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GENERALIZED CONVEXITY OF THE INVERSE HYPERBOLIC COSINE FUNCTION

YUE HE AND GENDI WANG

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Abstract. The generalized convexity of the inverse hyperbolic cosine function related to the hyperbolic metric is investigated in this paper.

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1. Introduction

The hyperbolic functions and their inverses play an important role in the study of the hyperbolic geometry and quasiconformal mappings [1, 4, 5, 8, 9, 11, 12]. For example, the explicit formulas for the hyperbolic metric in the unit disk $\mathbb{B}^2$ and the upper half plane $\mathbb{H}^2$ are given in terms of the inverse hyperbolic sine and cosine functions, respectively, as follows [5, p.35, p.40]:

$$\rho_{\mathbb{B}^2}(x,y) = 2\operatorname{arsh}\frac{|x-y|}{\sqrt{(1-|x|^2)(1-|y|^2)}},$$

$$\rho_{\mathbb{H}^2}(x,y) = \operatorname{arch}\left(1+\frac{|x-y|^2}{2\operatorname{Im}x\operatorname{Im}y}\right).$$

In recent papers [9, 11], the authors investigated the properties of hyperbolic Lambert quadrilaterals in the unit disk by studying the inverse hyperbolic tangent and sine functions.

The study of the convexity/concavity with respect to Hölder means, or simply $H_{p,q}$-convexity/concavity, of special functions has attracted attentions of many researchers, see [2, 6, 7, 9–11, 13–17]. In particular, the convexity/concavity of the inverse hyperbolic tangent and sine functions has been studied in [9] and [11], respectively. For the definition of the above so-called generalized convexity/concavity, the reader is referred to Section 2.

In this paper, we continue the work of [9, 11] to study the generalized convexity for the inverse hyperbolic cosine function. Our main result is stated in the following theorem.

Theorem 1. *For $p,q \in \mathbb{R}$, the inverse hyperbolic cosine function* arch *is strictly $H_{p,q}$-convex on $(1,+\infty)$ if and only if $(p,q) \in D_1$, while* arch *is strictly $H_{p,q}$-concave on $(1,+\infty)$ if and only if $(p,q) \in D_2 \cup D_3$, where*

$$D_1 = \{(p,q) | -\infty < p \le 0, 2 \le q < +\infty\},$$

$$D_2 = \{(p,q) | 0 \le p \le \frac{2}{3}, -\infty < q \le C(p)\},$$

$$D_3 = \{(p,q) | \frac{2}{3} < p < +\infty, -\infty < q \le 2\},$$

and $C(p)$ is the same as in Lemma 3(4) with $C(0) = 1$ and $C(\frac{2}{3}) = 2$. In particular, for all $x, y \in (1,+\infty)$, there hold

$$\operatorname{arch}\sqrt{xy} \le \sqrt{\frac{\operatorname{arch}^2 x + \operatorname{arch}^2 y}{2}} \le \operatorname{arch}\sqrt{\left(\frac{\sqrt[3]{x^2} + \sqrt[3]{y^2}}{2}\right)^3}, \tag{1.1}$$

with equalities if and only if $x = y$.

2. Preliminaries

For $r,s \in (0,+\infty)$, the *Hölder mean of order p* is defined by

$$H_p(r,s) = \left(\frac{r^p + s^p}{2}\right)^{\frac{1}{p}} \text{ for } p \ne 0, \quad H_0(r,s) = \sqrt{rs}.$$

For $p = 1$, we get the arithmetic mean $A = H_1$; for $p = 0$, the geometric mean $G = H_0$; and for $p = -1$, the harmonic mean $H = H_{-1}$. It is well known that $H_p(r,s)$ is continuous and increasing with respect to p.

A function $f\colon I \to J$ is called *$H_{p,q}$-convex (concave)* if it satisfies

$$f\Big(H_p(r,s)\Big) \le (\ge) H_q\Big(f(r), f(s)\Big)$$

for all $r,s \in I$, and strictly *$H_{p,q}$-convex (concave)* if the inequality is strict except for $r = s$.

The following monotone form of *l'Hôpital's rule* is of great use in deriving monotonicity properties and obtaining inequalities. See the extensive bibliography of [3].

Lemma 1 ([1, Theorem 1.25])**.** *For $-\infty < a < b < \infty$, let functions $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$, and be differentiable on (a,b). Let $g'(x) \ne 0$ on (a,b). If $f'(x)/g'(x)$ is increasing (deceasing) on (a,b), then so are*

$$\frac{f(x) - f(a)}{g(x) - g(a)} \quad and \quad \frac{f(x) - f(b)}{g(x) - g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

We prove the following three lemmas before giving the proof of Theorem 1.

Lemma 2. *Let* $r \in (1,+\infty)$.
(1)The function $f_1(r) \equiv \frac{\operatorname{arch} r}{\sqrt{r^2-1}}$ *is strictly decreasing with range* $(0,1)$*;*
(2)The function $f_2(r) \equiv 2 + \frac{\operatorname{arch} r + r\sqrt{r^2-1} - 2r^3\sqrt{r^2-1}}{(r^2-1)\left(\operatorname{arch} r + r\sqrt{r^2-1}\right)}$ *is strictly decreasing with range* $(0, \frac{2}{3})$.

Proof. (1) Let $f_{11}(r) = \operatorname{arch} r$ and $f_{12}(r) = \sqrt{r^2-1}$, then $f_{11}(1^+) = f_{12}(1^+) = 0$. By differentiation, we have

$$\frac{f_{11}'(r)}{f_{12}'(r)} = \frac{1}{r},$$

which is strictly decreasing. Hence by Lemma 1, the function f_1 is strictly decreasing with $f_1(1^+) = 1$ and $f_1(+\infty) = 0$.

(2) Let $f_{21} = \operatorname{arch} r + r\sqrt{r^2-1} - 2r^3\sqrt{r^2-1}$ and $f_{22} = (r^2-1)\left(\operatorname{arch} r + r\sqrt{r^2-1}\right)$, then $f_{21}(1^+) = f_{22}(1^+) = 0$. By differentiation, we have

$$\frac{f_{21}'(r)}{f_{22}'(r)} = \frac{-4}{2 + \frac{f_1(r)}{r}},$$

which is strictly decreasing by (1). Hence by Lemma 1, the function f_2 is strictly decreasing with $f_2(1^+) = \frac{2}{3}$ and $f_2(+\infty) = 0$. □

Lemma 3. *For* $p \in \mathbb{R}$ *and* $r \in (1,+\infty)$*, define*

$$h_p(r) = 1 + p\sqrt{r^2-1} \cdot \frac{\operatorname{arch} r}{r} + \frac{1}{\sqrt{r^2-1}} \cdot \frac{\operatorname{arch} r}{r}.$$

(1) If $p \geq \frac{2}{3}$*, then* h_p *is strictly increasing with range* $(2,+\infty)$.
(2) If $p < 0$*, then* h_p *is strictly decreasing with range* $(-\infty, 2)$.
(3) If $p = 0$*, then* h_p *is strictly decreasing with range* $(1,2)$.
(4) If $0 < p < \frac{2}{3}$*, then* h_p *is not monotone and the range of* h_p *is* $[C(p),+\infty)$*, where*

$$C(p) = \min_{r \in (1,+\infty)} h_p(r)$$

with $1 < C(p) < 2$.

Proof. By Lemma 2(1), it is easy to get

$$h_p(1^+) = 2 \quad \text{and} \quad h_p(+\infty) = \begin{cases} +\infty, & p > 0, \\ 1, & p = 0, \\ -\infty, & p < 0. \end{cases}$$

By differentiation, we have

$$h_p'(r) = \frac{1}{r}\left(1 + \frac{\operatorname{arch} r}{r\sqrt{r^2-1}}\right)(p - f_2(r)),$$

where $f_2(r)$ is the same as in Lemma 2(2).

By Lemma 2(2), we have (1)–(3).

(4) If $0 < p < \frac{2}{3}$, since the range of f_2 is $(0, \frac{2}{3})$, there exists one and only one point $r_p \in (1, +\infty)$ such that $p = f_2(r_p)$. Then h_p is strictly decreasing on $(1, r_p)$ and increasing on $(r_p, +\infty)$. Since h_p is continuous in r, there exists

$$C(p) = \min_{r \in (1,+\infty)} h_p(r)$$

and $1 < C(p) < 2$. □

Lemma 4. *Let $p, q \in \mathbb{R}$, $r \in (1, +\infty)$, and $C(p)$ be the same as in Lemma 3(4). Let*

$$g_{p,q}(r) = \frac{\operatorname{arch}^{q-1} r}{r^{p-1}\sqrt{r^2-1}}.$$

(1) If $p \geq \frac{2}{3}$, then $g_{p,q}$ is strictly decreasing for each $q \leq 2$, and $g_{p,q}$ is not monotone for any $q > 2$.
(2) If $p < 0$, then $g_{p,q}$ is strictly increasing for each $q \geq 2$, and $g_{p,q}$ is not monotone for any $q < 2$.
(3) If $p = 0$, then $g_{p,q}$ is strictly increasing for each $q \geq 2$, and $g_{p,q}$ is strictly decreasing for each $q \leq 1$, and $g_{p,q}$ is not monotone for any $1 < q < 2$.
(4) If $0 < p < \frac{2}{3}$, then $g_{p,q}$ is strictly decreasing for each $q \leq C(p)$, and $g_{p,q}$ is not monotone for any $q > C(p)$.

Proof. By logarithmic differentiation in r, we have

$$\frac{g'_{p,q}(r)}{g_{p,q}(r)} = \frac{1}{\sqrt{r^2-1}\cdot \operatorname{arch} r}\left(q - h_p(r)\right),$$

where $h_p(r)$ is the same as in Lemma 3. Hence the results immediately follow from Lemma 3. □

3. Proof of main result

We are now in a position to prove Theorem 1.

Proof of Theorem 1. Without loss of generality, we may assume that $1 < x \leq y < +\infty$. Let $t = H_p(x, y)$, then $x \leq t \leq y$ and

$$\frac{\partial t}{\partial x} = \frac{1}{2}\left(\frac{x}{t}\right)^{p-1}.$$

The proof is divided into the following four cases.
Case 1. $p \neq 0$ and $q \neq 0$.
Define

$$F(x, y) = \operatorname{arch}^q\left(H_p(x, y)\right) - \frac{\operatorname{arch}^q x + \operatorname{arch}^q y}{2}.$$

By differentiation, we have

$$\frac{\partial F}{\partial x} = \frac{q}{2}x^{p-1}\left(\frac{\operatorname{arch}^{q-1} t}{t^{p-1}\sqrt{t^2-1}} - \frac{\operatorname{arch}^{q-1} x}{x^{p-1}\sqrt{x^2-1}}\right) = \frac{q}{2}x^{p-1}\left(g_{p,q}(t) - g_{p,q}(x)\right),$$

where $g_{p,q}$ is defined in Lemma 4.

Case 1.1 $p \geq \frac{2}{3}$ and $q \leq 2$.

By Lemma 4(1), the function $g_{p,q}$ is strictly decreasing on $(1,+\infty)$.

Case 1.1.1 If $q > 0$, then $\frac{\partial F}{\partial x} \leq 0$. Hence $F(x,y)$ is strictly decreasing and $F(x,y) \geq F(y,y) = 0$. Namely,

$$\operatorname{arch}\left(H_p(x,y)\right) \geq \left(\frac{\operatorname{arch}^q x + \operatorname{arch}^q y}{2}\right)^{\frac{1}{q}} = H_q(\operatorname{arch} x, \operatorname{arch} y),$$

with equality if and only if $x = y$.

Case 1.1.2 If $q < 0$, then $\frac{\partial F}{\partial x} \geq 0$. Hence $F(x,y)$ is strictly increasing and $F(x,y) \leq F(y,y) = 0$. Namely,

$$\operatorname{arch}\left(H_p(x,y)\right) \geq \left(\frac{\operatorname{arch}^q x + \operatorname{arch}^q y}{2}\right)^{\frac{1}{q}} = H_q(\operatorname{arch} x, \operatorname{arch} y),$$

with equality if and only if $x = y$.

In conclusion, arch is strictly $H_{p,q}$-concave on the whole interval $(1,+\infty)$ for $(p,q) \in \{(p,q)|\frac{2}{3} \leq p < +\infty, 0 < q \leq 2\} \cup \{(p,q)|\frac{2}{3} \leq p < +\infty, q < 0\}$.

Case 1.2 $p \geq \frac{2}{3}$ and $q > 2$.

By Lemma 4(1), the function $g_{p,q}$ is not monotone on $(1,+\infty)$. With an argument similar to Case 1.1, it is easy to see that arch is neither $H_{p,q}$-concave nor $H_{p,q}$-convex on the whole interval $(1,+\infty)$ for $(p,q) \in \{(p,q)|p \geq \frac{2}{3}, q > 2\}$.

Case 1.3 $p < 0$ and $q \geq 2$.

By Lemma 4(2), the function $g_{p,q}$ is strictly increasing on $(1,+\infty)$ and hence $\frac{\partial F}{\partial x} \geq 0$. Then $F(x,y)$ is strictly increasing and $F(x,y) \leq F(y,y) = 0$. Namely,

$$\operatorname{arch}\left(H_p(x,y)\right) \leq \left(\frac{\operatorname{arch}^q x + \operatorname{arch}^q y}{2}\right)^{\frac{1}{q}} = H_q(\operatorname{arch} x, \operatorname{arch} y),$$

with equality if and only if $x = y$.

In conclusion, arch is strictly $H_{p,q}$-convex on the whole interval $(1,+\infty)$ for $(p,q) \in \{(p,q)|p < 0, q \geq 2\}$.

Case 1.4 $p < 0$ and $q < 2$.

By Lemma 4(2), the function $g_{p,q}$ is not monotone on $(1,+\infty)$. With an argument similar to Case 1.3, it is easy to see that arch is neither $H_{p,q}$-concave nor $H_{p,q}$-convex on the whole interval $(1,+\infty)$ for $(p,q) \in \{(p,q)|p < 0, q < 0\} \cup \{(p,q)|p < 0, 0 < q < 2\}$.

Case 1.5 $0 < p < \frac{2}{3}$ and $q \leq C(p)$.

By Lemma 4(4), the function $g_{p,q}$ is strictly decreasing on $(1,+\infty)$.

Case 1.5.1 If $q>0$, then $\frac{\partial F}{\partial x}\le 0$. Hence $F(x,y)$ is strictly decreasing and $F(x,y)\ge F(y,y)=0$. Namely,

$$\operatorname{arch}(H_p(x,y))\ge\left(\frac{\operatorname{arch}^q x+\operatorname{arch}^q y}{2}\right)^{\frac{1}{q}}=H_q(\operatorname{arch}x,\operatorname{arch}y),$$

with equality if and only if $x=y$.

Case 1.5.2 If $q<0$, then $\frac{\partial F}{\partial x}\ge 0$. Hence $F(x,y)$ is strictly increasing and $F(x,y)\le F(y,y)=0$. Namely,

$$\operatorname{arch}(H_p(x,y))\ge\left(\frac{\operatorname{arch}^q x+\operatorname{arch}^q y}{2}\right)^{\frac{1}{q}}=H_q(\operatorname{arch}x,\operatorname{arch}y),$$

with equality if and only if $x=y$.

In conclusion, arch is strictly $H_{p,q}$-concave on the whole interval $(1,+\infty)$ for $(p,q)\in\{(p,q)|0<p<\frac{2}{3},0<q\le C(p)\}\cup\{(p,q)|0<p<\frac{2}{3},q<0\}$.

Case 1.6 $0<p<\frac{2}{3}$ and $q>C(p)$.

By Lemma 4(4), the function $g_{p,q}$ is not monotone on $(1,+\infty)$. With an argument similar to Case 1.5, it is easy to see that arch is neither $H_{p,q}$-concave nor $H_{p,q}$-convex on the whole interval $(1,+\infty)$ for $(p,q)\in\{(p,q)|0<p<\frac{2}{3},q>C(p)\}$.

Case 2. $p\ne 0$ and $q=0$.

Define

$$F(x,y)=\frac{\operatorname{arch}^2(H_p(x,y))}{\operatorname{arch}x\cdot\operatorname{arch}y}.$$

By logarithmic differentiation, we obtain

$$\frac{1}{F}\frac{\partial F}{\partial x}=x^{p-1}(g_{p,0}(t)-g_{p,0}(x)),$$

where $g_{p,0}$ is defined in Lemma 4.

Case 2.1 $p\ge\frac{2}{3}$ and $q=0$.

By Lemma 4(1), the function $g_{p,q}$ is strictly decreasing on $(1,+\infty)$ and hence $\frac{\partial F}{\partial x}\le 0$. Then $F(x,y)$ is strictly decreasing and $F(x,y)\ge F(y,y)=1$. Namely,

$$\operatorname{arch}(H_p(x,y))\ge\sqrt{\operatorname{arch}x\cdot\operatorname{arch}y}=H_0(\operatorname{arch}x,\operatorname{arch}y),$$

with equality if and only if $x=y$.

In conclusion, arch is strictly $H_{p,q}$-concave on the whole interval $(1,+\infty)$ for $(p,q)\in\{(p,q)|p\ge\frac{2}{3},q=0\}$.

Case 2.2 $p<0$ and $q=0$.

By Lemma 4(2), the function $g_{p,q}$ is not monotone on $(1,+\infty)$. With an argument similar to Case 2.1, it is easy to see that arch is neither $H_{p,q}$-concave nor $H_{p,q}$-convex on the whole interval $(1,+\infty)$ for $(p,q)\in\{(p,q)|p<0,q=0)\}$.

Case 2.3 $0<p<\frac{2}{3}$ and $q=0$.

By Lemma 4(4), the function $g_{p,q}$ is strictly decreasing on $(1,+\infty)$ and hence $\frac{\partial F}{\partial x} \leq 0$. Then $F(x,y)$ is strictly decreasing and $F(x,y) \geq F(y,y) = 1$. Namely,

$$\operatorname{arch}(H_p(x,y)) \geq \sqrt{\operatorname{arch} x \cdot \operatorname{arch} y} = H_0(\operatorname{arch} x, \operatorname{arch} y),$$

with equality if and only if $x = y$.

In conclusion, arch is strictly $H_{p,q}$-concave on the whole interval $(1,+\infty)$ for $(p,q) \in \{(p,q)|0 < p < \frac{2}{3}, q = 0\}$.

Case 3. $p = 0$ and $q \neq 0$.

Define

$$F(x,y) = \operatorname{arch}^q(\sqrt{xy}) - \frac{\operatorname{arch}^q x + \operatorname{arch}^q y}{2}.$$

By differentiation, we obtain

$$\frac{\partial F}{\partial x} = \frac{q}{2x}(g_{0,q}(t) - g_{0,q}(x)),$$

where $g_{0,q}$ is defined in Lemma 4.

Case 3.1 $p = 0$ and $q \geq 2$.

By Lemma 4(3), the function $g_{p,q}$ is strictly increasing on $(1,+\infty)$ and hence $\frac{\partial F}{\partial x} \geq 0$. Then $F(x,y)$ is strictly increasing and $F(x,y) \leq F(y,y) = 0$. Namely,

$$\operatorname{arch}(H_0(x,y)) \leq \left(\frac{\operatorname{arch}^q x + \operatorname{arch}^q y}{2}\right)^{\frac{1}{q}} = H_q(\operatorname{arch} x, \operatorname{arch} y),$$

with equality if and only if $x = y$.

In conclusion, arch is strictly $H_{p,q}$-convex on the whole interval $(1,+\infty)$ for $(p,q) \in \{(p,q)|p = 0, q \geq 2\}$.

Case 3.2 $p = 0$ and $q \leq 1$.

By Lemma 4(3), the function $g_{p,q}$ is strictly decreasing on $(1,+\infty)$.

Case 3.2.1 If $0 < q \leq 1$, then $\frac{\partial F}{\partial x} \leq 0$. Hence $F(x,y)$ is strictly decreasing and $F(x,y) \geq F(y,y) = 0$. Namely,

$$\operatorname{arch}(H_0(x,y)) \geq \left(\frac{\operatorname{arch}^q x + \operatorname{arch}^q y}{2}\right)^{\frac{1}{q}} = H_q(\operatorname{arch} x, \operatorname{arch} y),$$

with equality if and only if $x = y$.

Case 3.2.2 If $q < 0$, then $\frac{\partial F}{\partial x} \geq 0$. Hence $F(x,y)$ is strictly increasing and $F(x,y) \leq F(y,y) = 0$. Namely,

$$\operatorname{arch}(H_0(x,y)) \geq \left(\frac{\operatorname{arch}^q x + \operatorname{arch}^q y}{2}\right)^{\frac{1}{q}} = H_q(\operatorname{arch} x, \operatorname{arch} y),$$

with equality if and only if $x = y$.

In conclusion, arch is strictly $H_{p,q}$-concave on the whole interval $(1,+\infty)$ for $(p,q) \in \{(p,q)|p = 0, 0 < q \leq 1\} \cup \{(p,q)|p = 0, q < 0\}$.

Case 3.3 $p = 0$ and $1 < q < 2$.

By Lemma 4(3), the function $g_{p,q}$ is not monotone on $(1,+\infty)$. With an argument similar to Case 3.2, it is easy to see that arch is neither $H_{p,q}$-concave nor $H_{p,q}$-convex on the whole interval $(1,+\infty)$ for $(p,q) \in \{(p,q) | p = 0,\ 1 < q < 2)\}$.

Case 4. $p = 0$ and $q = 0$.

By Case 2.3, for all $x, y \in (1,+\infty)$, we have

$$\operatorname{arch}(H_p(x,y)) \geq H_0(\operatorname{arch} x, \operatorname{arch} y) \qquad \text{for} \quad 0 < p < \frac{2}{3}.$$

By the continuity of H_p in p and arch in x, we have

$$\operatorname{arch}(H_0(x,y)) \geq H_0(\operatorname{arch} x, \operatorname{arch} y),$$

with equality if and only if $x = y$.

In conclusion, arch is strictly $H_{0,0}$-concave on the whole interval $(1,+\infty)$.

By Case 1.1 and Case 3.1, arch is strictly $H_{\frac{2}{3},2}$-concave and strictly $H_{0,2}$-convex on $(1,+\infty)$. Therefore, the inequalities (1.1) hold with equalities if and only if $x = y$.

This completes the proof of Theorem 1. □

Setting $p = 1 = q$ in Theorem 1, we easily obtain the concavity of arch.

Corollary 1. *The inverse hyperbolic cosine function* arch *is strictly concave on* $(1,+\infty)$.

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Authors' addresses

Yue He
School of Science, Zhejiang Sci-Tech University, Hangzhou 310018, China
E-mail address: yuehe_zstu@163.com

Gendi Wang
School of Science, Zhejiang Sci-Tech University, Hangzhou 310018, China
E-mail address: gendi.wang@zstu.edu.cn