



DERIVATIONS IN ŁUKASIEWICZ SEMIRINGS

IVAN CHAJDA AND HELMUT LÄNGER

Received 29 May, 2018

Abstract. An axiomatization of classical propositional logic is provided by means of Boolean algebras which are term equivalent to Boolean rings. This is important because rings form a classical part of algebra whose tools can be used for the investigations. The Łukasiewicz many-valued logic was axiomatized via so-called MV-algebras by C. C. Chang in 1950's. MV-algebras are successfully applied in the logic of quantum mechanics and hence they are considered as quantum structures. It is a natural question if also MV-algebras have their alter ego among classical structures. For this reason the so-called Łukasiewicz semirings were introduced by the first author and his collaborators in [3] – [4]. As shown, Łukasiewicz semirings are term equivalent to MV-algebras and we can use with advantage several developed tools for their study. In particular, we investigate derivations in semirings which were introduced formerly but here these semirings are enriched by an involution.

2010 *Mathematics Subject Classification:* 13N15; 16Y60; 06D35

Keywords: Łukasiewicz semiring, MV-algebra, derivation

1. INTRODUCTION

The concept of derivation in rings was introduced already in the 1960's. For semirings, it was defined in J. S. Golan's book ([9]). However, derivations were studied also in lattices (see e.g. [7] and [11]). Let us note that bounded distributive lattices are special semirings. Starting with 2010, several authors extended the study of derivations to certain algebras forming an algebraic semantic for non-classical logics including the logic of quantum mechanics. In this context let us mention the papers [1] and [10]. It was shown in [5] that every MV-algebra can be converted into a so-called Łukasiewicz semiring. The advantage of this approach is that we can apply also methods from the theory of semirings. This was done by the authors in [6].

The main difference between our approach to derivations in Łukasiewicz semirings and the approach by J. S. Golan in [9] is that in Łukasiewicz semirings we have a unary operation such that addition can be expressed by both multiplication and this

Support of the research of both authors by ÖAD, project CZ 04/2017, of the first author by IGA, project PpF 2018 012, and of the second author by the Austrian Science Fund (FWF), project I 1923-N25, is gratefully acknowledged.

unary operation. Moreover, Łukasiewicz semirings are additively idempotent and ordered by the induced semilattice ordering, see [1] – [4]. On the other hand, these semirings are not lattices and hence we cannot use results and methods from [7] or [11].

2. ŁUKASIEWICZ SEMIRINGS

We start with the following definition:

Definition 1. A *Łukasiewicz semiring* is an algebra $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that

- $(R, +, 0)$ is a commutative monoid,
- $(R, \cdot, 1)$ is a commutative monoid,
- $(x + y) \cdot z \approx x \cdot z + y \cdot z$,
- $x \cdot 0 \approx 0$,
- $x + x \approx x$ and $x + 1 \approx 1$,
- $(x' \cdot y)' \cdot y \approx (y' \cdot x)' \cdot x$,
- $x \leq y$ implies $y' \leq x'$,
- $(x')' \approx x$.

An algebra $(R, +, \cdot, 0, 1)$ of type $(2, 2, 0, 0)$ satisfying the first four conditions is called a *commutative semiring*. So $(R, +)$ can be considered a join-semilattice and we denote by \leq its corresponding partial order relation. We call it the *induced order* of \mathbf{R} . Put

$$x \wedge y := (x' + y')'$$

for all $x, y \in R$. Since $(R, +, 0, 1)$ is a bounded join-semilattice and $'$ is an antitone involution on (R, \leq) , $(R, +, \wedge, 0, 1)$ is a bounded lattice.

Example 1. The algebra $\mathbf{R} = ([0, 1], \vee, \odot, ', 0, 1)$ with

$$\begin{aligned} x \vee y &:= \max(x, y), \\ x \odot y &:= (x + y - 1) \vee 0, \\ x' &:= 1 - x \end{aligned}$$

for all $x, y \in [0, 1]$ is a Łukasiewicz semiring. Here $[0, 1]$ denotes the unit interval of the real numbers.

Example 2. For every positive integer n the algebra $\mathbf{R}_n = (\{0, 1, 2, \dots, n\}, \vee, \odot, ', 0, n)$ with

$$\begin{aligned} x \vee y &:= \max(x, y), \\ x \odot y &:= (x + y - n) \vee 0, \\ x' &:= n - x \end{aligned}$$

for all $x, y = 0, \dots, n$ is a Łukasiewicz semiring. Note that in this case the greatest element is n .

Example 3. Every Boolean algebra $\mathbf{B} = (B, \vee, \wedge, ', 0, 1)$ is a Łukasiewicz semiring.

Lemma 1. Let $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ be a Łukasiewicz semiring and $a, b, c \in R$. Then

- (i) $0' = 1$ and $1' = 0$,
- (ii) $a \cdot a' = 0$,
- (iii) $a \leq b$ implies $a \cdot c \leq b \cdot c$, and hence $a \cdot b \leq a \wedge b$, especially, $a \cdot a \leq a$,
- (iv) $a \cdot (a + b)' = 0$,
- (v) $a + b = ((a \cdot b')' \cdot b')'$,
- (vi) $a \leq b$ if and only if $a \cdot b' = 0$.

Proof.

- (i) This follows immediately from the fact that 0 is the smallest and 1 the greatest element of (R, \leq) and that $'$ is an antitone involution of (R, \leq) .
- (ii) According to (i) we have $a \cdot a' = (1 \cdot a')' \cdot a' = (a \cdot 0)' \cdot 0 = 0$.
- (iii) $a \leq b$ implies $a + b = b$ whence $a \cdot c + b \cdot c = (a + b) \cdot c = b \cdot c$, i.e. $a \cdot c \leq b \cdot c$. Since $a \cdot b \leq a \cdot 1 = a$ and $a \cdot b \leq 1 \cdot b = b$, we have $a \cdot b \leq a \wedge b$.
- (iv) According to (iii) and (ii) we have $a \cdot (a + b)' \leq (a + b) \cdot (a + b)' = 0$ and hence $a \cdot (a + b)' = 0$.
- (v) According to (iv), (i) and (ii) we have

$$a + b = ((a \cdot (a + b))' \cdot (a + b))' = (((a + b) \cdot a)' \cdot a)' = ((b \cdot a)' \cdot a)' = ((a \cdot b')' \cdot b')'.$$

- (vi) If $a \leq b$ then according to (iii) and (ii) we have $a \cdot b' \leq b \cdot b' = 0$ showing $a \cdot b' = 0$. If, conversely $a \cdot b' = 0$ then according to (v) and (i) we have $a \leq a + b = ((a \cdot b')' \cdot b')' = b$.

□

An element a of R is called *Boolean* if $a \cdot a = a$. This is equivalent to $a + a' = 1$ and to $a \wedge a' = 0$. Hence, a is Boolean if and only if a' has this property. The mentioned equivalence can be seen as follows: If $a \cdot a = a$ then $a + a' = ((a \cdot (a'))' \cdot (a'))' = ((a \cdot a)' \cdot a)' = (a' \cdot a)' = 0' = 1$. If, conversely, $a + a' = 1$ then $a \cdot a = a \cdot a + 0 = a \cdot a + a \cdot a' = a \cdot (a + a') = a \cdot 1 = a$. Let $\text{Bool}\mathbf{R}$ denote the set of all Boolean elements of R . It is easy to see that $\text{Bool}\mathbf{R}$ is a subuniverse of $(R, \cdot, ', 0, 1)$.

An *MV-algebra* is an algebra $(A, \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the following identities:

$$\begin{aligned} (x \oplus y) \oplus z &\approx x \oplus (y \oplus z), \\ x \oplus y &\approx y \oplus x, \\ x \oplus 0 &\approx x, \\ x \oplus 1 &\approx 1, \\ \neg(\neg x) &\approx x, \\ \neg(\neg x \oplus y) \oplus y &\approx \neg(\neg y \oplus x) \oplus x. \end{aligned}$$

Here and in the following $1 := \neg 0$. We define

$$\begin{aligned}x \otimes y &:= \neg(\neg x \oplus \neg y), \\x \vee y &:= \neg(\neg x \oplus y) \oplus y, \\x \wedge y &:= \neg(\neg x \vee \neg y)\end{aligned}$$

for all $x, y \in A$. Then \otimes is distributive with respect to \vee . Moreover, for $x, y \in A$ we define $x \leq y$ if $\neg x \oplus y = 1$. Then $(A, \vee, \wedge, 0, 1)$ is a bounded distributive lattice whose induced order coincides with \leq . Moreover $x \leq y$ implies $\neg y \leq \neg x$ ($x, y \in A$).

It is known that Łukasiewicz semirings are in a natural one-to-one correspondence with MV-algebras (cf. e.g. [2], [3] and [8]). For the convenience of the reader we provide a short direct proof.

Theorem 1. *If $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ is a Łukasiewicz semiring and*

$$x \oplus y := (x' \cdot y)'$$

for all $x, y \in R$ then $\mathbf{M}(\mathbf{R}) := (R, \oplus, ', 0)$ is an MV-algebra. If, conversely, $\mathbf{A} = (A, \oplus, \neg, 0)$ is an MV-algebra then $\mathbf{R}(\mathbf{A}) := (A, \vee, \otimes, \neg, 0, 1)$ is a Łukasiewicz semiring. This correspondence is one-to-one.

Proof. First assume $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ to be a Łukasiewicz semiring and define $x \oplus y := (x' \cdot y)'$ for all $x, y \in R$. Then we have

$$\begin{aligned}(x \oplus y) \oplus z &\approx (((x' \cdot y)')' \cdot z) \approx ((x' \cdot y) \cdot z)' \approx (x' \cdot (y' \cdot z))' \approx \\&\approx (x' \cdot ((y' \cdot z)'))' \approx x \oplus (y \oplus z), \\x \oplus y &\approx (x' \cdot y)' \approx (y' \cdot x)' \approx y \oplus x, \\x \oplus 0 &\approx (x' \cdot 0)' \approx (x' \cdot 1)' \approx (x')' \approx x, \\x \oplus 1 &\approx (x' \cdot 1)' \approx (x' \cdot 0)' \approx 0' \approx 1, \\(x')' &\approx x,\end{aligned}$$

$$\begin{aligned}(x' \oplus y)' \oplus y &\approx (((x')' \cdot y)')' \cdot y' \approx (((x')' \cdot y)' \cdot y)' \approx (((y')' \cdot x')' \cdot x')' \approx \\&\approx (((y')' \cdot x')')' \cdot x' \approx (y' \oplus x)' \oplus x.\end{aligned}$$

Hence $\mathbf{M}(\mathbf{R}) = (R, \oplus, ', 0)$ is an MV-algebra. Conversely, let $\mathbf{A} = (A, \oplus, \neg, 0)$ be an MV-algebra. Then $(A, \vee, 0)$ is a commutative monoid. Moreover,

$$\begin{aligned}(x \otimes y) \otimes z &\approx \neg(\neg(\neg(\neg x \oplus \neg y)) \oplus \neg z) \approx \neg((\neg x \oplus \neg y) \oplus \neg z) \approx \\&\approx \neg(\neg x \oplus (\neg y \oplus \neg z)) \approx \neg(\neg x \oplus \neg(\neg(\neg y \oplus \neg z))) \approx x \otimes (y \otimes z), \\x \otimes y &\approx \neg(\neg x \oplus \neg y) \approx \neg(\neg y \oplus \neg x) \approx y \otimes x, \\x \otimes 1 &\approx \neg(\neg x \oplus \neg 1) \approx \neg(\neg x \oplus 0) \approx \neg(\neg x) \approx x.\end{aligned}$$

Hence, $(A, \otimes, 1)$ is a commutative monoid, too. It is clear that the partial order relations in \mathbf{A} and $\mathbf{R}(\mathbf{A})$ coincide. Now for $a, b \in A$, $a \leq b$ implies $\neg b \leq \neg a$. Moreover,

we have

$$\begin{aligned}
(x \vee y) \otimes z &\approx (x \otimes z) \vee (y \otimes z), \\
x \otimes 0 &\approx \neg(\neg x \oplus \neg 0) \approx \neg(\neg x \oplus 1) \approx \neg 1 \approx 0, \\
x \vee x &\approx x, \\
x \vee 1 &\approx 1, \\
\neg(\neg x \otimes y) \otimes y &\approx \neg(\neg(\neg(\neg(\neg(\neg x) \oplus \neg y))) \oplus \neg y) \approx \neg(\neg(\neg(\neg x) \oplus \neg y) \oplus \neg y) \approx \\
&\approx \neg(\neg(\neg(\neg y) \oplus \neg x) \oplus \neg x) \approx \neg(\neg(\neg(\neg(\neg y) \oplus \neg x))) \oplus \neg x \approx \\
&\approx \neg(\neg y \otimes x) \otimes x.
\end{aligned}$$

This shows that $\mathbf{R}(\mathbf{A}) = (A, \vee, \otimes, \neg, 0, 1)$ is a Łukasiewicz semiring. If $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ is a Łukasiewicz semiring, $\mathbf{M}(\mathbf{R}) := (R, \oplus, ', 0)$ and $\mathbf{R}(\mathbf{M}(\mathbf{R})) = (R, \vee, \otimes, ', 0, 0')$ then

$$\begin{aligned}
x \vee y &\approx (x' \oplus y)' \oplus y \approx (((((x')' \cdot y')')')' \cdot y')' \approx ((x \cdot y)')' \cdot y')' \approx x + y, \\
x \otimes y &\approx (x' \oplus y')' \approx (((x')' \cdot (y')')')' \approx x \cdot y, \\
0' &\approx 1,
\end{aligned}$$

i.e. $\mathbf{R}(\mathbf{M}(\mathbf{R})) = \mathbf{R}$. If, finally, $\mathbf{A} = (A, \oplus, \neg, 0)$ is an MV-algebra, $\mathbf{R}(\mathbf{A}) := (A, \vee, \otimes, \neg, 0, 1)$ and $\mathbf{M}(\mathbf{R}(\mathbf{A})) = ((A, \oplus_1, \neg, 0)$ then

$$x \oplus_1 y \approx \neg(\neg x \otimes \neg y) \approx \neg(\neg(\neg(\neg x) \oplus \neg(\neg y))) \approx x \oplus y$$

and hence $\mathbf{M}(\mathbf{R}(\mathbf{A})) = \mathbf{A}$. □

Due to this correspondence, for every Łukasiewicz semiring $\mathbf{R} = (R, +, \cdot, ', 0, 1)$, $(R, +, \wedge, 0, 1)$ is a bounded distributive lattice.

We are going to show when an interval in a Łukasiewicz semiring can be converted into a bounded lattice with an antitone involution.

Lemma 2. *Let $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ be a Łukasiewicz semiring and $b \in \text{Bool}\mathbf{R}$ and put*

$$\begin{aligned}
x^* &:= x' \wedge b, \\
x \cap y &:= (x^* + y^*)^*
\end{aligned}$$

for all $x, y \in [0, b]$. Then $([0, b], +, \cap, *, 0, b)$ is a bounded lattice with an antitone involution.

Proof. Since $b \in \text{Bool}\mathbf{R}$, i.e. $b \wedge b' = 0$, we have

$$(x^*)^* = (x' \wedge b)' \wedge b = (x + b') \wedge b = (x \wedge b) + (b' \wedge b) = x + 0 = x$$

for all $x \in [0, b]$. Obviously, $*$ is antitone. The rest of the proof is clear. □

Lemma 3. *Let $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ be a Łukasiewicz semiring and $a \in \text{Bool } \mathbf{R}$ and put*

$$\begin{aligned} x^* &:= x' + a, \\ x \cap y &:= (x^* + y^*)^* \end{aligned}$$

for all $x, y \in [a, 1]$. Then $([a, 1], +, \cap, *, a, 1)$ is a bounded lattice with an antitone involution.

Proof. Since $a \in \text{Bool } \mathbf{R}$, i.e. $a + a' = 1$, we have

$$(x^*)^* = (x' + a)' + a = (x \wedge a') + a = (x + a) \wedge (a' + a) = x \wedge 1 = x$$

for all $x \in [a, 1]$. Obviously, $*$ is antitone. The rest of the proof is clear. \square

Theorem 2. *Let $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ be a Łukasiewicz semiring and $a, b \in \text{Bool } \mathbf{R}$, assume $a \leq b$ and put*

$$\begin{aligned} x^* &:= (x' + a) \wedge b, \\ x \cap y &:= (x^* + y^*)^* \end{aligned}$$

for all $x, y \in [a, b]$. Then $([a, b], +, \cap, *, a, b)$ is a bounded lattice with an antitone involution.

Proof. This follows from the fact that $(R, +, \wedge)$ is a distributive lattice. \square

Remark 1. Theorem 2 remains valid if one defines $x^* := (x' \wedge b) + a$ for all $x \in [a, b]$. Note that $(x' \wedge b) + a \leq (x' + a) \wedge b$ for all $x \in R$.

3. DERIVATIONS

Although Łukasiewicz semirings are in a one-to-one correspondence with MV-algebras, we do not define derivations like in [1] or [10] since there the MV-operation \oplus is used instead of $+$. Hence also our results differ essentially from those obtained in the mentioned papers.

Now we can adopt the definition of a derivation from [9].

Definition 2. Let $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ be a Łukasiewicz semiring. A *derivation* on \mathbf{R} is a mapping d from R to R satisfying the following identities:

$$\begin{aligned} d(x + y) &\approx dx + dy, \\ d(x \cdot y) &\approx (dx) \cdot y + x \cdot (dy). \end{aligned}$$

Let $\text{Der } \mathbf{R}$ denote the set of all derivations on \mathbf{R} , for every $a \in R$ let \bar{d}_a denote the mapping from R to R defined by $\bar{d}_a(x) := a \cdot x$ for all $x \in R$ and let id denote the identical mapping from R to R . We call a *derivation* d on \mathbf{R} *principal* if it is of the form \bar{d}_a for some $a \in R$. Hence $0 = \bar{d}_0$ and $\text{id} = \bar{d}_1$ are principal derivations. For any mapping d from R to R define $\text{Fix } d := \{x \in R \mid dx = x\}$.

Here and in the following we write dx instead of $d(x)$ if no confusion can arise.

Lemma 4. *Let $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ be a Łukasiewicz semiring, $a, b \in R$ and $d, d_1, d_2 \in \text{Der}\mathbf{R}$. Then*

- (i) $\bar{d}_a \in \text{Der}\mathbf{R}$,
- (ii) $d0 = 0$,
- (iii) $da \leq a$,
- (iv) $a \leq b$ implies $da \leq db$,
- (v) $a \cdot (d1) \leq da$,
- (vi) $d1 = 1$ if and only if $d = \text{id}$,
- (vii) $(d_1a) \cdot (d_2b) + (d_1b) \cdot (d_2a) \leq (d_1 \circ d_2)(a \cdot b)$ where \circ denotes the composition of mappings,
- (viii) $d_1 \circ d_2 \in \text{Der}\mathbf{R}$ if and only if for all $x, y \in R$ we have $(d_1x) \cdot (d_2y) + (d_1y) \cdot (d_2x) \leq ((d_1 \circ d_2)x) \cdot y + x \cdot ((d_1 \circ d_2)y)$.

Proof.

- (i) This is clear.
- (ii) We have $d0 = d(0 \cdot 0) = (d0) \cdot 0 + 0 \cdot (d0) = 0$.
- (iii) According to (ii) and Lemma 1 (ii) we have

$$(da) \cdot a' \leq d(a \cdot a') = d0 = 0$$

and hence $(da) \cdot a' = 0$ whence $da \leq a$ according to Lemma 1 (vi).

- (iv) $a \leq b$ implies $da \leq da + db = d(a + b) = db$.
- (v) We have $a \cdot (d1) \leq d(a \cdot 1) = da$.
- (vi) If $d1 = 1$ then $a = a \cdot (d1) \leq d(a \cdot 1) = da \leq a$ according to (iii) showing $d = \text{id}$. The converse is trivial.
- (vii) We have

$$\begin{aligned} & (d_1a) \cdot (d_2b) + (d_1b) \cdot (d_2a) \leq \\ & \leq (d_1a) \cdot (d_2b) + (d_1b) \cdot (d_2a) + ((d_1 \circ d_2)a) \cdot b + a \cdot ((d_1 \circ d_2)b) = \\ & = (d_1(d_2a)) \cdot b + (d_2a) \cdot (d_1b) + (d_1a) \cdot (d_2b) + a \cdot (d_1(d_2b)) = \\ & = d_1((d_2a) \cdot b) + d_1(a \cdot (d_2b)) = d_1((d_2a) \cdot b + a \cdot (d_2b)) = d_1(d_2(a \cdot b)) = \\ & = (d_1 \circ d_2)(a \cdot b). \end{aligned}$$

- (viii) According to the proof of (vii) we have

$$(d_1 \circ d_2)(a \cdot b) = (d_1a) \cdot (d_2b) + (d_1b) \cdot (d_2a) + ((d_1 \circ d_2)a) \cdot b + a \cdot ((d_1 \circ d_2)b).$$

□

Lemma 5. *Let $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ be a Łukasiewicz semiring and $d \in \text{Der}\mathbf{R}$. Then the following are equivalent:*

- (i) d is principal,
- (ii) $dx = x \cdot (d1)$ for all $x \in R$, i.e. $d = \bar{d}_{d1}$.

Moreover, if $x \wedge (d1) \leq x \cdot (d1)$ for all $x \in R$ then d is principal.

Proof. The equivalence of (i) and (ii) is clear. If, finally, $x \wedge (d1) \leq x \cdot (d1)$ for all $x \in R$ then

$$dx \leq x \wedge (d1) \leq x \cdot (d1) \leq d(x \cdot 1) = dx$$

for all $x \in R$ and hence $d = \bar{d}_{d1}$. □

Lemma 5 shows that the Łukasiewicz semiring from Example 3 has only principal derivations since in this semiring multiplication coincides with lattice meet.

It was shown in [10], Lemma 5, that basic algebras have only principal derivations. We are going to show that this is not the case for Łukasiewicz semirings. In these semirings there exist non-principal derivations, see the following example.

Example 4. Consider the Łukasiewicz semiring $\mathbf{R} = (\{0, a, b, c, 1\}, +, \cdot, ', 0, 1)$ with the following operations:

| | | | | | |
|---|---|---|---|---|---|
| + | 0 | a | b | c | 1 |
| 0 | 0 | a | b | c | 1 |
| a | a | a | b | c | 1 |
| b | b | b | b | c | 1 |
| c | c | c | c | c | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 |

| | | | | | |
|---|---|---|---|---|---|
| · | 0 | a | b | c | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 |
| a | 0 | 0 | 0 | 0 | a |
| b | 0 | 0 | 0 | a | b |
| c | 0 | 0 | a | b | c |
| 1 | 0 | a | b | c | 1 |

| | | | | | |
|----|---|---|---|---|---|
| x | 0 | a | b | c | 1 |
| x' | 1 | c | b | a | 1 |

The principal derivations look as follows:

| | | | | | |
|--------------|---|---|---|---|---|
| x | 0 | a | b | c | 1 |
| \bar{d}_0x | 0 | 0 | 0 | 0 | 0 |
| \bar{d}_ax | 0 | 0 | 0 | 0 | a |
| \bar{d}_bx | 0 | 0 | 0 | a | b |
| \bar{d}_cx | 0 | 0 | a | b | c |
| \bar{d}_1x | 0 | a | b | c | 1 |

However, there exists a derivation d of \mathbf{R} which is not principal, namely the following:

| | | | | | |
|----|---|---|---|---|---|
| x | 0 | a | b | c | 1 |
| dx | 0 | 0 | 0 | a | a |

The Łukasiewicz semiring considered in Example 4 is isomorphic to the Łukasiewicz semiring \mathbf{R}_4 from Example 2. The following proposition shows that the situation described in Example 4 can be generalized to \mathbf{R}_n for arbitrary $n > 1$.

Proposition 1. *For each $n > 1$ the Łukasiewicz semiring \mathbf{R}_n has the non-principal derivation d defined by*

$$d(x) := \begin{cases} 0 & \text{if } x < n - 1 \\ 1 & \text{otherwise.} \end{cases}$$

Proof. Let $a, b \in \{0, \dots, n\}$. Then $d(a \vee b) = (da) \vee (db)$ since $(\{0, 1, \dots, n\}, \leq)$ is a chain. Moreover,

$$\begin{aligned} d(a \odot b) &= \begin{cases} 0 & \text{if } a + b < 2n - 1 \\ 1 & \text{otherwise,} \end{cases} \\ (da) \odot b &= \begin{cases} 0 & \text{if } a < n - 1 \text{ or } b < n \\ 1 & \text{if } a \geq n - 1 \text{ and } b = n, \end{cases} \\ a \odot (db) &= \begin{cases} 0 & \text{if } a < n \text{ or } b < n - 1 \\ 1 & \text{if } a = n \text{ and } b \geq n - 1. \end{cases} \end{aligned}$$

Hence $d \in \text{Der}\mathbf{R}_n$. Since

$$\bar{d}_a(b) = \begin{cases} 0 & \text{if } b \leq n - a \\ b - (n - a) & \text{otherwise,} \end{cases}$$

d is not principal. \square

For the unary operation $'$, we can prove the following theorem:

Theorem 3. *Let $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ be a Łukasiewicz semiring and $d \in \text{Der}\mathbf{R}$. Then*

- (i) $dx' = (dx)'$ for all $x \in R$ if and only if $d = \text{id}$,
- (ii) $(dx) \cdot (dx') \approx x \cdot (dx') \approx x' \cdot (dx) \approx 0$ and hence $dx' \leq (dx)'$ for all $x \in R$,
- (iii) if $a \in \text{Bool}\mathbf{R}$ then $da + da' = d1$,
- (iv) $d((d1)') \leq (d1) \wedge (d1)'$,
- (v) if $d1 \in \text{Bool}\mathbf{R}$ then $d((d1)' \cdot x) = 0$ for each $x \in R$.

Proof.

- (i) If $dx' = (dx)'$ for all $x \in R$ then for all $x \in R$ we have $x = (x')' \leq (dx')' = ((dx)')' = dx \leq x$ according to Lemma 4 (iii) and hence $d = \text{id}$.
- (ii) Due to Lemma 4 (iii) and Lemma 1 (iii) and (ii) we have $(dx)(dx') \leq x \cdot (dx') \leq x \cdot x' = 0$ and $x' \cdot (dx) \leq x' \cdot x = 0$ for all $x \in R$. By Lemma 1 (vi), $dx' \leq (dx)'$ for all $x \in R$.
- (iii) If $a \in \text{Bool}\mathbf{R}$ then $da + da' = d(a + a') = d1$.
- (iv) Since $(d1)' \leq 1$, we have $d((d1)') \leq d1$ by Lemma 4 (iv). According to Lemma 4 (iii) we have $d((d1)') \leq (d1)'$. This shows $d((d1)') \leq (d1) \wedge (d1)'$.
- (v) If $d1 \in \text{Bool}\mathbf{R}$ then by (iv), Lemma 4 (iv) and Lemma 1 (ii) we have $d((d1)' \cdot x) = (d((d1)')) \cdot x + (d1)' \cdot (dx) = 0 \cdot x + (d1)' \cdot (dx) = (dx) \cdot (d1)' \leq (d1)(d1)' = 0$ for all $x \in R$. \square

Theorem 4. *Let $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ be a Łukasiewicz semiring, $a \in R$ and $d \in \text{Der}\mathbf{R}$. Then*

- (i) $(\text{Der}\mathbf{R}, +)$ is a join-semilattice with the least element \bar{d}_0 and the greatest element id ,

- (ii) $\text{Fix } d$ is a subuniverse of $(R, +, \cdot, 0)$,
- (iii) $(\text{Fix } d) \cap d^{-1}(\{0\}) = \{0\}$,
- (iv) if $a \in \text{Bool } \mathbf{R}$ then $\bar{d}_a \in \text{End}(R, +, \cdot, 0)$.

Proof.

- (i) It is easy to see that $\text{Der } \mathbf{R}$ is closed with respect to $+$. Hence $(\text{Der } \mathbf{R}, +)$ is a semilattice which is considered as a join-semilattice. We then have $d_1 \leq d_2$ if and only if $d_1 x \leq d_2 x$ for all $x \in R$. Obviously, $\bar{d}_0, \text{id} \in \text{Der } \mathbf{R}$ and \bar{d}_0 is the smallest element of $(\text{Der } \mathbf{R}, \leq)$. Because of Lemma 4 (iii), id is the greatest element of $(\text{Der } \mathbf{R}, \leq)$.
- (ii) If $a, b \in \text{Fix } d$ then $d(a + b) = da + db = a + b$ and

$$d(a \cdot b) = (da) \cdot b + a \cdot (db) = a \cdot b + a \cdot b = a \cdot b.$$

Moreover, $d0 = 0$ according to Lemma 4 (ii).

- (iii) $a \in (\text{Fix } d) \cap d^{-1}(\{0\})$ implies $a = da = 0$.
- (iv) This can be proved by a straightforward computation. □

For Łukasiewicz semirings we adopt the concept of an ideal as introduced for semirings in [9]. Hence, we define

Definition 3. Let $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ be a Łukasiewicz semiring. An *ideal* of \mathbf{R} is a subset I of R satisfying

- $0 \in I$,
- if $x, y \in I$ then $x + y \in I$,
- if $x \in I$ and $y \in R$ then $x \cdot y \in I$.

Let $\text{Id } \mathbf{R}$ denote the set of all ideals of \mathbf{R} .

It is well known and easy to prove that $(\text{Id } \mathbf{R}, \subseteq)$ is a complete lattice the with the smallest element $\{0\}$ and the greatest element R .

Proposition 2. Let $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ be a Łukasiewicz semiring, $d \in \text{Der } \mathbf{R}$ and $I \in \text{Id } \mathbf{R}$. Then

- (i) $d^{-1}(\{0\}) \in \text{Id } \mathbf{R}$,
- (ii) $d^{-1}(I) := \{x \in R \mid dx \in I\}$ is a subuniverse of $(R, +, \cdot, 0)$.

Proof.

- (i) We have $0 \in d^{-1}(\{0\})$. Moreover, $d^{-1}(\{0\})$ is closed with respect to $+$. If $a \in d^{-1}(\{0\})$ and $b \in R$ then $d(a \cdot b) \leq d(a \cdot 1) = da = 0$ according to Lemma 1 (iii) and Lemma 4 (iv) and hence $d(a \cdot b) = 0$, i.e. $a \cdot b \in d^{-1}(\{0\})$.
- (ii) The proof is straightforward. □

Let $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ be a Łukasiewicz semiring and $\Theta \in \text{Con } \mathbf{R}$. Denote by $[0]\Theta$ the class of 0 with respect to Θ , the so-called *kernel* of Θ . It is well-known that in any semiring, the kernel of any congruence is an ideal, but not every ideal is a kernel of some congruence, see e.g. [9]. Now let $d \in \text{Der } \mathbf{R}$. We are interested in the question if $d^{-1}([0]\Theta) \in \text{Id } \mathbf{R}$.

Theorem 5. *Let $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ be a Łukasiewicz semiring, $\Theta \in \text{Con } \mathbf{R}$ and $d \in \text{Der } \mathbf{R}$. Then*

- (i) $d^{-1}([0]\Theta) := \{x \in R \mid dx \in [0]\Theta\} \in \text{Id } \mathbf{R}$,
- (ii) $d([0]\Theta) \subseteq [0]\Theta$ and $d([0]\Theta)$ is a subuniverse of $(R, +, 0)$.

Proof.

- (i) If $I := d^{-1}([0]\Theta)$, $a, b \in I$ and $c \in R$ then $da, db \in [0]\Theta$ and hence

$$d(a + b) = da + db \in [0]\Theta,$$

$$d(a \cdot c) = d(a \cdot c) + 0 \in [d(a \cdot c) + da]\Theta = [da]\Theta = [0]\Theta$$

according to Lemma 1 (iii) and Lemma 4 (iv) showing $a + b, a \cdot c \in I$.

- (ii) Clearly, $0 = d0 \in d([0]\Theta)$. If $a, b \in d([0]\Theta)$ then there exist $c, e \in [0]\Theta$ with $dc = a$ and $de = b$ and hence

$$a = dc = dc + 0 \in [dc + c]\Theta = [c]\Theta = [0]\Theta,$$

$$a + b = dc + de = d(c + e) \in d([0]\Theta)$$

according to Lemma 4 (iii). □

Theorem 6. *Let $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ be a Łukasiewicz semiring, $d \in \text{Der } \mathbf{R}$, assume $d1 \in \text{Bool } \mathbf{R}$, put*

$$x^* := x' \wedge (d1),$$

$$x \cap y := (x^* + y^*)^*$$

for all $x, y \in d(R)$ and assume $x^* \in d(R)$ for all $x \in d(R)$. Then $(d(R), +, \cap, *, 0, d1)$ is a bounded lattice with an antitone involution.

Proof. Since $d1 \in \text{Bool } \mathbf{R}$, i.e. $(d1) \wedge (d1)' = 0$, we have

$$\begin{aligned} (x^*)^* &= (x' \wedge (d1))' \wedge (d1) = (x + (d1)') \wedge (d1) \\ &= (x \wedge (d1)) + ((d1)' \wedge (d1)) = x + 0 = x \end{aligned}$$

for all $x \in d(R)$. Obviously, $*$ is antitone. The rest of the proof is clear. □

Theorem 7. *Let $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ and $\mathbf{S} = (S, +, \cdot, ', 0, 1)$ be Łukasiewicz semirings. Then $\text{Der}(\mathbf{R} \times \mathbf{S}) = \text{Der } \mathbf{R} \times \text{Der } \mathbf{S}$.*

Proof. Clearly, $\text{Der } \mathbf{R} \times \text{Der } \mathbf{S} \subseteq \text{Der}(\mathbf{R} \times \mathbf{S})$. In order to prove the converse inclusion, let $d \in \text{Der}(\mathbf{R} \times \mathbf{S})$. Then for each $x \in R$ we have $d(x, 0) \leq (x, 0)$ according to Lemma 4 (iii) and hence there exists some mapping d_1 from R to R satisfying $(d_1x, 0) = d(x, 0)$ for all $x \in R$. Analogously, there exists some mapping d_2 from S to S satisfying $(0, d_2y) = d(0, y)$ for all $y \in S$. Now let $a, b \in R$ and $c \in S$. Then

$$\begin{aligned} (d_1(a+b), 0) &= d(a+b, 0) = d((a, 0) + (b, 0)) = d(a, 0) + d(b, 0) = \\ &= (d_1a, 0) + (d_1b, 0) = (d_1a + d_1b, 0), \end{aligned}$$

i.e. $d_1(a+b) = d_1a + d_1b$. Moreover,

$$\begin{aligned} (d_1(a \cdot b), 0) &= d(a \cdot b, 0) = d((a, 0) \cdot (b, 0)) = (d(a, 0)) \cdot (b, 0) + (a, 0) \cdot (d(b, 0)) = \\ &= (d_1a, 0) \cdot (b, 0) + (a, 0) \cdot (d_1b, 0) = ((d_1a) \cdot b, 0) + (a \cdot (d_1b), 0) = \\ &= ((d_1a) \cdot b + a \cdot (d_1b), 0), \end{aligned}$$

i.e. $d_1(a \cdot b) = (d_1a) \cdot b + a \cdot (d_1b)$. This shows $d_1 \in \text{Der } \mathbf{R}$. Analogously, $d_2 \in \text{Der } \mathbf{S}$. Finally,

$$d(a, c) = d((a, 0) + (0, c)) = d(a, 0) + d(0, c) = (d_1a, 0) + (0, d_2c) = (d_1a, d_2c)$$

proving $d \in \text{Der } \mathbf{R} \times \text{Der } \mathbf{S}$. Hence $\text{Der}(\mathbf{R} \times \mathbf{S}) \subseteq \text{Der } \mathbf{R} \times \text{Der } \mathbf{S}$ completing the proof of the theorem. \square

Proposition 3. Let $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ be a Łukasiewicz semiring and $a \in \text{Bool } \mathbf{R}$. Then $(\bar{d}_a(R), +, \cdot, 0, a)$ is a commutative semiring satisfying $x + x \approx x$ and $x + a \approx a$.

Proof. Since $\bar{d}_a(x + y) = \bar{d}_ax + \bar{d}_ay$, $\bar{d}_a(x \cdot y) = a \cdot (x \cdot y) = (a \cdot x) \cdot (a \cdot y) = (\bar{d}_ax) \cdot (\bar{d}_ay)$ for all $x, y \in R$ and $\bar{d}_a0 = a \cdot 0 = 0$ and $\bar{d}_a1 = a \cdot 1 = a$, \bar{d}_a is a homomorphism from $(R, +, \cdot, 0, 1)$ onto $(\bar{d}_a(R), +, \cdot, 0, a)$. \square

4. (f, g) -DERIVATIONS

The concept of an (f, g) -derivation is mentioned in the monograph [9]. For Łukasiewicz semirings, it is specified as follows:

Definition 4. Let $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ be a Łukasiewicz semiring and $f, g \in \text{End } \mathbf{R}$. An (f, g) -derivation on \mathbf{R} is a mapping d from R to R satisfying the following identities:

$$\begin{aligned} d(x + y) &\approx dx + dy, \\ d(x \cdot y) &\approx (dx) \cdot f(y) + g(x) \cdot (dy). \end{aligned}$$

Let $\text{Der}_{f,g} \mathbf{R}$ denote the set of all (f, g) -derivations on \mathbf{R} . Especially, $\text{Der}_{\text{id}, \text{id}} \mathbf{R} = \text{Der } \mathbf{R}$.

The following example shows that there exist (f, g) -derivations on Łukasiewicz semirings which are not derivations.

Example 5. Let M be a set with $|M| > 1$ and put $\mathbf{R} := (2^M, \cup, \cap, ', \emptyset, M)$. If $f \in \text{End } \mathbf{R}$ then $f \in \text{Der}_{f,f} \mathbf{R}$ since

$$f(A \cup B) = f(A) \cup f(B),$$

$$f(A \cap B) = f(A \cap B) \cup f(A \cap B) = (f(A) \cap f(B)) \cup (f(A) \cap f(B))$$

for all $A, B \in 2^M$. Now let $a \in M$ and g denote the mapping from 2^M to 2^M defined by

$$g(A) := \begin{cases} M & \text{if } a \in A, \\ \emptyset & \text{otherwise} \end{cases}$$

($A \in 2^M$). Then $g \in \text{End } \mathbf{R}$ (and hence $g \in \text{Der}_{g,g} \mathbf{R}$) as can be seen from the following table:

| A | B | $g(A)$ | $g(B)$ | $g(A \cup B)$ | $g(A \cap B)$ | $g(A')$ |
|-------------|-------------|-------------|-------------|---------------|---------------|-------------|
| $\ni a$ | $\ni a$ | M | M | M | M | \emptyset |
| $\ni a$ | $\not\ni a$ | M | \emptyset | M | \emptyset | \emptyset |
| $\not\ni a$ | $\ni a$ | \emptyset | M | M | \emptyset | M |
| $\not\ni a$ | $\not\ni a$ | \emptyset | \emptyset | \emptyset | \emptyset | M |

But $g \notin \text{Der } \mathbf{R}$ since

$$g(\{a\} \cap \{x\}) = \emptyset \neq \{x\} = (g(\{a\}) \cap \{x\}) \cup (\{a\} \cap g(\{x\}))$$

for all $x \in M \setminus \{a\}$.

Lemma 6. Let $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ be a Łukasiewicz semiring, $f, g \in \text{End } \mathbf{R}$, $a, b \in R$ and $d, d_1, d_2 \in \text{Der}_{f,g} \mathbf{R}$. Then

- (i) $\bar{d}_0 \in \text{Der}_{f,g} \mathbf{R}$,
- (ii) $d0 = 0$,
- (iii) $da \leq f(a) \wedge g(a)$,
- (iv) $a \leq b$ implies $da \leq db$,
- (v) $(f(a) + g(a)) \cdot (d1) \leq da$,
- (vi) $d1 = 1$ if and only if $d = f = g$,
- (vii) $(d_1(g(a))) \cdot f(d_2b) + (d_1(f(b))) \cdot g(d_2a) \leq (d_1 \circ d_2)(a \cdot b)$,
- (viii) if $f \circ f = f$ and $g \circ g = g$ then $d_1 \circ d_2 \in \text{Der}_{f,g} \mathbf{R}$ if and only if for all $x, y \in R$ we have $(d_1(g(x))) \cdot f(d_2y) + (d_1(f(y))) \cdot g(d_2x) \leq ((d_1 \circ d_2)x) \cdot f(y) + g(x) \cdot ((d_1 \circ d_2)y)$.

Proof.

- (i) This is clear.
- (ii) We have $d0 = d(0 \cdot 0) = (d0) \cdot f(0) + g(0) \cdot (d0) = (d0) \cdot 0 + 0 \cdot (d0) = 0$.
- (iii) According to (ii) and Lemma 1 (ii) we have

$$(da) \cdot (f(a))' = (da) \cdot f(a') \leq d(a \cdot a') = d0 = 0$$

and hence $(da) \cdot (f(a))' = 0$ whence $da \leq f(a)$ according to Lemma 1 (vi). Analogously, according to (ii) and Lemma 1 (ii) we have

$$(da) \cdot (g(a))' = g(a') \cdot (da) \leq d(a' \cdot a) = d0 = 0$$

and hence $(da) \cdot (g(a))' = 0$ whence $da \leq g(a)$ according to Lemma 1 (vi). (iv) can be proved exactly as Lemma 4 (iv).

(v) We have

$$(f(a) + g(a)) \cdot (d1) = (d1) \cdot f(a) + g(a) \cdot (d1) \leq d(1 \cdot a) + d(a \cdot 1) = da + da = da.$$

(vi) If $d1 = 1$ then

$$\begin{aligned} f(a) &= (d1) \cdot f(a) \leq d(1 \cdot a) = da \leq f(a), \\ g(a) &= g(a) \cdot (d1) \leq d(a \cdot 1) = da \leq g(a) \end{aligned}$$

according to (iii) showing $d = f = g$. The converse is trivial.

(vii) We have

$$\begin{aligned} &(d_1(g(a))) \cdot f(d_2b) + (d_1(f(b))) \cdot g(d_2a) \leq \\ &\leq (d_1(g(a))) \cdot f(d_2b) + (d_1(f(b))) \cdot g(d_2a) + \\ &+ ((d_1 \circ d_2)a) \cdot f(f(b)) + g(g(a)) \cdot ((d_1 \circ d_2)b) = \\ &= (d_1(d_2a)) \cdot f(f(b)) + g(d_2a) \cdot (d_1(f(b))) + \\ &+ (d_1(g(a))) \cdot f(d_2b) + g(g(a)) \cdot (d_1(d_2b)) = \\ &= d_1((d_2a) \cdot f(b)) + d_1(g(a) \cdot (d_2b)) = d_1((d_2a) \cdot f(b) + g(a) \cdot (d_2b)) = \\ &= d_1(d_2(a \cdot b)) = (d_1 \circ d_2)(a \cdot b). \end{aligned}$$

(viii) If $f \circ f = f$ and $g \circ g = g$ then according to the proof of (vii) we have

$$\begin{aligned} (d_1 \circ d_2)(a \cdot b) &= (d_1(g(a))) \cdot f(d_2b) + (d_1(f(b))) \cdot g(d_2a) + \\ &+ ((d_1 \circ d_2)a) \cdot f(b) + g(a) \cdot ((d_1 \circ d_2)b). \end{aligned}$$

□

For the unary operation $'$, we can prove the following theorem:

Theorem 8. *Let $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ be a Lukasiewicz semiring, $f, g \in \text{End } \mathbf{R}$ and $d \in \text{Der}_{f,g} \mathbf{R}$. Then*

- (i) $dx' = (dx)'$ for all $x \in R$ if and only if $d = f = g$,
- (ii) $(dx) \cdot (dx') \approx f(x) \cdot (dx') \approx f(x') \cdot (dx) \approx g(x) \cdot (dx') \approx g(x') \cdot (dx) \approx 0$
and hence $dx' \leq (dx)'$ for all $x \in R$,
- (iii) if $a \in \text{Bool } \mathbf{R}$ then $da + da' = d1$.

Proof.

(i) If $dx' = (dx)'$ for all $x \in R$ then for all $x \in R$ we have

$$\begin{aligned} f(x) &= (f(x'))' \leq (dx')' = dx \leq f(x), \\ g(x) &= (g(x'))' \leq (dx')' = dx \leq g(x) \end{aligned}$$

according to Lemma 6 (iii) and hence $d = f = g$.

(ii) Due to Lemma 6 (iii) and Lemma 1 (iii) and (ii) we have

$$\begin{aligned} (dx) \cdot (dx') &\leq f(x) \cdot (dx') \leq f(x) \cdot f(x') = f(x \cdot x') = f(0) = 0, \\ f(x') \cdot (dx) &\leq f(x') \cdot f(x) = f(x \cdot x') = f(0) = 0 \end{aligned}$$

for all $x \in R$. The proof for g is analogous. By Lemma 1 (vi), $dx' \leq (dx)'$ for all $x \in R$.

(iii) can be proved exactly as Theorem 3 (iii). □

Theorem 9. Let $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ be a Łukasiewicz semiring, $f, g \in \text{End } \mathbf{R}$ and $d \in \text{Der}_{f,g} \mathbf{R}$. Then

- (i) $(\text{Der}_{f,g} \mathbf{R}, +)$ is a join-semilattice with the least element \bar{d}_0 ,
- (ii) $(\text{Fix } d) \cap d^{-1}(\{0\}) = \{0\}$.

Proof.

- (i) It is easy to see that $\text{Der}_{f,g} \mathbf{R}$ is closed with respect to $+$. Hence $(\text{Der}_{f,g} \mathbf{R}, +)$ is a semilattice which is considered as a join-semilattice. We then have $d_1 \leq d_2$ if and only if $d_1 x \leq d_2 x$ for all $x \in R$. Obviously, $\bar{d}_0 \in \text{Der}_{f,g} \mathbf{R}$ and \bar{d}_0 is the smallest element of $(\text{Der}_{f,g} \mathbf{R}, \leq)$.
- (ii) can be proved exactly as Theorem 4 (iii). □

It is clear that Proposition 2, Theorem 5 (i) and Theorem 6 remain valid for (f, g) -derivations, too.

Theorem 10. Let $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ and $\mathbf{S} = (S, +, \cdot, ', 0, 1)$ be Łukasiewicz semirings, $f_1, g_1 \in \text{End } \mathbf{R}$ and $f_2, g_2 \in \text{End } \mathbf{S}$. Then $(f_1, f_2), (g_1, g_2) \in \text{End}(\mathbf{R} \times \mathbf{S})$ and

$$\text{Der}_{(f_1, f_2), (g_1, g_2)}(\mathbf{R} \times \mathbf{S}) = \text{Der}_{f_1, g_1} \mathbf{R} \times \text{Der}_{f_2, g_2} \mathbf{S}.$$

Proof. Clearly, $(f_1, f_2), (g_1, g_2) \in \text{End}(\mathbf{R} \times \mathbf{S})$ and

$$\text{Der}_{f_1, g_1} \mathbf{R} \times \text{Der}_{f_2, g_2} \mathbf{S} \subseteq \text{Der}_{(f_1, f_2), (g_1, g_2)}(\mathbf{R} \times \mathbf{S}).$$

In order to prove the converse inclusion, let $d \in \text{Der}_{(f_1, f_2), (g_1, g_2)}(\mathbf{R} \times \mathbf{S})$. Then for each $x \in R$ we have $d(x, 0) \leq (f_1(x), 0)$ according to Lemma 6 (iii) and hence there exists some mapping d_1 from R to R satisfying $(d_1 x, 0) = d(x, 0)$ for all $x \in R$.

Analogously, there exists some mapping d_2 from S to S satisfying $(0, d_2 y) = d(0, y)$ for all $y \in S$. Now let $a, b \in R$ and $c \in S$. Then

$$\begin{aligned}(d_1(a+b), 0) &= d(a+b, 0) = d((a, 0) + (b, 0)) = d(a, 0) + d(b, 0) = \\ &= (d_1 a, 0) + (d_1 b, 0) = (d_1 a + d_1 b, 0),\end{aligned}$$

i.e. $d_1(a+b) = d_1 a + d_1 b$. Moreover,

$$\begin{aligned}(d_1(a \cdot b), 0) &= d(a \cdot b, 0) = d((a, 0) \cdot (b, 0)) = \\ &= (d(a, 0)) \cdot (f_1(b), 0) + (g_1(a), 0) \cdot (d(b, 0)) = \\ &= (d_1 a, 0) \cdot (f_1(b), 0) + (g_1(a), 0) \cdot (d_1 b, 0) = \\ &= ((d_1 a) \cdot f_1(b), 0) + (g_1(a) \cdot (d_1 b), 0) = \\ &= ((d_1 a) \cdot f_1(b) + g_1(a) \cdot (d_1 b), 0),\end{aligned}$$

i.e. $d_1(a \cdot b) = (d_1 a) \cdot f_1(b) + g_1(a) \cdot (d_1 b)$. This shows $d_1 \in \text{Der}_{f_1, g_1} \mathbf{R}$. Analogously, $d_2 \in \text{Der}_{f_2, g_2} \mathbf{S}$. Finally,

$$d(a, c) = d((a, 0) + (0, c)) = d(a, 0) + d(0, c) = (d_1 a, 0) + (0, d_2 c) = (d_1 a, d_2 c)$$

proving $d \in \text{Der}_{f_1, g_1} \mathbf{R} \times \text{Der}_{f_2, g_2} \mathbf{S}$. Hence

$$\text{Der}_{(f_1, f_2), (g_1, g_2)}(\mathbf{R} \times \mathbf{S}) \subseteq \text{Der}_{f_1, g_1} \mathbf{R} \times \text{Der}_{f_2, g_2} \mathbf{S}$$

completing the proof of the theorem. \square

ACKNOWLEDGEMENT

We thank the referee for his valuable suggestions which made the paper better readable.

REFERENCES

- [1] N. O. Alshehri, "Derivations of MV-algebras," *Int. J. Math. Math. Sci.*, vol. 2010, p. 7, 2010, doi: [10.1155/2010/312027](https://doi.org/10.1155/2010/312027).
- [2] L. P. Belluce, A. Di Nola, and A. R. Ferraioli, "MV-semirings and their sheaf representations," *Order*, vol. 30, no. 1, pp. 165–179, 2013.
- [3] S. Bonzio, I. Chajda, and A. Ledda, "Representing quantum structures as near semirings," *Log. J. IGPL*, vol. 24, no. 5, pp. 719–742, 2016.
- [4] I. Chajda, D. Fazio, and A. Ledda, "A semiring-like representation of lattice pseudoeffect algebras," *Soft Computing*, 2018, doi: <https://doi.org/10.1007/s00500-018-3157-2>.
- [5] I. Chajda, D. Fazio, and A. Ledda, "On the structure theory of Łukasiewicz near semirings," *Log. J. IGPL*, vol. 26, no. 1, pp. 14–28, 2018, doi: <https://doi.org/10.1093/jigpal/jzx044>.
- [6] I. Chajda and H. Länger, "A representation of lattice effect algebras by means of right near semirings with involution," *Int. J. Theor. Phys.*, vol. 56, no. 12, pp. 3719–3726, 2017, doi: [10.1007/s10773-016-3191-8](https://doi.org/10.1007/s10773-016-3191-8).
- [7] L. Ferrari, "On derivations of lattices," *P.U.M.A., Pure Math. Appl.*, vol. 12, no. 4, pp. 365–382, 2001.
- [8] B. Gerla, "Many-valued logic and semirings," *Neural Network World*, vol. 13, pp. 467–480, 2003.
- [9] J. S. Golan, *Semirings and their applications*. Dordrecht: Kluwer Academic Publishers, 1999.

- [10] J. Krňávek and J. Kühr, “A note on derivations on basic algebras,” *Soft Comput.*, vol. 19, no. 7, pp. 1765–1771, 2015, doi: [10.1007/s00500-014-1586-0](https://doi.org/10.1007/s00500-014-1586-0).
- [11] G. Szasz, “Derivations of lattices,” *Acta Sci. Math.*, vol. 37, pp. 149–154, 1975.

Authors' addresses

Ivan Chajda

Palacký University Olomouc, Faculty of Science, Department of Algebra and Geometry, 17. listopadu 12, 771 46 Olomouc, Czech Republic

E-mail address: ivan.chajda@upol.cz

Helmut Länger

TU Wien, Faculty of Mathematics and Geoinformation, Institute of Discrete Mathematics and Geometry, Wiedner Hauptstraße 8-10, 1040 Vienna, Austria, and, Palacký University Olomouc, Faculty of Science, Department of Algebra and Geometry, 17. listopadu 12, 771 46 Olomouc, Czech Republic

E-mail address: helmut.laenger@tuwien.ac.at