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SECOND ORDER LINEAR RECURSIONS WHOSE SUBSCRIPTS ARE A POWER

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Abstract. We consider two kinds of second order linear recurrences whose subscripts are powers and present certain new identities including these recurrences. Furthermore, we derive first-order nonlinear homogeneous recurrence relations for these recurrences. Our results generalize earlier results as well as they provide new solutions for certain uncompleted cases of the literature.

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Keywords: Fibonacci and Lucas number, recurrence relation, polynomial

1. INTRODUCTION

Let p be a nonzero integer such that $\Delta = p^2+4 \neq 0$. Define the generalized Fibonacci type $\{u_n\}$ and Lucas type $\{v_n\}$ sequences as follows:

$$u_n = pu_{n-1} + u_{n-2}$$

and

$$v_n = pv_{n-1} + v_{n-2},$$

where $u_0 = 0, u_1 = 1$ and $v_0 = 2, v_1 = p$, respectively.

If the roots of $x^2 - px - 1 = 0$ are α and β, then the Binet forms of $\{u_n\}$ and $\{v_n\}$ are

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } v_n = \alpha^n + \beta^n.$$

If $p = 1$, then $u_n = F_n$ (nth Fibonacci number) and $v_n = L_n$ (nth Lucas number) respectively.

Usiskin [7, 8] suggested the following problems: For $n > 0$, show that

$$F_{3^n} = \prod_{k=0}^{n-1} L_{2.3^k} - 1 \tag{1.1}$$

and

$$L_{3^n} = \prod_{k=0}^{n-1} L_{2.3^k} + 1. \tag{1.2}$$

In [4], the author asked for the solution of the first order cubic recurrence relation:

$$a_{n+1} = 5a_n^3 - 3a_n \tag{1.3}$$

with $a_0 = 1$.

Then in [6], the solution is given as $a_n = F_{3^n}$. The same problem appeared as Problem 1809 in Crux Mathematicorum 20 (1994): 19-20.

In the same issue, there was a proposal to solve the recurrence

$$P_{n+1} = 25P_n^5 - 25P_n^3 + 5P_n, \quad P_0 = 1.$$

The solution was given as $P_n = 5^n$. Also the following recurrences and their solutions were commented by Murray S. Klamkin as an addendum to the solution of the problem given in [6]:

$$\begin{aligned} A_{n+1} &= A_n^2 - 2, \quad A_1 = 3, \\ B_{n+1} &= B_n^4 - 4B_n^2 + 2, \quad B_1 = 7, \\ C_{n+1} &= C_n^6 - 6C_n^4 + 9C_n^2 - 2, \; C_1 = 18, \end{aligned}$$

where $A_n = L_{2^n}$, $B_n = L_{4^n}$ and $C_n = L_{6^n}$.

In [1], the author presented some identities involving Fibonacci numbers of the form F_{k^n} for positive odd k and gave a first-order nonlinear homogeneous recurrence relation for F_{k^n}, which generalized (1.3), (1.1) and (1.2).

Recently in [2], *Helmut Prodinger* gave a general expansion formula for a sum of powers of Fibonacci numbers, as considered by Melham, as well as some extensions.

In this paper, we consider two generalized second order recursion sequences and then generalize the results of [1] for the odd k case as well as derive a new first-order nonlinear homogeneous recurrence relation for the sequence $\{u_{k^n}\}$ for possible even k. Further we present that the generalized Lucas numbers $v_{k^{n+1}}$ is a polynomial of generalized Fibonacci numbers u_{k^n} of degree k for even k.

2. Main Results

In this section, in order to derive a recurrence relation for both even and odd subscripted terms u_{k^n}, we start with the following result.

Proposition 1. *For $n \geq 1$ and even k,*

$$u_{k^n} = u_k \prod_{i=1}^{n-1} \left(\sum_{j=1}^{k/2} v_{(2j-1)k^i} \right).$$

Proof. Consider

$$u_{k^n} = u_k \frac{u_{k^2}}{u_k} \frac{u_{k^3}}{u_{k^2}} \cdots \frac{u_{k^n}}{u_{k^{n-1}}} = u_k \prod_{i=1}^{n-1} \frac{u_{k^{i+1}}}{u_{k^i}}. \tag{2.1}$$

From [5], we have that for even $k \geq 2$,

$$\frac{m^k - n^k}{m-k} = \sum_{j=0}^{k/2-1} (mn)^i \left(m^{k-2i-1} + n^{k-2i-1}\right) \tag{2.2}$$

When $m = \alpha^n, n = \beta^n$ in (2.2), we get

$$\frac{u_{kn}}{u_n} = \sum_{j=0}^{k/2-1} (\alpha\beta)^{jn} \left(\alpha^{(k-2j-1)n} + \beta^{(k-2j-1)n}\right) = \sum_{j=0}^{k/2-1} (-1)^{jn} v_{(k-2j-1)n}$$

$$\frac{u_{k^{i+1}}}{u_{k^i}} = \sum_{j=0}^{k/2-1} (-1)^{jk^i} v_{(k-2j-1)k^i} = \sum_{j=1}^{k/2} v_{(2j-1)k^i} \tag{2.3}$$

By (2.3), equation (2.1) equals to

$$u_{k^n} = u_k \prod_{i=1}^{n-1} \frac{u_{k^{i+1}}}{u_{k^i}} = u_k \prod_{i=1}^{n-1} \left(\sum_{j=1}^{k/2} v_{(2j-1)k^i} \right).$$

Thus the proof is complete. □

For the Fibonacci and Lucas case when $p = 1$, we also refer to [9].
For later use, we give the following two identities:

$$\begin{aligned}\sum_{i=1}^{r} (-1)^i v_{ai} &= \sum_{i=1}^{r} (-1)^i \alpha^{ai} + \sum_{i=1}^{r} (-1)^i \beta^{ai} \\ &= \frac{-\alpha^a + (-1)^r (\alpha^a)^{r+1}}{1+\alpha^a} + \frac{-\beta^a + (-1)^r (\beta^a)^{r+1}}{1+\beta^a} \\ &= \frac{(-1)^{a+r} v_{ar} + (-1)^r v_{a(r+1)} - v_a - 2(-1)^a}{v_a + 1 + (-1)^a}\end{aligned} \tag{2.4}$$

and

$$v_{a+b} - (-1)^b v_{a-b} = \Delta u_a u_b. \tag{2.5}$$

Proposition 2. *For odd k and $n \geq 1$,*

$$u_{k^n} = (-1)^{(n-1)(k-1)/2} u_k \prod_{i=1}^{n-1} \left(1 + \sum_{j=1}^{(k-1)/2} (-1)^j v_{2k^i j} \right).$$

Proof. If $(k-1)/2$ is even, we write by (2.4) and (2.5)

$$u_{k^n} = u_k \prod_{i=1}^{n-1} \frac{u_{k^{i+1}}}{u_{k^i}} = u_k \prod_{i=1}^{n-1} \left(1 + \frac{\Delta u_{k^{i+1}} u_{k^i}}{\Delta u_{k^i} u_{k^i}} - 1 \right)$$

$$
\begin{aligned}
&= u_k \prod_{i=1}^{n-1}\left(1+\frac{v_{k^{i+1}+k^i}-(-1)^{k^i} v_{k^{i+1}-k^i}}{v_{k^i+k^i}-(-1)^{k^i} v_{k^i-k^i}}-1\right)\\
&= u_k \prod_{i=1}^{n-1}\left(1+\frac{v_{2k^i\frac{k+1}{2}}+v_{2k^i\frac{k-1}{2}}-v_{2k^i}-2}{v_{2k^i}+2}\right)\\
&= u_k \prod_{i=1}^{n-1}\left(1+\sum_{j=1}^{(k-1)/2}(-1)^j v_{2k^i j}\right)\\
&= (-1)^{(n-1)(k-1)/2} u_k \prod_{i=1}^{n-1}\left(1+\sum_{j=1}^{(k-1)/2}(-1)^j v_{2k^i j}\right).
\end{aligned}
$$

If $(k-1)/2$ is odd, then we write

$$
\begin{aligned}
u_{k^n} &= (-1)^{n-1} u_k \prod_{i=1}^{n-1}\frac{u_{k^{i+1}}}{u_{k^i}} = (-1)^{n-1} u_k \prod_{i=1}^{n-1}\left(1-\frac{\Delta u_{k^{i+1}} u_{k^i}}{\Delta u_{k^i}}-1\right)\\
&= (-1)^{n-1} u_k \prod_{i=1}^{n-1}\left(1+\frac{-v_{2k^i\frac{k+1}{2}}-v_{2k^i\frac{k-1}{2}} - v_{2k^i}-2}{v_{2k^i}+2}\right)\\
&= (-1)^{n-1} u_k \prod_{i=1}^{n-1}\left(1+\sum_{k=1}^{(k-1)/2}(-1)^j v_{2k^i j}\right).
\end{aligned}
$$

So we have proved the conclusion for all cases. □

3. RECURRENCE RELATION FOR $\{u_{k^n}\}$

We shall derive recurrence relations for $\{u_{k^n}\}$ or $\{v_{k^n}\}$ for odd or even k. Thus we need the following result:

Lemma 1. *For $n,q \geq 0$,*

$$
\begin{aligned}
u_{(2q+1)n} &= u_n \sum_{k=0}^{q}(-1)^{n(q+k)}\frac{2q+1}{q+k+1}\left(p^2+4\right)^k\binom{q+k+1}{2k+1}u_n^{2k},\\
v_{2qn} &= \sum_{k=0}^{q}(-1)^{n(q+k)}\frac{2q}{q+k}\binom{q+k}{2k}\left(p^2+4\right)^k u_n^{2k},\\
v_{(2q+1)n} &= v_n \sum_{k=0}^{q}(-1)^{(n+1)(q+k)}\frac{2q+1}{q+k+1}\binom{q+k+1}{2k+1}v_n^{2k},
\end{aligned}
$$

$$v_{2qn} = \sum_{k=0}^{q} (-1)^{(n+1)(q+k)} \frac{2q}{q+k} \binom{q+k}{2k} v_n^{2k}.$$

Proof. The proof can be easily obtained from [5] by considering the classical binomial expansions for $a^n - b^n$ and $a^n + b^n$ where a and b are real numbers. □

We give a recurrence relation for the sequence $\{u_{k^n}\}$ for odd k.

Proposition 3. *For odd $k > 0$ and $n \geq 0$,*

$$u_{k^{n+1}} = \triangle^{(k-1)/2} u_{k^n}^k - \sum_{i=0}^{(k-3)/2} \triangle^i C_{i,k} u_{k^n}^{2i+1}$$

where the coefficients $C_{i,k}$ are given by

$$C_{i,k} = (-1)^{(k+1)/2+i} \frac{2k}{k+2i+1} \binom{(k+1)/2+i}{2i+1}$$

for $0 \leq i \leq (k-3)/2$

Proof. Consider

$$u_{k^n}^k = \frac{1}{\triangle^{k/2}} \sum_{j=0}^{k} \binom{k}{j} (-1)^j \beta^{jk^n} \alpha^{(k-j)k^n} \tag{3.1}$$

$$= \frac{1}{\triangle^{(k-1)/2}} \left(u_{k^{n+1}} + \sum_{j=1}^{(k-1)/2} \binom{k}{j} u_{(k-2j)k^n} \right),$$

where Δ is defined as before. By (3.1), we obtain for odd k,

$$u_{k^{n+1}} = \triangle^{(k-1)/2} u_{k^n}^k - \sum_{j=1}^{(k-1)/2} \binom{k}{j} u_{(k-2j)k^n}. \tag{3.2}$$

Then by (3.2) and Lemma 1, we write

$$u_{k^{n+1}} = \triangle^{(k-1)/2} u_{k^n}^k - \sum_{j=1}^{\frac{k-1}{2}} \sum_{i=0}^{\frac{k-1}{2}-j} (-1)^{\frac{k-1}{2}+i-j} \triangle^i \binom{k}{j} \binom{\frac{k+1}{2}+i-j}{2i+1} \frac{k-2j}{\frac{k+1}{2}+i-j} u_{k^n}^{2i+1}$$

which, after reversing the order of summation, can be rewritten as

$$u_{k^{n+1}} = \triangle^{(k-1)/2} u_{k^n}^k - \sum_{i=0}^{(k-1)/2} \triangle^i A_{i,k} u_{k^n}^{2i+1}, \tag{3.3}$$

where

$$A_{i,k}=\sum_{j=1}^{(k-1)/2-i}(-1)^{(k-1)/2+i-j}\binom{k}{j}\binom{(k+1)/2+i-j}{2i+1}\frac{k-2j}{(k+1)/2+i-j}.$$

Since $A_{(k-1)/2,k}=0$, the equality (3.3) becomes

$$u_{k^{n+1}}=\triangle^{(k-1)/2}u_{k^n}^{k}-\sum_{i=0}^{(k-3)/2}\triangle^{i}A_{i,k}u_{k^n}^{2i+1}.$$

From (pp. 58, [3]), we have the combinatorial identity: for $1\le m\le(k-3)/2$

$$\sum_{j=1}^{m}(-1)^{j}\frac{k-2j}{k-m-j}\binom{k}{j}\binom{k-m-j}{m-j}=(-1)^{m}\frac{k}{k-m}\binom{k-m}{m}. \tag{3.4}$$

If we replace m by $\frac{k-1}{2}-i$ in (3.4), then we obtain $C_{i,k}=A_{i,k}$. Thus the proof is complete. □

In a similar manner, we may give the following result:

Proposition 4. *For $n>0$ and odd $k>1$,*

$$u_{k^{n+1}}=\Delta^{\frac{k-1}{2}}u_{k^n}^{k}-\sum_{i=0}^{\frac{k-3}{2}}(-1)^{\frac{k+1}{2}+i}\frac{2k}{k-2i-1}\binom{\frac{k-1}{2}+i}{2i+1}\Delta^{i}u_{k^n}^{2i+1}.$$

Proof. For odd k, we get

$$u_{k^n}^{k}=\frac{1}{\Delta^{k}}\sum_{j=0}^{k}\binom{k}{j}(-1)^{j}\beta^{jk^n}\alpha^{(k-j)k^n}$$

$$=\frac{1}{\Delta^{\frac{k-1}{2}}}\left(u_{k^{n+1}}+\sum_{j=1}^{\frac{k-1}{2}}\binom{k}{j}u_{(k-2j)k^n}\right).$$

So we write

$$u_{k^{n+1}}=\Delta^{\frac{k-1}{2}}u_{k^n}^{k}-\sum_{j=1}^{\frac{k-1}{2}}\binom{k}{j}u_{(k-2j)k^n}.$$

Using Lemma 1 and reversing the order of summation, we write

$$u_{k^{n+1}}=\Delta^{\frac{k-1}{2}}u_{k^n}^{k}-u_{k^n}\sum_{j=1}^{\frac{k-1}{2}}\sum_{i=0}^{\frac{k-1}{2}-j}(-1)^{\frac{k-1}{2}+i-j}$$
$$\times\frac{2(k-2j)}{k-2j+2i+1}\binom{k}{j}\binom{\frac{k-1}{2}+i-j+1}{2i+1}\Delta^{i}u_{k^n}^{2i}$$

$$= \Delta^{\frac{k-1}{2}} u_{k^n}^k - u_{k^n} \sum_{i=0}^{\frac{k-3}{2}} \sum_{j=1}^{\frac{k-1}{2}-i} (-1)^{\frac{k-1}{2}+i-j} \times \frac{2(k-2j)}{k-2j+2i+1} \binom{k}{j} \binom{\frac{k-1}{2}+i-j+1}{2i+1} \Delta^i u_{k^n}^{2i}.$$

By simplifying, we derive

$$u_{k^{n+1}} = \Delta^{\frac{k-1}{2}} u_{k^n}^k - u_{k^n} \sum_{i=0}^{\frac{k-3}{2}} (-1)^{\frac{k+1}{2}+i} \frac{2k}{k-2i-1} \binom{\frac{k-1}{2}+i}{2i+1} \Delta^i u_{k^n}^{2i}.$$

□

We give a recurrence relation for the sequence $\{v_{k^n}\}$ for odd k.

Proposition 5. *For $n > 0$ and odd $k > 1$,*

$$v_{k^{n+1}} = v_{k^n}^k - \sum_{i=0}^{\frac{k-1}{2}} E_{i,k} v_{k^n}^{2i+1},$$

where

$$E_{i,k} = \frac{2k}{-k+2i+1} \binom{\frac{k-1}{2}+i}{2i+1}$$

for $0 \le i < (k-1)/2$.

Proof. By the Binet formula of $\{v_n\}$ and the binomial expansion, we write

$$v_{k^n}^k = \sum_{j=0}^{k} \binom{k}{j} \beta^{jk^n} \alpha^{(k-j)k^n}$$
$$= v_{k^{n+1}} + \sum_{j=1}^{\frac{k-1}{2}} (-1)^j \binom{k}{j} v_{(k-2j)k^n}.$$

By Lemma 1, we write

$$v_{k^{n+1}} = v_{k^n}^k - \sum_{j=1}^{\frac{k-1}{2}} \sum_{i=0}^{\frac{k-1}{2}-j} (-1)^j \binom{k}{j} \binom{\frac{k+1}{2}+i-j}{2i+1} \frac{k-2j}{\frac{k+1}{2}+i-j} v_{k^n}^{2i+1}$$

and by reversing the order of summation, we get

$$v_{k^{n+1}} = v_{k^n}^k - \sum_{i=0}^{\frac{k-1}{2}} \sum_{j=1}^{\frac{k-1}{2}-i} (-1)^j \binom{k}{j} \binom{\frac{k+1}{2}+i-j}{2i+1} \frac{k-2j}{\frac{k+1}{2}+i-j} v_{k^n}^{2i+1}$$

which, by the definition of $C_{i,k}$, gives

$$v_{k^{n+1}}=v_{k^n}^k-\sum_{i=0}^{\frac{k-3}{2}}\sum_{j=1}^{\frac{k-1}{2}-i}(-1)^j\binom{k}{j}\binom{\frac{k+1}{2}+i-j}{2i+1}\frac{k-2j}{\frac{k+1}{2}+i-j}v_{k^n}^{2i+1}.$$

If we take $m=\frac{k-1}{2}-i$ in (3.4), we get

$$v_{k^{n+1}}=v_{k^n}^k-\sum_{i=0}^{\frac{k-3}{2}}E_{i,k}v_{k^n}^{2i+1},$$

where

$$E_{i,k}=\frac{2k}{-k+2i+1}\binom{\frac{k-1}{2}+i}{2i+1}.$$

□

For example, when $k=5$,

$$v_{5^{n+1}}=v_{5^n}^5+5v_{5^n}^3+5v_{5^n}^1. \tag{3.5}$$

We give a recurrence relation for the sequence $\{v_{k^n}\}$ for even k.

Proposition 6. *For n and even $k>0$,*

$$v_{k^{n+1}}=v_{k^n}^k-\sum_{i=0}^{\frac{k-2}{2}}H_{i,k}v_{k^n}^{2i},$$

where for $1\le m\le k/2$

$$H_{i,k}=(-1)^{(\frac{k}{2}+i)}\binom{\frac{k}{2}+i-1}{2i}\frac{2k}{-k+2i}.$$

Proof. It is easy to see that

$$\begin{aligned}v_{k^n}^k&=\sum_{j=0}^{k}\binom{k}{j}\beta^{jk^n}\alpha^{(k-j)k^n}\\&=v_{k^{n+1}}-\binom{k}{\frac{k}{2}}+\sum_{j=1}^{\frac{k}{2}}\binom{k}{j}v_{(k-2j)k^n},\end{aligned}$$

Then by Lemma 1 and reversing the order of summation, we get

$$v_{k^n}^k=v_{k^{n+1}}-\binom{k}{\frac{k}{2}}+\sum_{j=1}^{\frac{k}{2}}\binom{k}{j}v_{(k-2j)k^n}$$

$$
\begin{aligned}
&= v_{k^{n+1}} + \binom{k}{\frac{k}{2}} + \sum_{j=1}^{\frac{k}{2}} \sum_{i=0}^{\frac{k}{2}-j} (-1)^{\left(\frac{k}{2}-j+i\right)} \binom{k}{j} \binom{\frac{k}{2}-j+i}{2i} \frac{2\left(\frac{k}{2}-j\right)}{\frac{k}{2}-j+i} v_{k^n}^{2i} \\
&= v_{k^{n+1}} + \sum_{i=0}^{\frac{k-2}{2}} \sum_{j=1}^{\frac{k}{2}-i} (-1)^{\left(\frac{k}{2}-j+i\right)} \binom{k}{j} \binom{\frac{k}{2}-j+i}{2i} \frac{2\left(\frac{k}{2}-j\right)}{\frac{k}{2}-j+i} v_{k^n}^{2i} \\
&= v_{k^{n+1}} + \sum_{i=0}^{\frac{k-2}{2}} (-1)^{\left(\frac{k}{2}+i\right)} \binom{\frac{k}{2}+i-1}{2i} \frac{2k}{-k+2i} v_{k^n}^{2i}.
\end{aligned}
$$

Thus the proof is complete. □

When $k = 6$, we get

$$
v_{6^{n+1}} = v_{6^n}^6 - 6v_{6^n}^4 + 9v_{6^n}^2 - 2v_{6^n}^0. \tag{3.6}
$$

Here we note that the coefficients of the formulas in (3.5) and (3.6) with adjusted sings appears to be the terms of the sequence *A034807* in the OEIS.

4. A Polynomial Representations for $v_{k^{n+1}}$

In this section, we show that the generalized Lucas numbers $v_{k^{n+1}}$ are polynomials of the generalized Fibonacci numbers u_{k^n} of degree k for even k.

Proposition 7. *For even $k > 0$ and $n \geq 0$,*

$$
v_{k^{n+1}} = \sum_{i=0}^{\frac{k-2}{2}} D_{i,k} \Delta^i u_{k^n}^{2i},
$$

where $D_{i,k}$ is given by

$$
D_{i,k} = \frac{2k}{k+2i} \binom{i+\frac{k}{2}}{2i}
$$

for $1 \leq m < k/2$.

Proof. Consider

$$
\begin{aligned}
u_{k^n}^k &= \frac{1}{\Delta^{\frac{k}{2}}} \sum_{j=0}^{k} \binom{k}{j} (-1)^j \beta^{jk^n} \alpha^{(k-j)k^n} \\
&= \frac{1}{\Delta^{\frac{k}{2}}} \left((-1)^{\frac{k}{2}} \binom{k}{k/2} + v_{k^{n+1}} - (-1)^{\frac{k}{2}} \binom{k}{k/2} v_0 \right. \\
&\quad \left. + \sum_{j=1}^{\frac{k}{2}} (-1)^j \binom{k}{j} v_{(k-2j)k^n} \right)
\end{aligned}
$$

$$= \frac{1}{\triangle^{\frac{k}{2}}}\left(v_{k^{n+1}} - (-1)^{\frac{k}{2}}\binom{k}{k/2} + \sum_{j=1}^{\frac{k}{2}} (-1)^j \binom{k}{j} v_{(k-2j)k^n}\right)$$

After using Lemma 1 and reversing the order of summation, we get for even k,

$$u_{k^n}^k = \frac{1}{\triangle^{\frac{k}{2}}}\left(v_{k^{n+1}} + (-1)^{\frac{k}{2}}\binom{k}{k/2}\right.$$
$$\left. + \sum_{j=1}^{\frac{k-2}{2}} \sum_{i=0}^{\frac{k}{2}-j} \left((-1)^j \frac{2\left(\frac{k}{2}-j\right)}{\left(\frac{k}{2}-j\right)+i}\binom{k}{j}\binom{\left(\frac{k}{2}-j\right)+i}{2i}\triangle^i u_{k^n}^{2i}\right)\right)$$

which becomes

$$u_{k^n}^k = \frac{1}{\triangle^{\frac{k}{2}}}\left(v_{k^{n+1}} + \sum_{i=0}^{\frac{k-2}{2}} \sum_{j=1}^{\frac{k}{2}-i} \left((-1)^j \frac{2\left(\frac{k}{2}-j\right)}{\left(\frac{k}{2}-j\right)+i}\binom{k}{j}\binom{\left(\frac{k}{2}-j\right)+i}{2i}\triangle^i u_{k^n}^{2i}\right)\right).$$

If we take $m = \frac{k}{2} - i$ in (3.4) for $1 \le m \le k/2$, the last equation takes the form:

$$u_{k^n}^k = \frac{1}{\triangle^{\frac{k}{2}}}\left(v_{k^{n+1}} + \sum_{i=0}^{\frac{k-2}{2}} D_{i,k} \triangle^i u_{k^n}^{2i}\right)$$

where $D_{i,k}$ is the right hand side of (3.4) for $m = \frac{k}{2} - i$, that is,

$$D_{i,k} = \left(\frac{-2k}{k+2i}\right)\binom{i+\frac{1}{2}k}{2i}.$$

Thus we have proved the conclusion. □

When $k = 6$, we have that

$$v_{6^{n+1}} = 125u_{6^n}^6 + 150u_{6^n}^4 + 45u_{6^n}^2 + 2. \tag{4.1}$$

Note that the coefficients of the formula in (4.1) with adjusted sings appears to be the terms of the sequence *A104064* in the OEIS.

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