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INFINITELY MANY WEAK SOLUTIONS FOR A FOURTH-ORDER EQUATION WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract. Existence results of infinitely many solutions for a fourth-order differential equation are established. This equation depends on two real parameters. The approach is based on an infinitely many critical points theorem.

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1. INTRODUCTION

The aim of the present paper is to investigate the existence of infinitely many solutions for the following fourth-order problem

$$\begin{cases} u^{(iv)}(x) = \lambda\, \alpha(x) f(x,u(x)) + h(x,u(x)), \quad x \in [0,1], \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \quad u'''(1) = \mu\, g(u(1)) \end{cases} \tag{1.1}$$

where λ and μ are positive parameters, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is L^1-Carathéodory function, $g : \mathbb{R} \to \mathbb{R}$ is a continuous function ,$\alpha \in L^\infty([0,1])$, $\alpha(x) \geq 0$, for a.e. $x \in \mathbb{R}$,$\alpha \not\equiv 0$ and $h : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function , there exists $0 < L < 1$, such that $h(x,t) \leq L|t|$ for each $x \in [0,1]$ and $t \in \mathbb{R}$.
The problem (1.1) is related to the deflections of elastic beams based on nonlinear elasticity . In relation with the problem (1.1), there is an interesting physical description.
Suppose an elastic beam of length $d = 1$, which is clamped at its left side $x = 0$, and resting on a kind of elastic bearing at its right side $x = 1$ is given by μg . Along its length, a load $\lambda\alpha f + h$, is added to cause deformations. If $u = u(x)$ denotes the configuration of the deformed beam, then since $u'''(1)$ represents the shear force at $x = 1$, the condition $u'''(1) = \mu g(u(1))$ means that the vertical force is equal to $\mu g(u(1))$, which denotes a relation, possibly nonlinear, between the vertical force and the displacement $u(1)$. In addition , since $u''(1) = 0$ indicates that there is no

bending moment at $x = 1$, the beam is resting on the bearing μg.
Different models and their applications for problems such as (1.1) can be derived from [4]. We refer the reader to the references [14] and [10] for a physical justification of this model. There is increasing interest in studying fourth-order boundary value problems. Because the change of the static form beam or the support of rigid body can be described by a fourth-order equation. Also a model to study travelling waves in suspension bridges can be furnished by nonlinear fourth-order equations (for instance , see [5]).

When the boundary conditions are nonzero or nonlinear, fourth-order equations can model beams resting on elastic bearings located in their extremities(see for instance [1,3,6–8,11–13] and the references therein).

For example, using a variational methods, the existence of three solutions for special cases of problem (1.1) has been established in [13] and [12]. In [7] the author obtained the existence of at least two positive solutions for the problem

$$\begin{cases} u^{(iv)}(x) = f(x,u(x)), & x \in [0,1], \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \quad u'''(1) = g(u(1)), \end{cases} \tag{1.2}$$

based on variational methods and maximum principle. It should be noted that the function f is assumed to be continuous. By assuming appropriate conditions on f and g, the author guarantees positive solutions to problem (1.2). Also existence and multiplicity results for this kind of problems were considered in [1,3,6] . In all these works the critical point theory is applied.

Moreover in [8] authors, considered numerical solutions for problem (1.2) with nonlinear boundary conditions.

In particular, using a variational methods the existence of non-zero solutions for problem

$$\begin{cases} u^{(iv)}(x) = f(x,u(x)), & x \in [0,1], \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \quad u'''(1) = \mu g(u(1)), \end{cases} \tag{1.3}$$

has been established in [1].
In the present paper, using one kind of infinitely many critical points theorem obtained in [2] , we establish the existence of infinitely many weak solutions for the problem (1.1). The paper is organized as follows.
In section 2 we establish all the preliminary results that we need, and in section 3 we present our main results.

2. Preliminaries

The following theorem is a smooth version of Theorem 2.1 of [2] which is a more precise version of Ricceri's Variational Principle [9] and it is the main tool of the next section .

Theorem 1. *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous, and coercive and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, let us put*

$$\varphi(r) := \inf_{u \in \Phi^{-1}(]-\infty, r[)} \frac{\sup_{v \in \Phi^{-1}(]-\infty, r[)} \Psi(v) - \Psi(u)}{r - \Phi(u)}$$

and

$$\gamma := \liminf_{r \to +\infty} \varphi(r), \quad \delta := \liminf_{r \to (\inf_X \Phi)^+} \varphi(r).$$

Then, one has

(a) *for every $r > \inf_X \Phi$ and every $\lambda \in]0, \frac{1}{\varphi(r)}[$, the restriction of the functional $I_\lambda = \Phi - \lambda \Psi$ to $\Phi^{-1}(]-\infty, r[)$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.*

(b) *If $\gamma < +\infty$ then, for each $\lambda \in]0, \frac{1}{\gamma}[$, the following alternative holds:*
either
(b_1) *I_λ possesses a global minimum,*
or
(b_2) *there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that*

$$\lim_{n \to +\infty} \Phi(u_n) = +\infty.$$

(c) *If $\delta < +\infty$ then, for each $\lambda \in]0, \frac{1}{\delta}[$, the following alternative holds:*
either
(c_1) *there is a global minimum of Φ which is a local minimum of I_λ,*
or
(c_2) *there is a sequence of pairwise distinct critical points (local minima) of I_λ which weakly converges to a global minimum of Φ.*

Now, we recall some basic facts and introduce the needed notations.

Definition 1. A function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is said to be a Carathéodory function if
(C_1) the function $x \to f(x,t)$ is measurable for every $t \in \mathbb{R}$;
(C_2) the function $t \to f(x,t)$ is continuous for a.e. $x \in [0,1]$.
And $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is said to be a L^1-Carathéodory function if, in addition to conditions (C_1) and (C_2) ,the following condition is also satisfied :
(C_3) for every $\rho > 0$ there is function $l_\rho \in L^1([0,1])$ such that $\sup_{|t| \le \rho} |f(x,t)| \le l_\rho(x)$ for almost every $x \in [0,1]$.

Denote

$$X := \{u \in H^2([0,1]) | u(0) = u'(0) = 0\},$$

where $H^2([0,1])$ is the Sobolev space of all functions $u : [0,1] \to \mathbb{R}$ such that u and its distributional derivative u' are absolutely continuous and u'' belongs to $L^2([0,1])$. X is a Hilbert space with the usual norm

$$\|u\|_X = \left(\int_0^1 (|u''(x)|^2 + |u'(x)|^2 + |u(x)|^2) dx \right)^{1/2}$$

which is equivalent to the norm

$$\|u\| = \left(\int_0^1 |u''(x)|^2 dx \right)^{1/2}.$$

Also the embedding $X \hookrightarrow C^1([0,1])$ is compact and we have

$$\|u\|_{C^1([0,1])} = \max\{\|u\|_\infty, \|u'\|_\infty\} \leq \|u\| \tag{2.1}$$

for each $u \in X$ (see [12]) .

Put

$$F(x,t) = \int_0^t f(x,\xi) d\xi \quad \text{for all } (x,t) \in [0,1] \times \mathbb{R},$$

$$G(t) = \int_0^t g(\xi) d\xi \quad \text{for all } t \in \mathbb{R},$$

and

$$H(x,t) = \int_0^t h(x,\xi) d\xi \quad \text{for all } (x,t) \in [0,1] \times \mathbb{R}.$$

Let Φ, $\Psi : X \to \mathbb{R}$ be defined by

$$\Phi(u) = \frac{1}{2} \int_0^1 |u''(x)|^2 dx - \int_0^1 H(x,u(x)) dx = \frac{1}{2} \|u\|^2 - \int_0^1 H(x,u(x)) dx \tag{2.2}$$

and

$$\Psi(u) = \int_0^1 \alpha(x) F(x,u(x)) dx - \frac{\mu}{\lambda} G(u(1)) \tag{2.3}$$

for every $u \in X$.
Now according to (2.1) we observe that

$$\frac{(1-L)}{2} \|u\|^2 \leq \Phi(u) \leq \frac{(1+L)}{2} \|u\|^2 \tag{2.4}$$

for every $u \in X$. Similar to [[1]-page 3], Ψ is a differentiable functional whose differential at the point $u \in X$ is

$$\Psi'(u)(v) = \int_0^1 \alpha(x) f(x,u(x)) v(x) dx - \frac{\mu}{\lambda} g(u(1)) v(1),$$

and, Φ is continuously Gâteaux differentiable functional whose differential at the point $u \in X$ is

$$\Phi'(u)(v) = \int_0^1 u''(x)v''(x)dx - \int_0^1 h(x,u(x))v(x)dx$$

for every $v \in X$.

Definition 2. A function $u \in X$ is a weak solution to the problem (1.1) if

$$\int_0^1 u''(x)v''(x)dx - \lambda \int_0^1 \alpha(x)f(x,u(x))v(x)dx + \mu g(u(1))v(1) - \int_0^1 h(x,u(x))v(x)dx = 0 \tag{2.5}$$

for every $v \in X$.

3. Main results

Let

$$G_\eta = \min_{|t|\le\eta} G(t) = \inf_{|t|\le\eta} G(t) \quad \text{for all } \eta > 0,$$

$$\tau := \frac{1-L}{8\pi^4(\frac{2}{3})^3(1+L)}, \tag{3.1}$$

$$C := \liminf_{\xi\to+\infty} \frac{\int_0^1 \sup_{|t|\le\xi} \alpha(x)F(x,t)dx}{\xi^2} \tag{3.2}$$

and

$$D := \limsup_{\xi\to+\infty} \frac{\int_{\frac{3}{4}}^1 \alpha(x)F(x,\xi)dx}{\xi^2}. \tag{3.3}$$

Now we formulate our main result as follows.

Theorem 2. *Assume that*

(A_1) $\int_0^{\frac{3}{4}} \alpha(x)F(x,t)dx \ge 0$ *for all* $t \ge 0$.

(A_2) $C < \tau D$, *where* τ , C *and* D *are given by* (3.1) , (3.2) *and* (3.3) .

Then, setting

$$\lambda_1 := \frac{4\pi^4(\frac{2}{3})^3(1+L)}{D}, \quad \lambda_2 := \frac{1-L}{2C},$$

for each $\lambda \in (\lambda_1, \lambda_2)$,*for every arbitrary continuous function* $g : \mathbb{R} \to \mathbb{R}$ *whose potential*

$$G(t) = \int_0^t g(\xi)d\xi \quad \textit{for all } t \in \mathbb{R},$$

is a non-positive function satisfying the condition

$$G^* := -\frac{2}{1-L} \lim_{\xi \to +\infty} \frac{G_\xi}{\xi^2} < +\infty \tag{3.4}$$

and for every $\mu \in (0, \mu_{g,\lambda})$ *where* $\mu_{g,\lambda} = \frac{1}{G^*}\left(1 - \frac{\lambda}{\lambda_2}\right)$, *the problem* (1.1) *has unbounded sequence of weak solution in* X.

Proof. From (A_2) we see that $\lambda_1 < \lambda_2$. Our aim is to apply Theorem 1 , to problem (1.1). Fix $\bar{\lambda} \in (\lambda_1, \lambda_2)$ and let g be a function satisfies the condition (3.4). Since $\bar{\lambda} < \lambda_2$, we have $\mu_{g,\bar{\lambda}} = \frac{1}{G^*}\left(1 - \frac{\bar{\lambda}}{\lambda_2}\right) > 0$. Now fix $\bar{\mu} \in (0, \mu_{g,\bar{\lambda}})$ and put $z_1 = \lambda_1$ and $z_2 = \frac{\lambda_2}{1 + \frac{\bar{\mu}}{\bar{\lambda}} \lambda_2 G^*}$. If $G^* = 0$, clearly, $z_1 = \lambda_1$, $z_2 = \lambda_2$ and $\bar{\lambda} \in (z_1, z_2)$. If $G^* \neq 0$, since $\bar{\mu} < \mu_{g,\bar{\lambda}}$, we obtain $\frac{\bar{\lambda}}{\lambda_2} + \bar{\mu} G^* < 1$, and so $\frac{\lambda_2}{1 + \frac{\bar{\mu}}{\bar{\lambda}} \lambda_2 G^*} > \bar{\lambda}$, namely, $\bar{\lambda} < z_2$. Hence, since $\bar{\lambda} > \lambda_1 = z_1$, one has $\bar{\lambda} \in (z_1, z_2)$.

Take X , Φ and Ψ as in the previous section.

For each $u \in X$, we let the functional $I_{\bar{\lambda}} : X \to \mathbb{R}$ be defined

$$I_{\bar{\lambda}}(u) = \Phi(u) - \bar{\lambda}\Psi(u).$$

Φ is a sequentially weakly lower semicontinuous on X. Indeed, consider an arbitrary $u \in X$ and $\{u_n\}_{n=1}^{\infty} \subset X$ such that $u_n \rightharpoonup u$ in X. Due to the compact embedding X into $C([0,1])$, we have that $u_n \to u$ in $C([0,1])$. This implies

$$\int_0^1 H(x, u_n(x))dx \to \int_0^1 H(x, u(x))dx, \tag{3.5}$$

and the weakly sequentially lower semicontinuous property of the $\|.\|$ implies

$$\liminf_{n \to +\infty} \|u_n\|^2 \geq \| u \|^2 . \tag{3.6}$$

From (3.5)-(3.6) we have

$$\liminf_{n \to +\infty} \Phi(u_n) \geq \Phi(u).$$

Since

$$\Phi(u) \geq (\frac{1-L}{2})\|u\|^2,$$

taking the condition $0 < L < 1$ into account we observe Φ is coercive. Moreover, Φ is strongly continuous. On the other hand, the compact embedding X into $C[0,1]$

implies that the functional Ψ is continuously differentiable and with compact derivative. Hence Ψ is sequentially weakly upper semicontinuous (see [15, Corollary 41.9]).Therefore we observe that the regularity assumptions of Theorem 1 on Φ and Ψ are satisfied . Note that the weak solution of problem (1.1) are exactly the critical points of $I_{\bar{\lambda}}$(in particular , see [12, Lemma 2.1]).
Let $\{\xi_n\}$ be a sequence of positive numbers such that $\xi_n \to +\infty$ as $n \to +\infty$ and

$$\begin{aligned}\lim_{n\to+\infty}&\left(\frac{\int_0^1 \sup_{|t|\le\xi_n} \alpha(x)F(x,t)dx}{\xi_n^2} - \frac{\bar{\mu}}{\bar{\lambda}}\frac{G_{\xi_n}}{\xi_n^2}\right)\\ &= \liminf_{\xi\to+\infty}\left(\frac{\int_0^1 \sup_{|t|\le\xi} \alpha(x)F(x,t)dx}{\xi^2} - \frac{\bar{\mu}}{\bar{\lambda}}\frac{G_{\xi}}{\xi^2}\right). \qquad (3.7)\end{aligned}$$

According to (A_2) and (3.4), it is obvious that the limit on the right-hand side in (3) is finite and therefore we have

$$\lim_{n\to+\infty}\left(\frac{\int_0^1 \sup_{|t|\le\xi_n} \alpha(x)F(x,t)dx}{\xi_n^2} - \frac{\bar{\mu}}{\bar{\lambda}}\frac{G_{\xi_n}}{\xi_n^2}\right) < +\infty. \qquad (3.8)$$

Now for all $n \in \mathbb{N}$, put $r_n = \left(\frac{1-L}{2}\right)\xi_n^2$. Since $\left(\frac{1-L}{2}\right)\|u\|^2 \le \Phi(u)$, for each $u \in X$ and bearing (2.1) in mind, we see that

$$\begin{aligned}\Phi^{-1}(]-\infty, r_n[) =&\{u \in X;\ \Phi(u) < r_n\}\\ \subseteq&\left\{u \in X;\ (\frac{1-L}{2})\|u\|^2 < r_n\right\}\\ \subseteq&\{u \in X;\ |u(x)| \le \xi_n \text{ for each } x \in [0,1]\}.\end{aligned}$$

Note that $\Phi(0) = \Psi(0) = 0$. Hence, for all $n \in \mathbb{N}$, one has

$$\begin{aligned}\varphi(r_n) =& \inf_{u\in\Phi^{-1}(]-\infty,r_n[)} \frac{\sup_{v\in\Phi^{-1}(]-\infty,r_n[)}\Psi(v) - \Psi(u)}{r_n - \Phi(u)}\\ \le&\frac{\sup_{v\in\Phi^{-1}(]-\infty,r_n[)}\Psi(v)}{r_n}\\ =&\frac{\sup_{v\in\Phi^{-1}(]-\infty,r_n[)}\left(\int_0^1 \alpha(x)F(x,v(x))dx - \frac{\bar{\mu}}{\bar{\lambda}}G(v(1))\right)}{r_n} \le\end{aligned}$$

$$\frac{\left(\sup_{v\in\Phi^{-1}(]-\infty,r_n[)}\int_0^1 \alpha(x)F(x,v(x))dx\right)-\frac{\bar{\mu}}{\bar{\lambda}}\left(\inf_{v\in\Phi^{-1}(]-\infty,r_n[)} G(v(1))\right)}{r_n}\leq$$

$$\frac{\left(\int_0^1 \sup_{|t|\leq\xi_n}\alpha(x)F(x,t)dx\right)-\frac{\bar{\mu}}{\bar{\lambda}}\left(\inf_{|t|\leq\xi_n} G(t)\right)}{r_n}=$$

$$\frac{2}{1-L}\left(\frac{\int_0^1 \sup_{|t|\leq\xi_n}\alpha(x)F(x,t)dx}{\xi_n^2}-\frac{\bar{\mu}}{\bar{\lambda}}\frac{G_{\xi_n}}{\xi_n^2}\right).$$

Therefore, from (3.8) we obtain

$$\gamma\leq\liminf_{n\to+\infty}\varphi(r_n)\leq\lim_{n\to+\infty}\frac{2}{1-L}\left(\frac{\int_0^1 \sup_{|t|\leq\xi_n}\alpha(x)F(x,t)dx}{\xi_n^2}-\frac{\bar{\mu}}{\bar{\lambda}}\frac{G_{\xi_n}}{\xi_n^2}\right)<+\infty. \tag{3.9}$$

Now we show that $\bar{\lambda}<\dfrac{1}{\gamma}$. Since $\bar{\mu}\in(0,\mu_{g,\bar{\lambda}})$ then $\bar{\mu}<\frac{1}{G^*}(1-\frac{\bar{\lambda}}{\lambda_2})=\frac{1}{G^*}(1-\frac{2C\bar{\lambda}}{1-L})$. Hence from (3) and (3.9) we have

$$\gamma\leq\liminf_{n\to+\infty}\varphi(r_n)\leq\lim_{n\to+\infty}\frac{2}{1-L}\left(\frac{\int_0^1 \sup_{|t|\leq\xi_n}\alpha(x)F(x,t)dx}{\xi_n^2}-\frac{\bar{\mu}}{\bar{\lambda}}\frac{G_{\xi_n}}{\xi_n^2}\right)=$$

$$\liminf_{\xi\to+\infty}\frac{2\left[\int_0^1 \sup_{|t|\leq\xi}\alpha(x)F(x,t)dx-\frac{\bar{\mu}}{\bar{\lambda}}G_\xi\right]}{(1-L)\xi^2}<$$

$$\frac{2C}{1-L}-\lim_{\xi\to+\infty}\frac{2(1-\frac{2C\bar{\lambda}}{1-L})G_\xi}{\bar{\lambda}(1-L)G^*\xi^2}=\frac{2C}{1-L}+\frac{1-\dfrac{2C\bar{\lambda}}{1-L}}{\bar{\lambda}}=\frac{1}{\bar{\lambda}}.$$

This implies that $\bar{\lambda}<\dfrac{1}{\gamma}$. So we proved that

$$\bar{\lambda}\in(z_1,z_2)\subseteq(\lambda_1,\lambda_2)\subseteq\left(0,\frac{1}{\gamma}\right).$$

For the fixed $\bar{\lambda}$, the inequality (3.9) concludes that the condition (b) of Theorem 1 can be applied and either $I_{\bar{\lambda}}$ has a global minimum or there exists a sequence $\{u_n\}$ of weak solutions of the problem (1.1) such that $\lim_{n\to+\infty}\|u_n\|=+\infty$.

The other step is to prove that for the fixed $\overline{\lambda}$ the functional $I_{\overline{\lambda}}$ has no global minimum. Let us show that the functional $I_{\overline{\lambda}}$ is unbounded from below. Since $\bar{\lambda} > \lambda_1$ then we have

$$\frac{1}{\bar{\lambda}} < \frac{1}{4\pi^4(\frac{2}{3})^3(1+L)} \limsup_{\xi\to+\infty} \frac{\int_{\frac{3}{4}}^{1} \alpha(x)F(x,\xi)dx}{\xi^2}$$

and so there exists a sequence $\{\eta_n\}$ of positive numbers and a constant θ such that $\eta_n \to \infty$ as $n \to \infty$ and

$$\frac{1}{\bar{\lambda}} < \theta < \frac{1}{4\pi^4(\frac{2}{3})^3(1+L)} \frac{\int_{\frac{3}{4}}^{1} \alpha(x)F(x,\eta_n)dx}{\eta_n^2} \tag{3.10}$$

for each $n \in \mathbb{N}$ large enough. For all $n \in \mathbb{N}$ define

$$w_n(x) := \begin{cases} 0 & \text{if } x \in [0, \frac{3}{8}] \\ \eta_n \cos^2(\frac{4\pi x}{3}) & \text{if } x \in]\frac{3}{8}, \frac{3}{4}] \\ \eta_n & \text{if } x \in]\frac{3}{4}, 1]. \end{cases}$$

We observe that $w_n \in X$ and $\|w_n\|^2 = 8\pi^4(\frac{2}{3})^3\eta_n^2$ and so from (2.4) we see that

$$\Phi(w_n) \le \frac{(1+L)}{2}\|w_n\|^2 = 4\pi^4(\frac{2}{3})^3(1+L)\eta_n^2. \tag{3.11}$$

On the other hand, bearing (A_1) in mind and since G is non-positive, we have

$$\Psi(w_n) \ge \int_{\frac{3}{4}}^{1} \alpha(x)F(x,\eta_n)dx. \tag{3.12}$$

It follows from (3.10)-(3.12) that

$$\begin{aligned} I_{\bar{\lambda}}(w_n) =& \Phi(w_n) - \bar{\lambda}\Psi(w_n) \\ \le& 4(1+L)\pi^4(\frac{2}{3})^3\eta_n^2 - \bar{\lambda}\int_{\frac{3}{4}}^{1} \alpha(x)F(x,\eta_n)dx \\ <& 4\pi^4(\frac{2}{3})^3(1+L)\eta_n^2 - 4\pi^4(\frac{2}{3})^3(1+L)\,\eta_n^2\,\bar{\lambda}\,\theta = \end{aligned}$$

$$4\pi^4(\frac{2}{3})^3(1+L)\,\eta_n^2(1-\bar{\lambda}\theta)$$

for every $n \in \mathbb{N}$ large enough. Since $\bar{\lambda}\theta > 1$ and $\eta_n \to +\infty$ as $n \to +\infty$, we have

$$\lim_{n\to+\infty} I_{\bar{\lambda}}(w_n) = -\infty$$

and it follows that $I_{\bar{\lambda}}$ has no global minimum. Therefore, taking the fact

$$\Phi(u) \leq \frac{1+L}{2}\|u\|^2$$

into account, by (b) from Theorem 1 , there exist a sequence $\{u_n\}$ of critical points of $I_{\bar{\lambda}}$ such that $\lim_{n\to\infty}\|u_n\| = +\infty$, and the conclusion is achieved. □

Now we present the following example to illustrate the result.

Example 1. Let $F : [0,1]\times\mathbb{R} \to \mathbb{R}$ be the function defined as

$$F(x,t) := \begin{cases} e^{x^2}t^3\Big(1-\cos(\ln(|t|))\Big) & \text{if } (x,t) \in [0,1]\times(\mathbb{R}-\{0\}) \\ 0 & \text{if } (x,t) \in [0,1]\times\{0\} \end{cases}$$

and

$$\alpha(x) := \begin{cases} \frac{1}{4} & \text{if } x \in [0,\frac{1}{4}] \\ 0 & \text{if } x \in (\frac{1}{4},1]. \end{cases}$$

Consider the problem

$$\begin{cases} u^{(iv)} = \lambda\alpha(x)\, e^{x^2}\Bigg(3u^2\Big(1-\cos(\ln(|u|))\Big)+u^2\sin(\ln(|u|))\Bigg)+\frac{1}{2}\sin u, \\ \qquad\qquad\qquad\qquad\qquad\qquad x\in(0,1), \\ u(0)=u'(0)=0, \\ u''(1)=0, \quad u'''(1)=\mu\, g(u(1)). \end{cases} \tag{3.13}$$

Let $f(x,u) = e^{x^2}\Bigg(3u^2\Big(1-\cos(\ln(|u|))\Big)+u^2\sin(\ln(|u|))\Bigg)$, $h(x,u) = \frac{1}{2}\sin u$ with $L = \frac{1}{2}$, $g(u) = -u$.

We observe that

$$C := \liminf_{\xi\to+\infty}\frac{\int_0^1 \sup_{|t|\leq\xi}\alpha(x)F(x,t)dx}{\xi^2} = 0$$

$$D := \limsup_{\xi\to+\infty}\frac{\int_{\frac{3}{4}}^1 \alpha(x)F(x,\xi)dx}{\xi^2} = +\infty$$

and

$$G^* := -\frac{2}{1-L}\lim_{\xi\to+\infty}\frac{G_\xi}{\xi^2} = 2.$$

Hence, by Theorem 2, for every $\lambda \in (0,+\infty)$ and $\mu \in (0,\frac{1}{2})$ the problem (3.13) has a sequence of generalized solutions which is unbounded in X.

Theorem 3. *Assume that assumption (A_1) in Theorem 2 holds and*

$$D > 4\pi^4(\frac{2}{3})^3(1+L) \quad \textit{and} \quad C < \frac{1-L}{2}$$

and for every arbitrary continuous function $g : \mathbb{R} \to \mathbb{R}$ whose potential $G(t) = \int_0^t g(\xi)d\xi$ for all $t \in \mathbb{R}$, is a non-positive function, satisfying the condition $\lim_{\xi\to+\infty} \frac{G_\xi}{\xi^2} = 0$. Then, the problem

$$\begin{cases} u^{iv} = f(x,u) + h(x,u), & x \in (0,1), \\ u(0) = u'(0) = 0, & \\ u''(1) = 0, \quad u'''(1) = \mu g(u(1)) & \end{cases} \tag{3.14}$$

for every $\mu \in (0,+\infty)$ has an unbounded sequence of weak solution in X.

Proof. Theorem is an immediately consequence of Theorem 2 when $\lambda = 1$. $\square$

Remark 1. In Theorem 2, if we assume that the function f is non-negative, the assumption (A_2) can be written as

$$\liminf_{\xi\to+\infty} \frac{\int_0^1 \alpha(x)F(x,\xi)dx}{\xi^2} < \tau \limsup_{\xi\to+\infty} \frac{\int_{\frac{3}{4}}^1 \alpha(x)F(x,\xi)dx}{\xi^2}$$

as well as $\mu_{g,\lambda} = \frac{1}{G^*}\Big(1 - \frac{2\lambda}{1-L} \liminf_{\xi\to+\infty} \frac{\int_0^1 \alpha(x)F(x,\xi)dx}{\xi^2}\Big)$. Moreover, in the autonomous case, putting $F(t) = \int_0^t f(\xi)d\xi$ for all $t \in \mathbb{R}$, and $\alpha(x) = 1$ for a.e. $x \in \mathbb{R}$, the assumption (A_2) assumes the form

$$\liminf_{\xi\to+\infty} \frac{F(\xi)}{\xi^2} < \frac{\tau}{4} \limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^2},$$

and in this case, we have

$$\lambda_1 = \frac{16\pi^4(\frac{2}{3})^3(1+L)}{\limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^2}} \qquad \text{and} \qquad \lambda_2 = \frac{1-L}{2\liminf_{\xi\to+\infty} \frac{F(\xi)}{\xi^2}}$$

and $\mu_{g,\lambda} = \frac{1}{G^*}\Big(1 - \frac{2\lambda}{1-L} \liminf_{\xi\to+\infty} \frac{F(\xi)}{\xi^2}\Big)$.

Remark 2. Replacing $\xi \to +\infty$ with $\xi \to 0^+$ and also put

$$C := \liminf_{\xi\to 0^+} \frac{\int_0^1 \sup_{|t|\le\xi} \alpha(x)F(x,t)dx}{\xi^2} \tag{3.15}$$

and

$$D := \limsup_{\xi \to 0^+} \frac{\int_{\frac{3}{4}}^{1} \alpha(x) F(x,\xi)dx}{\xi^2} \tag{3.16}$$

and

$$G^* := -\frac{2}{1-L} \lim_{\xi \to 0^+} \frac{G_\xi}{\xi^2} < +\infty \tag{3.17}$$

in Theorem 2, by the same way as in the proof of Theorem 2 but using conclusion (c) of Theorem 2 instead of (b), we can obtain a sequence of pairwise distinct weak solutions to the problem (1.1) which converges uniformly to zero.

We present the following example to illustrate Remark 2.

Example 2. Let $\beta > 1847$ be a real number and $F : \mathbb{R} \to \mathbb{R}$ be a function defined by putting

$$F(t) := \begin{cases} t^2\Big(1 + \beta \sin^2(\frac{1}{t})\Big) & \text{if } t \in]0, +\infty[\\ 0 & \text{if } t \in]-\infty, 0]. \end{cases}$$

We see that $F(t) = \displaystyle\int_0^t f(\xi)d\xi$, where $f(t) = 2t + 2\beta t \sin^2(\frac{1}{t}) - \beta \sin(\frac{2}{t})$.

Put $g(t) = -t^3$, $\alpha(x) = 1$ and $h(x,t) = \frac{1}{3} x t \sin t$ for every $x \in (0,1)$ and $t \in \mathbb{R}$. We can consider $L = \frac{1}{3}$ and in this case $\tau = \frac{27}{128\pi^4}$.
Let $a_n = \frac{1}{n\pi}$, $b_n = \frac{1}{\frac{2n+1}{2}\pi}$. For every $n \in \mathbb{N}$, we have

$$C := \liminf_{\xi \to 0^+} \frac{F(\xi)}{\xi^2} \leq \lim_{n \to \infty} \frac{F(a_n)}{a_n^2} = 1, \tag{3.18}$$

and

$$D := \limsup_{\xi \to 0^+} \frac{F(\xi)}{\xi^2} \geqslant \lim_{n \to \infty} \frac{F(b_n)}{b_n^2} = \beta + 1, \tag{3.19}$$

and by (3.18) and (3.19)one has

$$\liminf_{\xi \to 0^+} \frac{F(\xi)}{\xi^2} < \frac{\tau}{4} \limsup_{\xi \to 0^+} \frac{F(\xi)}{\xi^2},$$

as well as

$$\left(\frac{512\pi^4}{81(\alpha+1)}, \frac{1}{3}\right) \subset \left(\frac{16\pi^4(\frac{2}{3})^3(1+L)}{\limsup_{\xi \to 0^+} \frac{F(\xi)}{\xi^2}}, \frac{1-L}{2\liminf_{\xi \to 0^+} \frac{F(\xi)}{\xi^2}}\right).$$

Therefore, by applying Remark 1 and Remark 2, for every $\beta > 1847$, and every $\lambda \in \left(\frac{512\pi^4}{81(\alpha+1)}, \frac{1}{3}\right)$ and $\mu \in (0, +\infty)$ the problem

$$\begin{cases} u^{iv} = \lambda\Big(2u + 2\beta u \sin^2(\frac{1}{u}) - \beta \sin(\frac{2}{u})\Big) + \frac{1}{3} x\, u\, \sin u, & x \in (0,1), \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \quad u'''(1) = \mu\, g(u(1)) \end{cases} \tag{3.20}$$

has a sequence of pairwise distinct weak solutions which converges uniformly to zero.

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