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NOTES ON TWO KINDS OF SPECIAL VALUES FOR THE BELL POLYNOMIALS OF THE SECOND KIND

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Abstract. In the paper, by methods and techniques in combinatorial analysis and the theory of special functions, the authors discuss two kinds of special values for the Bell polynomials of the second kind for two special sequences, find a relation between these two kinds of special values for the Bell polynomials of the second kind, and derive an identity involving the combinatorial numbers.

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1. Motivation

In [1, Definition 11.2] and [2, p. 134, Theorem A], the Bell polynomials of the second kind, denoted by $\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})$ for $n\ge k\ge 0$, are defined by

$$\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})=\sum_{\substack{1\le i\le n,\ell_i\in\{0\}\cup\mathbb{N}\\ \sum_{i=1}^{n}i\ell_i=n\\ \sum_{i=1}^{n}\ell_i=k}}\frac{n!}{\prod_{i=1}^{n-k+1}\ell_i!}\prod_{i=1}^{n-k+1}\left(\frac{x_i}{i!}\right)^{\ell_i}.$$

For more information on the Bell polynomials of the second kind $\mathrm{B}_{n,k}$, please refer to the monographs and handbooks [1–3] and closely related references therin.

In [1, p. 451], the formulas

$$\mathrm{B}_{2r,k}(0,2!,\dots,0,(2r)!)=\frac{(2r)!}{k!}\binom{r-1}{k-1},$$
$$\mathrm{B}_{2r-1,k}(0,2!,\dots,(2r-2)!,0)=0,$$
$$\mathrm{B}_{2r,2s}(1!,0,\dots,(2r-1)!,0)=\frac{(2r)!}{(2s)!}\binom{r+s-1}{2s-1},$$
$$\mathrm{B}_{2r,2s-1}(1!,0,\dots,(2r-1)!,0)=0,$$

$$\mathrm{B}_{2r-1,2s-1}(1!,0,\dots,(2r-1)!,0)=\frac{(2r-1)!}{(2s-1)!}\binom{r+s-2}{2s-2},$$
$$\mathrm{B}_{2r-1,2s}(1!,0,\dots,0,(2r-1)!)=0$$

were stated, but no proof was supplied for them there.

For simplicity, we denote

$$\lambda(n,k)=\mathrm{B}_{n,k}\left(0,2!,0,4!,\dots,(n-k+1)!\frac{1+(-1)^{n-k+1}}{2}\right)$$

and

$$\mu(n,k)=\mathrm{B}_{n,k}\left(1!,0,3!,0,\dots,(n-k+1)!\frac{1-(-1)^{n-k+1}}{2}\right).$$

In terms of these notations, the above claims in [1, p. 451] can be restated as

$$\lambda(2r,k)=\frac{(2r)!}{k!}\binom{r-1}{k-1}, \qquad \lambda(2r-1,k)=0,$$
$$\mu(2r,2s)=\frac{(2r)!}{(2s)!}\binom{r+s-1}{2s-1}, \qquad \mu(2r,2s-1)=0,$$
$$\mu(2r-1,2s-1)=\frac{(2r-1)!}{(2s-1)!}\binom{r+s-2}{2s-2}, \qquad \mu(2r-1,2s=0.$$

In this paper, by methods and techniques in combinatorial analysis and the theory of special functions, we will provide alternative proofs for the above six formulas, find a relation between them, and derive an identity involving combinatorial numbers.

2. Main results

We first derive an identity involving combinatorial numbers, which will be useful in next proofs of our main results.

Theorem 1. *For $k\ge 1$ and $n\ge 0$, we have*

$$\sum_{q=0}^{n}\frac{(-1)^q}{k+q}\binom{n}{q}=\frac{1}{k\binom{k+n}{k}}. \tag{2.1}$$

Proof. Let

$$f(x)=\sum_{q=0}^{n}\frac{(-1)^q}{k+q}\binom{n}{q}x^{k+q}, \quad x\in[-1,1].$$

Then $f(0)=0$ and

$$f'(x)=\sum_{q=0}^{n}(-1)^q\binom{n}{q}x^{k+q-1}=x^{k-1}\sum_{q=0}^{n}(-1)^q\binom{n}{q}x^q=x^{k-1}(1-x)^n.$$

Integrating from 0 to $t \in (0,1]$ on both sides of the above equality yields

$$f(t) = \int_0^t x^{k-1}(1-x)^n \,\mathrm{d}x = B(t;k,n+1),$$

where $B(z;a,b)$ denotes the incomplete beta function, see [3, p. 183]. Therefore, we obtain

$$\begin{aligned} f(1) &= B(1;k,n+1) = B(k,n+1) \\ &= \frac{\Gamma(k)\Gamma(n+1)}{\Gamma(k+n+1)} = \frac{(k-1)!n!}{(k+n)!} = \frac{1}{k}\frac{k!n!}{(k+n)!} = \frac{1}{k\binom{n+k}{k}}, \end{aligned}$$

where $B(a,b)$ and $\Gamma(z)$ denote the beta function and the classical Euler gamma function respectively, see [3, p. 142] and [3, Chapter 5]. The formula (2.1) is thus proved. The proof of Theorem 1 is complete. □

We are now in a position to state and prove our main results.

Theorem 2. *For $n \ge k \ge 0$, we have the relation*

$$\frac{\mu(n,k)}{n!} = \frac{\lambda(n+k,k)}{(n+k)!} \tag{2.2}$$

and two explicit formulas

$$\lambda(n,k) = \frac{1+(-1)^n}{2}\frac{n!}{k!}\binom{\frac{n}{2}-1}{k-1} \tag{2.3}$$

and

$$\mu(n,k) = \frac{1+(-1)^{n+k}}{2}\frac{n!}{k!}\binom{\frac{n+k}{2}-1}{k-1}. \tag{2.4}$$

Proof. It is known [2, 10] that the quantities

$$\langle x\rangle_n = \prod_{\ell=0}^{n-1}(x-\ell) = \begin{cases} x(x-1)\cdots(x-n+1), & n \ge 1 \\ 1, & n = 0 \end{cases}$$

and

$$(x)_n = \prod_{\ell=0}^{n-1}(x+\ell) = \begin{cases} x(x+1)\cdots(x+n-1), & n \ge 1 \\ 1, & n = 0 \end{cases}$$

are called the falling and rising factorials respectively. In [2, p. 133], it was listed that

$$\frac{1}{k!}\left(\sum_{m=1}^{\infty} x_m \frac{t^m}{m!}\right)^k = \sum_{n=k}^{\infty} \mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})\frac{t^n}{n!} \tag{2.5}$$

for $k \ge 0$. Hence, we have

$$\sum_{n=k}^{\infty} \lambda(n,k)\frac{t^n}{n!} = \frac{1}{k!}\left[\sum_{m=1}^{\infty} m!\frac{1+(-1)^m}{2}\frac{t^m}{m!}\right]^k = \frac{1}{k!}\left(\frac{t^2}{1-t^2}\right)^k$$

which is equivalent to

$$\sum_{n=0}^{\infty} \lambda(n+k,k)\frac{1}{\binom{n+k}{n}}\frac{t^n}{n!} = \left(\frac{t}{1-t^2}\right)^k.$$

Since the function $\frac{t}{1-t^2}$ is odd, we derive $\lambda(2r-1,k) = 0$. Further computation yields

$$\begin{aligned}
\lambda(n+k,k) &= \binom{n+k}{n}\lim_{t\to 0}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(\frac{t}{1-t^2}\right)^k \\
&= \binom{n+k}{n}\frac{1}{2^k}\lim_{t\to 0}\left[\left(\frac{1}{1-t}-\frac{1}{1+t}\right)^k\right]^{(n)} \\
&= \binom{n+k}{n}\frac{1}{2^k}\lim_{t\to 0}\left[\sum_{\ell=0}^{k}\binom{k}{\ell}\left(\frac{1}{1-t}\right)^{\ell}(-1)^{k-\ell}\left(\frac{1}{1+t}\right)^{k-\ell}\right]^{(n)} \\
&= \binom{n+k}{n}\frac{1}{2^k}\sum_{\ell=0}^{k}(-1)^{k-\ell}\binom{k}{\ell}\lim_{t\to 0}\sum_{q=0}^{n}\binom{n}{q}\left[\left(\frac{1}{1-t}\right)^{\ell}\right]^{(q)}\left[\left(\frac{1}{1+t}\right)^{k-\ell}\right]^{(n-q)} \\
&= \binom{n+k}{n}\frac{1}{2^k}\sum_{\ell=0}^{k}(-1)^{k-\ell}\binom{k}{\ell}\lim_{t\to 0}\sum_{q=0}^{n}\binom{n}{q} \\
&\quad\times\langle-\ell\rangle_q(-1)^q\left(\frac{1}{1-t}\right)^{\ell+q}\langle-(k-\ell)\rangle_{n-q}\left(\frac{1}{1+t}\right)^{k-\ell+(n-q)} \\
&= \frac{(-1)^k}{2^k}\binom{n+k}{n}\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}\sum_{q=0}^{n}(-1)^q\binom{n}{q}\langle-\ell\rangle_q\langle\ell-k\rangle_{n-q} \\
&= \frac{(-1)^{n+k}}{2^k}\binom{n+k}{n}\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}\sum_{q=0}^{n}(-1)^q\binom{n}{q}(\ell)_q(k-\ell)_{n-q}.
\end{aligned}$$

In summary, we obtain

$$\lambda(n,k) = \frac{(-1)^k}{2^k}\binom{n}{k}\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}\sum_{q=0}^{n-k}(-1)^q\binom{n-k}{q}\langle-\ell\rangle_q\langle\ell-k\rangle_{n-k-q}$$

$$= \frac{(-1)^n}{2^k}\binom{n}{k}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\sum_{q=0}^{n-k}(-1)^q\binom{n-k}{q}(\ell)_q(k-\ell)_{n-k-q}. \quad (2.6)$$

By similar argument as above, it follows that

$$\sum_{n=k}^{\infty}\mu(n,k)\frac{t^n}{n!} = \frac{1}{k!}\left[\sum_{m=1}^{\infty}m!\frac{1-(-1)^m}{2}\frac{t^m}{m!}\right]^k = \frac{(-1)^k}{k!2^k}\left(\frac{1}{t-1}+\frac{1}{t+1}\right)^k$$

which is equivalent to

$$\sum_{n=0}^{\infty}\mu(n+k,k)\frac{1}{\binom{n+k}{n}}\frac{t^n}{n!} = \frac{(-1)^k}{2^k}\frac{1}{t^k}\left(\frac{1}{t-1}+\frac{1}{t+1}\right)^k = \left(\frac{1}{1-t^2}\right)^k. \quad (2.7)$$

Since $\frac{1}{1-t^2}$ is even, we deduce immediately that $\mu(2r-1,2s)=0$ and $\mu(2r,2s-1)=0$. By the L'Hôspital rule and the identity (2.1) in Theorem 1, it follows that

$$\mu(n+k,k) = \binom{n+k}{n}\frac{(-1)^k}{2^k}\lim_{t\to0}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left[\frac{1}{t^k}\left(\frac{1}{t-1}+\frac{1}{t+1}\right)^k\right]$$

$$= \binom{n+k}{n}\frac{(-1)^k}{2^k}\lim_{t\to0}\sum_{q=0}^{n}\binom{n}{q}\frac{\langle -k\rangle_q}{t^{k+q}}\left[\left(\frac{1}{t-1}+\frac{1}{t+1}\right)^k\right]^{(n-q)}$$

$$= \binom{n+k}{n}\frac{(-1)^k}{2^k}\sum_{q=0}^{n}\binom{n}{q}\frac{\langle -k\rangle_q}{(k+q)!}\lim_{t\to0}\left[\left(\frac{1}{t-1}+\frac{1}{t+1}\right)^k\right]^{(n+k)}$$

$$= \binom{n+k}{n}\frac{(-1)^k}{2^k}\sum_{q=0}^{n}\binom{n}{q}\frac{(-1)^q(k+q-1)!}{(k-1)!(k+q)!}$$

$$\times\lim_{t\to0}\left[\sum_{\ell=0}^{k}\binom{k}{\ell}\left(\frac{1}{t-1}\right)^\ell\left(\frac{1}{t+1}\right)^{k-\ell}\right]^{(n+k)}$$

$$= \binom{n+k}{n}\frac{(-1)^k}{2^k(k-1)!}\sum_{q=0}^{n}\frac{(-1)^q}{k+q}\binom{n}{q}\sum_{\ell=0}^{k}\binom{k}{\ell}$$

$$\times\lim_{t\to0}\sum_{m=0}^{n+k}\binom{n+k}{m}\left[\left(\frac{1}{t-1}\right)^\ell\right]^{(m)}\left[\left(\frac{1}{t+1}\right)^{k-\ell}\right]^{(n+k-m)}$$

$$= \frac{(-1)^k}{2^k k!}\sum_{\ell=0}^{k}\binom{k}{\ell}\lim_{t\to0}\sum_{m=0}^{n+k}\binom{n+k}{m}$$

$$\times\langle -\ell\rangle_m\left(\frac{1}{t-1}\right)^{\ell+m}\langle \ell-k\rangle_{n+k-m}\left(\frac{1}{t+1}\right)^{(k-\ell)+(n+k-m)}$$

$$= \frac{(-1)^k}{2^k k!} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \sum_{m=0}^{n+k} (-1)^m \binom{n+k}{m} \langle -\ell \rangle_m \langle \ell - k \rangle_{n+k-m}.$$

In short, we obtain

$$\mu(n,k) = \frac{(-1)^k}{2^k k!} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \sum_{m=0}^{n} (-1)^m \binom{n}{m} \langle -\ell \rangle_m \langle \ell - k \rangle_{n-m}$$
$$= \frac{(-1)^{n+k}}{2^k k!} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \sum_{m=0}^{n} (-1)^m \binom{n}{m} (\ell)_m (k-\ell)_{n-m}. \quad (2.8)$$

Comparing between (2.6) and (2.8) reveals

$$\mu(n,k) = \frac{(-1)^{n+k}}{2^k k!} \frac{2^k}{(-1)^{n+k}} \frac{1}{\binom{n+k}{k}} \lambda(n+k,k)$$

which can be rearranged as (2.2).

The Faà di Bruno formula, see [1, Theorem 11.4] and [2, p. 139, Theorem C], can be described in terms of $\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})$ by

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} f \circ h(x) = \sum_{k=0}^{n} f^{(k)}(h(x)) \mathrm{B}_{n,k}\big(h'(x), h''(x), \dots, h^{(n-k+1)}(x)\big). \quad (2.9)$$

In [1, p. 412] and [2, p. 135], one can find the identity

$$\mathrm{B}_{n,k}\big(abx_1, ab^2x_2, \dots, ab^{n-k+1}x_{n-k+1}\big)$$
$$= a^k b^n \mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1}) \quad (2.10)$$

for $n \ge k \ge 0$ and $a,b \in \mathbb{C}$. In [8, Theorem 4.1] and [12, Section 3], it was set up little by little that

$$\mathrm{B}_{n,k}(x,1,0,\dots,0) = \frac{1}{2^{n-k}} \frac{n!}{k!} \binom{k}{n-k} x^{2k-n}, \quad (2.11)$$

where $\binom{0}{0} = 1$ and $\binom{p}{q} = 0$ for $q > p \ge 0$. For detailed information on applications of the formula (2.11), please refer to the papers [4–9, 11–14] and closely related references therein. Then it follows from (2.7), (2.9), (2.10), and (2.11) that, when denoting $u = u(t) = 1 - t^2$,

$$\mu(n+k,k) = \binom{n+k}{n} \lim_{t\to 0} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \left(\frac{1}{1-t^2}\right)^k$$
$$= \binom{n+k}{n} \lim_{t\to 0} \sum_{\ell=0}^{n} \left(\frac{1}{u^k}\right)^{(\ell)} \mathrm{B}_{n,\ell}(-2t,-2,0,\dots,0)$$

$$
\begin{aligned}
&= \binom{n+k}{n} \lim_{t\to 0} \sum_{\ell=0}^{n} \langle -k\rangle_\ell \frac{1}{u^{k+\ell}} (-2)^\ell \mathrm{B}_{n,\ell}(t,1,0,\dots,0) \\
&= \binom{n+k}{n} \lim_{t\to 0} \sum_{\ell=0}^{n} \langle -k\rangle_\ell \frac{1}{(1-t^2)^{k+\ell}} (-2)^\ell \frac{1}{2^{n-\ell}} \frac{n!}{\ell!} \binom{\ell}{n-\ell} t^{2\ell-n} \\
&= \binom{n+k}{n} \lim_{t\to 0} \sum_{\ell=0}^{n} (k)_\ell \frac{n!}{\ell!} \binom{\ell}{n-\ell} \frac{(2t)^{2\ell-n}}{(1-t^2)^{k+\ell}} \\
&= \begin{cases} 0, & n=2m-1 \\ \dbinom{2m+k}{2m} (k)_m \dfrac{(2m)!}{m!} \dbinom{m}{2m-m}, & n=2m \end{cases} \\
&= \begin{cases} 0, & n=2m-1 \\ \dfrac{(2m+k)!}{k!} \dbinom{m+k-1}{k-1}, & n=2m \end{cases} \\
&= \begin{cases} 0, & n=2m-1 \\ \dfrac{(n+k)!}{k!} \dbinom{\frac{n+2k}{2}-1}{k-1}, & n=2m \end{cases}
\end{aligned}
$$

for $m \in \mathbb{N}$. The formula in (2.4) is thus proved.

Substituting the formula in (2.4) into (2.2) and simplifying lead readily to the formula in (2.3). The proof of Theorem 2 is complete. □

3. REMARKS

In this section, we state several remarks on something related.

Remark 1. In [8, Theorem 2.1], it was proved that

$$
\mathrm{B}_{n,k}\left(1,0,1,\dots,\frac{1-(-1)^{n-k+1}}{2}\right) = \frac{1}{2^k k!} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} (k-2\ell)^n
$$

and

$$
\mathrm{B}_{n,k}\left(0,1,0,\dots,\frac{1+(-1)^{n-k+1}}{2}\right) = \frac{1}{2^k k!} \sum_{\ell=0}^{2k} (-1)^\ell \binom{2k}{\ell} (k-\ell)^n
$$

for $n \ge k \ge 0$, where 0^0 is regarded as 1. In [8, Section 3], basing on numerical calculation, the authors guessed that

$$
\mathrm{B}_{2\ell-1,k}\left(0,1,0,\dots,\frac{1+(-1)^{(2\ell-1)-k+1}}{2}\right) = 0, \quad 2\ell-1 \ge k \ge 0, \tag{3.1}
$$

$$\mathrm{B}_{2\ell,k}\left(0,1,0,\dots,\frac{1-(-1)^{2\ell-k+1}}{2}\right)=0, \quad 2\ell>k>\ell, \tag{3.2}$$

$$\mathrm{B}_{2\ell,k}\left(0,1,0,\dots,\frac{1-(-1)^{2\ell-k+1}}{2}\right)\ne 0, \quad \ell\ge k\ge 1, \tag{3.3}$$

$$\mathrm{B}_{k+2\ell,k}(1,0,1,\dots,1,0,1)\ne 0, \quad k,\ell\in\mathbb{N}, \tag{3.4}$$

and

$$\mathrm{B}_{k+2\ell-1,k}(1,0,1,\dots,0,1,0)=0, \quad k,\ell\in\mathbb{N}. \tag{3.5}$$

In [8, Theorem 3.1], the identity (3.5) was proved to be true. In fact, the identities (3.1) and (3.5) can be concluded readily from the proof of [8, Theorem 2.1] as follows. From the formula (2.5), it followed that

$$\sum_{n=k}^{\infty}\mathrm{B}_{n,k}\left(0,1,0,\dots,\frac{1+(-1)^{n-k+1}}{2}\right)\frac{t^n}{n!}=\frac{(\cosh t-1)^k}{k!} \tag{3.6}$$

and

$$\sum_{n=k}^{\infty}\mathrm{B}_{n,k}\left(1,0,1,\dots,\frac{1-(-1)^{n-k+1}}{2}\right)\frac{t^n}{n!}=\frac{\sinh^k t}{k!}. \tag{3.7}$$

In (3.6), the function $\cosh t-1$ is even, so the identity (3.1) is clearly valid. Since the function $\sinh t$ in (3.7) is odd, then

$$\mathrm{B}_{2n,2k-1}(1,0,1,\dots,1,0)=0 \quad \text{and} \quad \mathrm{B}_{2n-1,2k}(1,0,1,\dots,1,0)=0$$

which are equivalent to the identity (3.5). However, the validity of the identities (3.2), (3.3), and (3.4) has not been verified yet.

Remark 2. The formula (2.8) can be rewritten as

$$\mu(n,k)=\frac{(-1)^{n+k}n!}{2^k k!} \times\sum_{\ell=0}^{k}\sum_{m=0}^{n}(-1)^{\ell+m}\binom{k}{\ell}\binom{\ell+m-1}{\ell-1}\binom{(k-\ell)+(n-m)-1}{k-\ell-1}. \tag{3.8}$$

Then combining the formulas (2.8) and (3.8) with the formula in (2.4) and rearranging arrive at

$$\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}\sum_{m=0}^{n}(-1)^{m}\binom{n}{m}(\ell)_m(k-\ell)_{n-m} =[1+(-1)^{n+k}]2^{k-1}n!\binom{\frac{n+k}{2}-1}{k-1}$$

and

$$\sum_{m=0}^{n}\sum_{\ell=0}^{k}(-1)^{\ell+m}\binom{k}{\ell}\binom{\ell+m-1}{\ell-1}\binom{(k-\ell)+(n-m)-1}{k-\ell-1} = [1+(-1)^{n+k}]2^{k-1}\binom{\frac{n+k}{2}-1}{k-1}.$$

Comparing these identities with the Vandermonde convolution formula

$$\sum_{k=0}^{n}\binom{n}{k}\langle x\rangle_k\langle y\rangle_{n-k} = \langle x+y\rangle_n$$

in [1, Theorem 3.1] and [2, p. 44] motivates us to ask a question: is the quantity

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\langle x\rangle_k\langle y\rangle_{n-k}$$

summable?

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References

[1] C. A. Charalambides, *Enumerative combinatorics*, ser. CRC Press Series on Discrete Mathematics and its Applications. Chapman & Hall/CRC, Boca Raton, FL, 2002.

[2] L. Comtet, *Advanced combinatorics*, enlarged ed. D. Reidel Publishing Co., Dordrecht, 1974, the art of finite and infinite expansions.

[3] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, Eds., *NIST handbook of mathematical functions*. U.S. Department of Commerce, National Institute of Standards and Technology, Washington, DC; Cambridge University Press, Cambridge, 2010, with 1 CD-ROM (Windows, Macintosh and UNIX).

[4] F. Qi, "Integral representations for multivariate logarithmic polynomials," *J. Comput. Appl. Math.*, vol. 336, pp. 54–62, 2018, doi: 10.1016/j.cam.2017.11.047.

[5] F. Qi and V. Cernanova, "Some discussions on a kind of improper integrals," *Int. J. Anal. Appl.*, vol. 11, pp. 101–109, 2016.

[6] F. Qi, V. Cernanova, X.-T. Shi, and B.-N. Guo, "Some properties of central Delannoy numbers," *J. Comput. Appl. Math.*, vol. 328, pp. 101–115, 2018, doi: 10.1016/j.cam.2017.07.013.

[7] F. Qi and B.-N. Guo, "Explicit and recursive formulas, integral representations, and properties of the large Schröder numbers," *Kragujevac J. Math.*, vol. 41, no. 1, pp. 121–141, 2017, doi: 10.5937/KgJMath1701121F.

[8] F. Qi and B.-N. Guo, "Explicit formulas for special values of the Bell polynomials of the second kind and for the Euler numbers and polynomials," *Mediterr. J. Math.*, vol. 14, no. 3, pp. Art. 140, 14, 2017, doi: 10.1007/s00009-017-0939-1.

[9] F. Qi, X.-T. Shi, and B.-N. Guo, "Two explicit formulas of the Schröder numbers," *Integers*, vol. 16, pp. Paper No. A23, 15, 2016.

[10] F. Qi, X.-T. Shi, and F.-F. Liu, “Several identities involving the falling and rising factorials and the Cauchy, Lah, and Stirling numbers,” *Acta Univ. Sapientiae Math.*, vol. 8, no. 2, pp. 282–297, 2016, doi: 10.1515/ausm-2016-0019.

[11] F. Qi, X.-T. Shi, F.-F. Liu, and D. V. Kruchinin, “Several formulas for special values of the Bell polynomials of the second kind and applications,” *J. Appl. Anal. Comput.*, vol. 7, no. 3, pp. 857–871, 2017, doi: 10.11948/2017054.

[12] F. Qi and M.-M. Zheng, “Explicit expressions for a family of the Bell polynomials and applications,” *Appl. Math. Comput.*, vol. 258, pp. 597–607, 2015, doi: 10.1016/j.amc.2015.02.027.

[13] C.-F. Wei and F. Qi, “Several closed expressions for the Euler numbers,” *J. Inequal. Appl.*, pp. 2015:219, 8, 2015, doi: 10.1186/s13660-015-0738-9.

[14] J.-L. Zhao and F. Qi, “Two explicit formulas for the generalized Motzkin numbers,” *J. Inequal. Appl.*, pp. Paper No. 44, 8, 2017, doi: 10.1186/s13660-017-1313-3.

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