\quad + w_n(t) l_{n(|n|-1)} D_{2^{|n|+1}} - w_n(t) \varepsilon_2(n) l_{n(1)} D_{2^2}(t).
\end{aligned}$$

Hence, from (1.1) we obtain

$$\begin{aligned}
\|H_n^{(1)}\|_1 &\leq \sum_{j=2}^{|n|-1} |\varepsilon_j(n) l_{n(j-1)} - \varepsilon_{j+1}(n) l_{n(j)}| + c|n| \\
&\leq \sum_{j=2}^{|n|-1} |\varepsilon_j(n) - \varepsilon_{j+1}(n)| l_{n(j-1)} \\
&\quad + \sum_{j=2}^{|n|-1} \varepsilon_{j+1}(n) (l_{n(j)} - l_{n(j-1)}) + c|n| \\
&\leq \sum_{j=2}^{|n|} |\varepsilon_j(n) - \varepsilon_{j+1}(n)| l_{n(j-1)} + c|n|.
\end{aligned}$$

Now, we find lower estimation for $\|H_n^{(1)}\|_1$. Let a_i and $b_i, i = 1, \dots, s$ be strictly increasing sequences, i. e.

$$0 \leq a_1 \leq b_1 < a_2 \leq b_2 < \dots < a_s \leq b_s < a_{s+1} = \infty$$

for which

$$\varepsilon_j(n) = \begin{cases} 1, & a_i \leq j \leq b_i \\ 0, & b_i < j < a_{i+1} \end{cases} \quad (3.10)$$

Then it is evident that

$$b_j + 1 < a_{j+1}. \quad (3.11)$$

Set

$$A_k := \left(\frac{1}{2^{a_k+1}}, \frac{1}{2^{a_k}} \right), B_k := \left(\frac{1}{2^{b_k+2}}, \frac{1}{2^{b_k+1}} \right), k = 1, \dots, s.$$

Let $x \in A_k$. Then we can write

$$\begin{aligned}
|H_n^{(1)}(t)| &= \left| \sum_{j=2}^{|n|} \varepsilon_j(n) (D_{2^{j+1}}(t) - D_{2^j}(t)) l_{n(j-1)} \right| \\
&= \left| \sum_{i=1}^{k-1} \sum_{j=a_i}^{b_i} (D_{2^{j+1}}(t) - D_{2^j}(t)) l_{n(j-1)} \right|
\end{aligned}$$

$$\begin{aligned}
& + \sum_{j=a_k}^{b_k} (D_{2^{j+1}}(t) - D_{2^j}(t)) l_{n(j-1)} \Bigg| \\
& = \left| \sum_{i=1}^{k-1} \sum_{j=a_i}^{b_i} 2^j l_{n(j-1)} - 2^{a_k} l_{n(a_k-1)} \right|.
\end{aligned}$$

From (3.11) we can write

$$\begin{aligned}
\sum_{i=1}^{k-1} \sum_{j=a_i}^{b_i} 2^j l_{n(j-1)} & \leq l_{n(b_{k-1}-1)} \sum_{i=1}^{k-1} (2^{b_i+1} - 2^{a_i}) \\
& \leq l_{n(b_{k-1}-1)} \sum_{i=1}^{k-1} (2^{b_i+1} - 2^{b_{i-1}+1}) \\
& \leq 2^{b_{k-1}+1} l_{n(b_{k-1}-1)} \\
& \leq 2^{b_{k-1}+1} l_{n(a_k-1)}.
\end{aligned}$$

Consequently,

$$\left| H_n^{(1)}(t) \right| \geq 2^{a_k} l_{n(a_k-1)} - 2^{b_{k-1}+1} l_{n(a_k-1)} \geq 2^{a_k-1} l_{n(a_k-1)}.$$

Integrating on A_k we get

$$\int_{A_k} \left| H_n^{(1)}(t) \right| dt \geq \int_{A_k} 2^{a_k-1} l_{n(a_k-1)} dt = \frac{l_{n(a_k-1)}}{4}. \quad (3.12)$$

On the interval B_k we have

$$\begin{aligned}
\left| H_n^{(1)}(t) \right| & = \left| \sum_{i=1}^k \sum_{j=a_i}^{b_i} (D_{2^{j+1}}(t) - D_{2^j}(t)) l_{n(j-1)} \right| \\
& = \sum_{i=1}^k \sum_{j=a_i}^{b_i} 2^j l_{n(j-1)} \geq l_{n(b_k-1)} 2^{b_k}.
\end{aligned}$$

Hence,

$$\int_{B_k} \left| H_n^{(1)}(t) \right| dt \geq \int_{B_k} l_{n(b_k-1)} 2^{b_k} dt = \frac{l_{n(b_k-1)}}{4}. \quad (3.13)$$

Since $A_i, B_i, i = 1, \dots, s$ are pairwise disjoint from (3.12) and (3.13) we have

$$\begin{aligned} \int_{\mathbb{I}} |H_n^{(1)}(t)| dt &\geq \sum_{k=1}^s \left(\int_{A_k} |H_n^{(1)}(t)| dt + \int_{B_k} |H_n^{(1)}(t)| dt \right) \\ &\geq \frac{1}{4} \sum_{k=1}^s (l_{n(a_k-1)} + l_{n(b_k-1)}). \end{aligned} \quad (3.14)$$

Since (see (3.10))

$$n(a_k - 1) = n(a_k - 2)$$

and

$$|\varepsilon_j(n) - \varepsilon_{j+1}(n)| = \begin{cases} 1, & j = a_k - 1 \text{ or } b_k, k = 1, 2, \dots, s \\ 0 & \text{otherwise} \end{cases}$$

we conclude that

$$\int_{\mathbb{I}} |H_n^{(1)}(t)| dt \geq \frac{1}{4} \sum_{k=1}^{|n|} |\varepsilon_k(n) - \varepsilon_{k+1}(n)| l_{n(k-1)}.$$

Combining (3.4)-(3.10) and (3.14) we complete the proof of Theorem 1. \square

4. ALMOST EVERYWHERE CONVERGENCE OF LOGARITHMIC MEANS

For a non-negative integer n let us denote

$$V_S(n) := \sum_{i=0}^{\infty} |\varepsilon_i(n) - \varepsilon_{i+1}(n)| + \varepsilon_0(n)$$

and

$$V_L(n) := \frac{1}{|n|} \sum_{k=1}^{|n|} |\varepsilon_k(n) - \varepsilon_{k+1}(n)| l_{n(k-1)}.$$

It is known that if $n_j < n_{j+1}$,

$$\sup_j V_S(n_j) < \infty, \quad (4.1)$$

then a. e. $S_{n_j}(f) \rightarrow f$. On the other hand, Konyagin [14] proved that the condition (4.1) is not necessary for a. e. convergence of subsequence of partial sums. Moreover, he gave negative answer to the question of Balashov and proved the validity of the following theorem.

Theorem K (Konyagin). *Suppose $\{n_A\}$ is an increasing sequence of natural numbers, $k_A := \lfloor \log_2 n_A \rfloor + 1$, and 2^{k_A} is a divider of n_{A+1} for all A . Then $S_{n_A}(f) \rightarrow f$ a. e. for any function $f \in L_1(\mathbb{I})$.*

For instance, a sequence $\{n_A\}, n_A := 2^{A^2} \sum_{i=0}^A 4^i$, such that $\sup_{n_A} V(n_A) = \infty$, satisfies the hypotheses of the theorem.

Almost everywhere convergence of $\{t_{2^A}(f) : A \geq 1\}$ with respect to Walsh-Paley system was studied by author [11]. In particular, the following is proved

Theorem G. *Let $f \in L_1(\mathbb{I})$. Then $t_{2^A}(x, f) \rightarrow f(x)$ as $A \rightarrow \infty$ a. e. $x \in \mathbb{I}$.*

Nagy in [18] established a similar result for the Walsh-Kaczmarz system. However, a divergence on the set with positive measure for the whole sequence $\{t_n(f) : n \geq 1\}$ was proved by Gát and Goginava [7]. Memić [16] improved Theorem G and proved that the following is true.

Theorem M. *Let $f \in L_1(\mathbb{I})$ and*

$$\sup_A \frac{1}{|m_A|} \sum_{k=1}^{|m_A|} \varepsilon_k(m_A) l_{m_A(k-1)} < \infty. \quad (4.2)$$

Then $t_{m_A}(x, f) \rightarrow f(x)$ as $A \rightarrow \infty$ for a. e. $x \in \mathbb{I}$.

In this paper we are going to replace condition (4.2) with more weaker condition

$$\sup_A \frac{1}{|m_A|} \sum_{k=1}^{|m_A|} |\varepsilon_k(m_A) - \varepsilon_{k+1}(m_A)| l_{m_A(k-1)} < \infty. \quad (4.3)$$

It is easy to see that condition (4.2) imply condition (4.3), on other hand, for the sequence $\{2^A - 1 : A \in \mathbb{N}\}$ condition (4.2) does not holds and condition (4.3) holds. So, we prove that the following is valid.

Theorem 2. *Let $f \in L_1(\mathbb{I})$ and condition (4.3) is holds. Then $t_{m_A}(x, f) \rightarrow f(x)$ as $A \rightarrow \infty$ for a. e. $x \in \mathbb{I}$.*

Proof. From (3.4) we have

$$f * (l_{m_A} F_{m_A})(x) = f * H_{m_A}^{(1)}(x) + f * H_{m_A}^{(2)}(x) + f * H_{m_A}^{(3)}(x). \quad (4.4)$$

It is easy to see that

$$\left\| \sup_A \left| f * H_{m_A}^{(3)} \right| \right\|_1 \lesssim \|f\|_1. \quad (4.5)$$

From (3.5) we can write

$$f * H_{m_A}^{(2)}(x) = f * H_{m_A}^{(21)}(x) + f * H_{m_A}^{(22)}(x) + f * H_{m_A}^{(22)}(x). \quad (4.6)$$

Using (1.1) we have

$$\left| f * H_{m_A}^{(21)}(x) \right| \leq \sum_{j=1}^{|m_A|-1} (|f| * D_{2^j})(x) \leq |m_A| E^*(x, f).$$

Hence

$$\sup_A \frac{|f * H_{m_A}^{(21)}(x)|}{|m_A|} \leq E^*(x, f). \quad (4.7)$$

We can write

$$\begin{aligned} |f * H_{m_A}^{(22)}(x)| &\leq \sum_{j=1}^{|m_A|-1} \varepsilon_j(m_A) (l_{m_A(j)} - l_{m_A(j-1)}) (|f| * D_{2^j}(x)) \\ &\lesssim |m_A| E^*(x, f), \\ \sup_A \frac{|f * H_{m_A}^{(22)}(x)|}{|m_A|} &\lesssim E^*(x, f). \end{aligned} \quad (4.8)$$

It is proved in [7]

$$\sup_{\lambda} \lambda \left| \left\{ \sup_A \frac{|f * H_{m_A}^{(23)}|}{|m_A|} > \lambda \right\} \right| \lesssim \|f\|_1. \quad (4.9)$$

Since $\sup_{\lambda} \lambda |\{E^*(f) > \lambda\}| \lesssim \|f\|_1$ from (4.7)- (4.9) we get

$$\sup_{\lambda} \lambda \left| \left\{ \sup_A \frac{|f * H_{m_A}^{(2)}|}{|m_A|} > \lambda \right\} \right| \lesssim \|f\|_1. \quad (4.10)$$

Now, we estimate $|f * H_{m_A}^{(1)}(x)|$. We have

$$\begin{aligned} |f * H_{m_A}^{(1)}(x)| &\lesssim \sum_{j=1}^{|m_A|-1} (\varepsilon_j(m_A) l_{m_A(j-1)} - \varepsilon_{j+1}(m_A) l_{m_A(j)}) (|f| * D_{2^j}(x)) \\ &\quad + l_{(|m_A|-1)} (|f| * D_{2^{|m_A|+1}}(x)) \\ &\lesssim E^*(x, f) \left(l_{(|m_A|-1)} + \sum_{j=1}^{|m_A|-1} (\varepsilon_j(m_A) l_{m_A(j-1)} - \varepsilon_{j+1}(m_A) l_{m_A(j)}) \right) \\ &\lesssim E^*(x, f) \left(l_{(|m_A|-1)} + \sum_{j=1}^{|m_A|-1} |\varepsilon_j(m_A) - \varepsilon_{j+1}(m_A)| l_{m_A(j-1)} \right. \\ &\quad \left. + \sum_{j=1}^{|m_A|-1} \varepsilon_{j+1}(m_A) (l_{m_A(j)} - l_{m_A(j-1)}) \right) \end{aligned}$$

$$\lesssim E^*(x, f) \left(l_{(|m_A|-1)} + \sum_{j=1}^{|m_A|-1} |\varepsilon_j(m_A) - \varepsilon_{j+1}(m_A)| l_{m_A(j-1)} \right).$$

From the condition of the Theorem we can write

$$\sup_A \frac{|f * H_{m_A}^{(1)}(x)|}{l_{m_A}} \lesssim E^*(x, f) V_L(m_A)$$

and

$$\sup_{\lambda} \lambda \left| \left\{ \sup_A \frac{|f * H_{m_A}^{(1)}|}{|m_A|} > \lambda \right\} \right| \lesssim \|f\|_1. \quad (4.11)$$

Combining (3.5), (4.5), (4.10) and (4.11) we conclude that

$$\sup_{\lambda} \lambda \left| \left\{ \sup_A \frac{|f * F_{m_A}|}{|m_A|} > \lambda \right\} \right| \lesssim \|f\|_1.$$

By the well-known density argument we complete the proof of Theorem 2. \square

5. UNIFORM AND L -CONVERGENCE OF LOGARITHMIC MEANS

Denote by $C_w(\mathbb{I})$ the space of uniformly continuous functions on \mathbb{I} , with the supremum norm

$$\|f\|_{C_w} := \sup_{x \in \mathbb{I}} |f(x)| \quad (f \in C_w(\mathbb{I})).$$

Let $X = X(\mathbb{I})$ be either the space $L_1(\mathbb{I})$, or the space of uniformly continuous functions, that is, $C_w(\mathbb{I})$. The corresponding norm is denoted by $\|\cdot\|_X$.

For Walsh-Fourier series Fine [2] has obtained a sufficient condition for the uniform convergence which is in a complete analogy with the Dini-Lipshitz condition (see also [20]). Similar results are true for the space of integrable functions $L_1(\mathbb{I})$ [19]. Gulicev [13] has estimated the rate of uniform convergence of a Walsh-Fourier series using Lebesgue constant and modulus of continuity. Uniform convergence of Walsh-Fourier series of the functions of classes of generalized bounded variation was investigated by author [10]. This problem has been considered for Vilenkin group by Fridli [3] and Gát [5]. Lukomskii [15] considered uniform and L_1 -convergence of subsequence of partial sums of Walsh-Fourier series. In particular, he proved that the condition $\sup_A V_S(m_A) < \infty$ is necessary and sufficient condition for uniform and

L_1 -convergence of subsequence of partial sums $S_{m_A}(f)$ of Walsh-Fourier series. In Móricz and Siddiqi [17] investigated approximation properties of Nörlund means

$\frac{1}{Q_n} \sum_{k=0}^{n-1} q_{n-k} S_k f$. The case when we have $q_k := 1/k$ differs from the types discussed by Móricz and Siddiqi in [17]. His method is not applicable for logarithmic

means. In [6] it is proved that Theorem of Móricz does not hold for L_1 , C_w and $q_k := 1/k$.

In [12] it is investigated X -norm convergence of subsequence of logarithmic means of Walsh-Fourier series. In particular, the following are proved.

Theorem GT. a) Let $f \in X(\mathbb{I})$ and

$$\sup_A \frac{\log(m_A - 2^{|m_A|} + 1)}{\log m_A} \left\| \frac{1}{l_{m_A - 2^{|m_A|}}} \sum_{j=1}^{m_A - 2^{|m_A|} - 1} \frac{D_{n-j}}{j} \right\|_1 < \infty. \quad (5.1)$$

Then subsequence of Nörlund logarithmic means $t_{m_A}(f)$ converges in the norm of space $X(\mathbb{I})$.

b) If the condition (5.1) does not holds then we can find a function from the space $X(\mathbb{I})$ for which the convergence of logarithmic means $L_{m_A}(f)$ in the norm of space $X(\mathbb{I})$ does not holds.

Since

$$\begin{aligned} & \sup_A \frac{\log(m_A - 2^{|m_A|} + 1)}{\log m_A} \left\| \frac{1}{l_{m_A - 2^{|m_A|}}} \sum_{j=1}^{m_A - 2^{|m_A|} - 1} \frac{D_{n-j}}{j} \right\|_1 \\ & \sim \sup_A \frac{\log(m_A - 2^{|m_A|} + 1)}{\log m_A} \frac{1}{|m_A - 2^{|m_A|}|} \sum_{k=1}^{|m_A - 2^{|m_A|}|} |\varepsilon_k(n) - \varepsilon_{k+1}(n)| l_{n(k-1)} \\ & \sim \sup_A \frac{1}{|m_A|} \sum_{k=1}^{|m_A|} |\varepsilon_k(n) - \varepsilon_{k+1}(n)| l_{n(k-1)}, \end{aligned}$$

from Theorem GT and Theorem 1 we can prove necessary and sufficient condition for norm convergence of subsequence of Nörlund logarithmic means

Theorem 3. Let $f \in X(\mathbb{I})$. Then the condition $\sup_A V_L(m_A) < \infty$ is necessary and sufficient for convergence subsequence of Nörlund logarithmic means of Walsh-Fourier series in norm of space $X(\mathbb{I})$.

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Author’s address

Ushangi Goginava

U. Goginava, I. Vekua Institute of Applied Mathematics and Faculty of Exact and Natural Sciences of I. Javakhishvili Tbilisi State University, Tbilisi 0186, 2 University str., Georgia

E-mail address: zazagoginava@gmail.com