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ON (p,q)–EXTENSION OF FURTHER MEMBERS OF BESSEL–STRUVE FUNCTIONS CLASS

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Abstract. In [10] (p,q)–extensions of the modified Bessel and the modified Struve functions of the first kind are presented. This article companion to [10] contains the (p,q)–extension of modified Struve function of the second kind $\mathbf{M}_{\nu,p,q}$ and the Bessel-Struve kernel function $S_{\nu,p,q}$. Systematic investigation of its properties, among integral representation, Mellin transform, Laguerre polynomial representation for both introduced special functions, while additional differential–difference equation, log–convexity property and Turán–type inequalities are realized for the latter.

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1. INTRODUCTION AND PRELIMINARIES

The theory of special functions has been one of the most rapidly growing research subjects in Mathematical Analysis due to their diverse applications in many different areas of mathematical, physical, statistical, and engineering sciences. Further, extensions of the Eulerian Gamma function enable to define important generalizations not only of the higher transcendental hypereoetric functions, but the Bessel and alike functions class members. So, bearing in mind that

$$\frac{4^{-n}}{n!\,\Gamma(n+\nu+1)} = \frac{\mathrm{B}\left(n+\frac{1}{2}, \nu+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)(2n)!}, \qquad 2\nu > -1,\, n \in \mathbb{N}_0\,,$$

the Bessel function of the first kind can be rewritten into

$$J_\nu(z) = \sum_{n\ge 0} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n+\nu}}{n!\,\Gamma(n+\nu+1)} = \frac{\left(\frac{z}{2}\right)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \sum_{n\ge 0} \mathrm{B}\left(n+\tfrac{1}{2},\, \nu+\tfrac{1}{2}\right) \frac{(-z^2)^n}{(2n)!}\,.$$

Then making use of the (p,q)–extended Beta function [7] (see also, [18])

$$\mathrm{B}(x,y;p,q) = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{e}^{-\frac{p}{t}-\frac{q}{1-t}}\,\mathrm{d}t\,,$$

when $\min\{\Re(x),\Re(y)\}>0; \min\{\Re(p),\Re(q)\}\geq 0$ to $J_\nu(z)$ several extensions, generalizations and unifications of various special functions of (p,q)–variant, and in turn the (p,p)–, that is the p–variant have been studied widely together with the set of related higher transcendental hypergeometric type special functions by several authors, consult for instance [3–6, 12, 15, 18]). In particular, Maširević *et al.* [10] applied the (p,q)–extended Beta in introducing the (p,q)–Bessel function of the first kind

$$J_{\nu,p,q}(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac12)} \sum_{n\geq 0} \mathrm{B}\left(n+\frac12,\, \nu+\frac12; p,q\right) \frac{(-z^2)^n}{(2n)!}. \tag{1.1}$$

These publication also contains the newly defined the (p,q)–extended modified Bessel $I_{\nu,p,q}$ together with the modified Struve $\mathbf{L}_{\nu,p,q}$ functions of the first kind of order ν in the form:

$$I_{\nu,p,q}(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac12)} \sum_{n\geq 0} \mathrm{B}\left(n+\frac12,\, \nu+\frac12; p,q\right) \frac{z^{2n}}{(2n)!}$$

$$\mathbf{L}_{\nu,p,q}(z) = \frac{2\left(\frac{z}{2}\right)^{\nu+1}}{\sqrt{\pi}\,\Gamma(\nu+\frac12)} \sum_{n\geq 0} \mathrm{B}\left(n+1,\, \nu+\frac12; p,q\right) \frac{z^{2n}}{(2n+1)!},$$

where $\min\{p,q\}\geq 0$ while for $p=0=q$ necessarily $\Re(\nu)>-\frac12$. Among other findings they reported about the integral representations [10]

$$I_{\nu,p,q}(z) = \frac{2\left(\frac{z}{2}\right)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac12)} \int_0^1 (1-t^2)^{\nu-\frac12}\, \cosh(zt)\, \mathrm{e}^{-\frac{p}{t^2}-\frac{q}{1-t^2}}\, \mathrm{d}t, \tag{1.2}$$

$$\mathbf{L}_{\nu,p,q}(z) = \frac{2\left(\frac{z}{2}\right)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac12)} \int_0^1 (1-t^2)^{\nu-\frac12}\, \sinh(zt)\, \mathrm{e}^{-\frac{p}{t^2}-\frac{q}{1-t^2}}\, \mathrm{d}t. \tag{1.3}$$

Clearly, for $p=0=q$ all these extensions reduce to their classical ancestors.

Here our aim is to introduce and investigate, in a rather systematic manner, (p,q)–extended modified Struve function $\mathbf{M}_{\nu,p,q}(x)$ and the so–called (p,q)–extended Bessel–Struve kernel function $S_{\nu,p,q}(x)$ presenting their various analytical properties.

2. THE (p,q)–EXTENDED MODIFIED STRUVE FUNCTION $\mathbf{M}_{\nu,p,q}(x)$

The series definition of the Fox-Wright function ${}_p\Psi_q(z)$ [17] reads

$${}_p\Psi_q\left[\begin{array}{c} (\alpha_1,A_1),\cdots,(\alpha_p,A_p) \\ (\beta_1,B_1),\cdots,(\beta_q,B_q)\end{array}\middle|\, z\right] = \sum_{n\geq 0} \frac{\prod_{j=1}^p \Gamma(\alpha_j+A_j n)}{\prod_{j=1}^q \Gamma(\beta_k+B_k n)} \frac{z^n}{n!}, \tag{2.1}$$

where $A_j, B_k > 0$ $j = 1, \cdots, p; k = 1, \cdots, q$ and the series converges for all $z \in \mathbb{C}$ when $\Delta = 1 + \sum_{k=1}^{q} B_k - \sum_{j=1}^{p} A_j > 0$; in the case $\Delta = 0$ the convergence occurs inside the open disc

$$|z| < \nabla = \left(\prod_{j=1}^{p} A_j^{-A_j} \right) \left(\prod_{j=1}^{q} B_j^{B_j} \right).$$

The modifed Struve functions of the second kind [2, p. 1388, Eq. (2)], [13, Eq. 11.2.6]

$$\mathbf{M}_\nu(x) = \mathbf{L}_\nu(x) - I_\nu(x),$$

having integral form [13, Eq. 11.5.4]

$$\mathbf{M}_\nu(x) = -\frac{2\left(\frac{x}{2}\right)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \int_0^1 (1-t^2)^{\nu-\frac{1}{2}} \mathrm{e}^{-xt}\,\mathrm{d}t, \qquad 2\nu+1>0, x>0.$$

By routine calculation we infer the related power series representation:

Theorem 1. *For all* $2\nu+1>0, x>0$ *we have*

$$\mathbf{M}_\nu(x) = -\frac{2^{-\nu}x^\nu}{\Gamma(\nu+\frac{1}{2})}\, {}_1\Psi_1\left[\begin{matrix} (\frac{1}{2},\frac{1}{2}) \\ (\nu+1,\frac{1}{2}) \end{matrix} \,\middle|\, -x \right].$$

The proof is straightforward mentioning only that being $\Delta = 1$, the series converges for all $x > 0$.

We now introduce the (p,q)–extended modified Struve function of the second kind as

$$\mathbf{M}_{\nu,p,q}(x) = \mathbf{L}_{\nu,p,q}(x) - I_{\nu,p,q}(x).$$

Applying this definition by taking (1.2) and (1.3), we obtain the related integral representation

$$\mathbf{M}_{\nu,p,q}(x) = -\frac{2\left(\frac{x}{2}\right)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \int_0^1 (1-t^2)^{\nu-\frac{1}{2}} \mathrm{e}^{-xt-\frac{p}{t^2}-\frac{q}{1-t^2}}\,\mathrm{d}t\,, \tag{2.2}$$

with $\min\{p,q\} \geq 0$ and $\Re(\nu) > -\frac{1}{2}$ when $p = q = 0$. Clearly, the latter case coincides with the original definition of modified Struve function $\mathbf{M}_\nu(x)$ of second kind.

The double Mellin transforms [14, p. 293, Eq. (7.1.6)] of a suitable class of $\mathrm{L}_1(\mathbb{R}_+^2)$–integrable function $f(x,y)$ with respect to the indices r and s is defined by

$$\mathscr{M}\{f(x,y)\}(r,s) = \int_0^\infty \int_0^\infty x^{r-1} y^{s-1}\, f(x,y)\,\mathrm{d}x\,\mathrm{d}y, \tag{2.3}$$

provided that the improper integral in (2.3) exists. Now, we examine the existence and the establish the shape of the double Mellin transform of $\mathbf{M}_{\nu,p,q}(x)$ when the arguments are the extension parameters p,q.

Theorem 2. *For all* $\min\{\Re(r),\,\Re(s),\,2\Re(\nu)+1\}>0$ *and* $x>0$ *the Mellin transform of* $\mathbf{M}_{\nu,p,q}(x)$ *with respect to* $p,q\ge 0$ *reads as follows*

$$\mathscr{M}\{\mathbf{M}_{\nu,p,q}(x)\}(r,s) = -\frac{\left(\frac{x}{2}\right)^{\nu}\,\Gamma(r)\Gamma(s)\Gamma(\nu+s+\frac12)}{\sqrt{\pi}\,\Gamma(\nu+\frac12)} \times {}_1\Psi_1\left[\begin{array}{c} (\frac12+r,\frac12) \\ (\nu+r+s+1,\frac12) \end{array}\middle| -x\right].$$

Proof. By the definition of Mellin transform we find from (2.2) that

$$\begin{aligned}\mathscr{M}\left\{\mathbf{M}_{\nu,p,q}(x)\right\}(r,s) &= \int_0^\infty\int_0^\infty p^{r-1}q^{s-1}\mathbf{M}_{\nu,p,q}(x)\,\mathrm{d}p\,\mathrm{d}q \\ &= -\frac{2\left(\frac{x}{2}\right)^{\nu}}{\sqrt{\pi}\,\Gamma(\nu+\frac12)}\int_0^1 (1-t^2)^{\nu-\frac12}\,\mathrm{e}^{-xt} \\ &\quad\times\left(\int_0^\infty p^{r-1}\mathrm{e}^{-\frac{p}{t^2}}\,\mathrm{d}p\right)\left(\int_0^\infty q^{s-1}\mathrm{e}^{-\frac{q}{1-t^2}}\,\mathrm{d}q\right)\mathrm{d}t,\end{aligned}$$

where we have interchanged the integration order. Applying the well-known Gamma function formula [13, p.136, Eq. (5.2.1)]

$$\Gamma(\eta)\,\xi^{-\eta} = \int_0^\infty e^{-\xi t}t^{\eta-1}\,\mathrm{d}t, \qquad \Re(\xi)>0,\ \Re(\eta)>0$$

to the inner p- and q-integrals we get

$$\begin{aligned}\mathscr{M}\left\{\mathbf{M}_{\nu,p,q}(x)\right\}(r,s) &= -\frac{2\left(\frac{x}{2}\right)^{\nu}\Gamma(r)\Gamma(s)}{\sqrt{\pi}\,\Gamma(\nu+\frac12)}\int_0^1 t^{2r}\,(1-t^2)^{\nu+s-\frac12}\,\mathrm{e}^{-xt}\mathrm{d}t \\ &= -\frac{2\left(\frac{x}{2}\right)^{\nu}\Gamma(r)\Gamma(s)}{\sqrt{\pi}\,\Gamma(\nu+\frac12)}\sum_{n\ge0}\frac{(-x)^n}{n!}\int_0^1 t^{2r+n}\,(1-t^2)^{\nu+s-\frac12}\,\mathrm{d}t \\ &= -\frac{\left(\frac{x}{2}\right)^{\nu}\Gamma(r)\Gamma(s)}{\sqrt{\pi}\,\Gamma(\nu+\frac12)}\sum_{n\ge0}\frac{(-x)^n}{n!}\int_0^1 u^{r+\frac{n}{2}-\frac12}\,(1-u)^{\nu+s-\frac12}\,\mathrm{d}u \\ &= -\frac{\left(\frac{x}{2}\right)^{\nu}\Gamma(r)\Gamma(s)}{\sqrt{\pi}\,\Gamma(\nu+\frac12)}\sum_{n\ge0}\frac{\Gamma(r+\frac{n}{2}+\frac12)\,\Gamma(\nu+s+\frac12)}{\Gamma(\nu+r+s+1+\frac{n}{2})}\frac{(-x)^n}{n!},\end{aligned}$$

taking into account the series e^{-xt}. Thus, (2.1) finishes the proof. □

Remark 1. If we set $r=1=s$ in Theorem 2 we get

$$\int_0^\infty\int_0^\infty \mathbf{M}_{\nu,p,q}(x)\,\mathrm{d}p\,\mathrm{d}q = -\frac{\left(\frac{x}{2}\right)^{\nu}(2\nu+1)}{2\sqrt{\pi}}\,{}_1\Psi_1\left[\begin{array}{c} (\frac32,\frac12) \\ (\nu+3,\frac12) \end{array}\middle| -x\right], \qquad x>0.$$

Next, we report on the Laguerre polynomial representation for $\mathbf{M}_{\nu,p,q}(x)$. The Laguerre polynomial $L_n^{(\alpha)}(p)$ of the degree $n \in \mathbb{N}_0$ is given by the generating function [16, p. 202]

$$\frac{1}{(1-t)^{\alpha+1}} \exp\left(-\frac{pt}{1-t}\right) = \sum_{n\geq 0} L_n^{(\alpha)}(p)\, t^n, \qquad \alpha \in \mathbb{C};\ |t|<1.$$

Theorem 3. *For all* $\min\{\Re(p),\ \Re(q)\} > 0,\ \Re(\nu) > -\frac{1}{2}$ *and* $\alpha \in \mathbb{C}$*, for which* $\nu+\alpha > -\frac{3}{2}$*, the following Laguerre polynomial representation holds true*

$$\mathbf{M}_{\nu,p,q}(x) = -\frac{(\nu+\frac{1}{2})_{\alpha+1}\, x^\nu}{\sqrt{\pi}\, 2^\nu\, \mathrm{e}^{p+q}} \sum_{m,n\geq 0} (\nu+\alpha+\tfrac{3}{2})_n\, L_m^\alpha(p)\, L_n^\alpha(q)$$
$$\times {}_1\Psi_1\left[\begin{array}{c} (\alpha+n+\frac{3}{2},\frac{1}{2}) \\ (\nu+2\alpha+m+n+3,\frac{1}{2}) \end{array}\middle| -x\right].$$

Proof. Simple transformation gives

$$\exp\left(-\frac{p}{t^2}\right) = \frac{t^{2\alpha+2}}{\mathrm{e}^p} \sum_{m\geq 0} L_m^{(\alpha)}(p)\,(1-t^2)^m;$$
$$\exp\left(-\frac{q}{1-t^2}\right) = \frac{(1-t^2)^{\alpha+1}}{\mathrm{e}^q} \sum_{n\geq 0} L_n^{(\alpha)}(q)\, t^{2n},$$

that is

$$\mathrm{e}^{-\frac{p}{t^2}-\frac{q}{1-t^2}} = \mathrm{e}^{-p-q}\, t^2 (1-t^2)^{\alpha+1} \sum_{m,n\geq 0} L_m^{(\alpha)}(p)\, L_n^{(\alpha)}(q)\, t^{2n}\,(1-t^2)^m. \tag{2.4}$$

By inserting (2.4) into (2.2) and changing the order of integration and summation, we conclude

$$\mathbf{M}_{\nu,p,q}(x) = -\frac{2\left(\frac{x}{2}\right)^\nu \mathrm{e}^{-p-q}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \sum_{m,n\geq 0} L_m^\alpha(p)\, L_n^\alpha(q)$$
$$\times \int_0^1 t^{2(\alpha+n+1)}\,(1-t^2)^{\nu+\alpha+m+\frac{1}{2}}\,\mathrm{e}^{-xt}\,\mathrm{d}t,$$

and then series expansion of e^{-xt} is used. The rest is obvious. □

We close this section with a certain differential–difference equation for real argument $\mathbf{M}_{\nu,p,q}(x)$.

Theorem 4. *For all* $\min\{\Re(p),\Re(q)\} > 0$ *and for* $\Re(\nu) > -\frac{1}{2}$ *when* $p=q=0$*, we have*

$$x^2 w_\nu'' - 2\nu x\, w_\nu' - [x^2 - \nu(\nu+1)]\, w_\nu = -(2\nu+1)x\, w_{\nu+1}, \qquad x>0, \tag{2.5}$$

where $w_\nu = w_\nu(x) \equiv \mathbf{M}_{\nu,p,q}(x)$.

Proof. Upon differentiating both sides of (2.2) with respect to x, we find that

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 (x^{-\nu}\mathbf{M}_{\nu,p,q}(x)) = -\frac{2^{1-\nu}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^1 t^2\,(1-t^2)^{\nu-\frac{1}{2}}\,\mathrm{e}^{-xt-\frac{p}{t^2}-\frac{q}{1-t^2}}\,\mathrm{d}t$$

$$= -\frac{2^{1-\nu}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^1 \{1-(1-t^2)\}\,(1-t^2)^{\nu-\frac{1}{2}}\,\mathrm{e}^{-xt-\frac{p}{t^2}-\frac{q}{1-t^2}}\,\mathrm{d}t\,,$$

which can be rewritten as follows:

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 (x^{-\nu}\mathbf{M}_{\nu,p,q}(x)) &= -\frac{2^{1-\nu}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^1 (1-t^2)^{\nu-\frac{1}{2}}\,\mathrm{e}^{-xt-\frac{p}{t^2}-\frac{q}{1-t^2}}\,\mathrm{d}t \\ &\quad + \frac{2^{1-\nu}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^1 (1-t^2)^{\nu+1}\,\mathrm{e}^{-xt-\frac{p}{t^2}-\frac{q}{1-t^2}}\,\mathrm{d}t \\ &= \frac{1}{x^{\nu}}\mathbf{M}_{\nu,p,q}(x) - \frac{2\nu+1}{x^{\nu+1}}\mathbf{M}_{\nu+1,p,q}(x),\end{aligned}$$

where we use the definition (2.2). Expanding the left-hand-side second derivative and reducing the material we arrive at (2.5). □

3. THE (p,q)–BESSEL–STRUVE KERNEL FUNCTION

The so-called Bessel-Struve kernel function S_ν is defined by the series [1, p. 1845]

$$S_\nu(x) = \frac{\Gamma(\nu+1)}{\sqrt{\pi}}\sum_{n\ge 0}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+\nu+1\right)}\frac{x^n}{n!}, \qquad \nu > -1,\, x \in \mathbb{R}.$$

Consider the initial value problem (see [8, 9, 11]):

$$\mathcal{L}_\nu\, u(x) = \lambda^2 u(x), \;\; u(0) = 1, \qquad u'(0) = \frac{\lambda\Gamma(\nu+1)}{\sqrt{\pi}\;\Gamma\left(\nu+\frac{3}{2}\right)},$$

where for $\nu > -\frac{1}{2}$, $\mathcal{L}_\nu$ stands for the Bessel-Struve operator defined by

$$\mathcal{L}_\nu\, u(x) = \frac{d^2u}{dx^2}(x) + \frac{2\nu+1}{x}\left(\frac{du}{dx}(x) - \frac{du}{dx}(0)\right)$$

with some $u \in \mathrm{C}^\infty(\mathbb{R})$. The Bessel–Struve kernel function becomes the particular case of the unique solution $S_\nu(\lambda x)$ of the above initial value problem.

The function S_ν is expressible in terms of the Beta function *viz.*

$$S_\nu(x) = \frac{\Gamma(\nu+1)}{\sqrt{\pi}\;\Gamma(\nu+\frac{1}{2})}\sum_{n\ge 0}\mathrm{B}\left(\frac{n}{2}+\frac{1}{2},\,\nu+\frac{1}{2}\right)\frac{x^n}{n!},$$

and also as a specific Fox–Wright function

$$S_\nu(x) = {}_1\Psi_1\left[\begin{array}{c}(\frac{1}{2},\frac{1}{2})\\ (\nu+1,\frac{1}{2})\end{array}\middle|\, x\right], \qquad x \in \mathbb{R},$$

mentioning that $S_\nu(0) = 1$ and $\Delta = 1$ which ensures the convergence of this series on the whole real x axis.

Here, our aim is introducing and studying the novel (p,q)–extended Bessel-Struve kernel function $S_{\nu,p,q}(x)$ by presenting functional bound, integral representation, Mellin transform and other properties.

Analogously to (1.1), we introduce the (p,q)–extended Bessel-Struve kernel function $S_{\nu,p,q}(x)$ in the form

$$S_{\nu,p,q}(x) = \frac{\Gamma(\nu+1)}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \sum_{n\geq 0} \mathrm{B}\left(\frac{n}{2}+\frac{1}{2}, \nu+\frac{1}{2}; p,q\right) \frac{x^n}{n!}. \tag{3.1}$$

with $\min\{p,q\} \geq 0$ and $\Re(\nu) > -\frac{1}{2}$ when $p = q = 0$.

Firstly, we derive an integral representation and a functional upper bound for $S_{\nu,p,q}(x)$.

Theorem 5. *For all* $\min\{\Re(p),\Re(q)\} > 0$ *and for* $\Re(\nu) > -\frac{1}{2}$ *when* $p = q = 0$, *we have*

$$S_{\nu,p,q}(x) = \frac{2\,\Gamma(\nu+1)}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \int_0^1 (1-t^2)^{\nu-\frac{1}{2}}\, \mathrm{e}^{xt-\frac{p}{t^2}-\frac{q}{1-t^2}}\, \mathrm{d}t, \qquad x \geq 0. \tag{3.2}$$

Proof. By virtue of the integral expression [10]

$$\mathrm{B}(x,y;p,q) = 2\int_0^1 t^{2x-1}(1-t^2)^{y-1}\, \mathrm{e}^{-\frac{p}{t^2}-\frac{q}{1-t^2}}\, \mathrm{d}t\,,$$

valid for all $\min\{\Re(x),\Re(y)\} > 0; \min\{\Re(p),\Re(q)\} \geq 0$, from (3.1) one yields

$$\begin{aligned} S_{\nu,p,q}(x) &= \frac{2\,\Gamma(\nu+1)}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \sum_{n\geq 0} \frac{x^n}{n!} \int_0^1 t^n\,(1-t^2)^{\nu-\frac{1}{2}}\, \mathrm{e}^{-\frac{p}{t^2}-\frac{q}{1-t^2}}\, \mathrm{d}t \\ &= \frac{2\,\Gamma(\nu+1)}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \int_0^1 (1-t^2)^{\nu-\frac{1}{2}}\, \mathrm{e}^{-\frac{p}{t^2}-\frac{q}{1-t^2}} \sum_{n\geq 0} \frac{(xt)^n}{n!}\, \mathrm{d}t, \end{aligned}$$

with interchanged order of summation and integration. This is equivalent to the assertion. □

Now, making use of the estimate [15, Lemma 2]

$$\sup_{0<t<1} \mathrm{e}^{-\frac{p}{t^2}-\frac{q}{1-t^2}} = \mathrm{e}^{-(\sqrt{p}+\sqrt{q})^2}, \qquad \min\{p,q\} \geq 0\,,$$

in (3.2) we deduce

Corollary 1. *For all* $\min\{p,q\} > 0$ *and for* $\nu > -\frac{1}{2}$ *if* $p = q = 0$ *we have*

$$S_{\nu,p,q}(x) \leq \mathrm{e}^{-(\sqrt{p}+\sqrt{q})^2}\, S_\nu(x)\,, \qquad x \geq 0.$$

Theorem 6. *For all* $\min\{\Re(p),\Re(q)\}>0$, *for* $\Re(\nu)>-\frac{1}{2}$ *if* $p=q=0$ *and for all* $x\in\mathbb{R}\setminus\{0\}$ *we have*

$$S_{\nu,p,q}(x)=\Gamma(\nu+1)\left(\frac{2}{x}\right)^{\nu}\big(J_{\nu,p,q}(\mathrm{i}x)-\mathrm{i}\,\mathbf{H}_{\nu,p,q}(\mathrm{i}x)\big),$$
$$S_{\nu,p,q}(x)=\Gamma(\nu+1)\left(\frac{2}{x}\right)^{\nu}\big(I_{\nu,p,q}(x)+\mathbf{L}_{\nu,p,q}(x)\big).$$

Proof. Using the known integral representations [10]

$$J_{\nu,p,q}(x)=\frac{2\left(\frac{x}{2}\right)^{\nu}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^1(1-t^2)^{\nu-\frac{1}{2}}\cos(xt)\,\mathrm{e}^{-\frac{p}{t^2}-\frac{q}{1-t^2}}\,\mathrm{d}t,$$
$$\mathbf{H}_{\nu,p,q}(x)=\frac{2\left(\frac{x}{2}\right)^{\nu}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^1(1-t^2)^{\nu-\frac{1}{2}}\sin(xt)\,\mathrm{e}^{-\frac{p}{t^2}-\frac{q}{1-t^2}}\,\mathrm{d}t,$$

in right-hand side of the first stated relation we recover $S_{\nu,p,q}(x)$. Similarly, with the aid of integral forms [10]

$$I_{\nu,p,q}(x)=\frac{2\left(\frac{x}{2}\right)^{\nu}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^1(1-t^2)^{\nu-\frac{1}{2}}\cosh(xt)\,\mathrm{e}^{-\frac{p}{t^2}-\frac{q}{1-t^2}}\,\mathrm{d}t,$$
$$\mathbf{L}_{\nu,p,q}(x)=\frac{2\left(\frac{x}{2}\right)^{\nu}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^1(1-t^2)^{\nu-\frac{1}{2}}\sinh(xt)\,\mathrm{e}^{-\frac{p}{t^2}-\frac{q}{1-t^2}}\,\mathrm{d}t\,,$$

we finish the proof of the second formula. □

Our next goal is the Mellin transform result.

Theorem 7. *For all* $\min\{\Re(r),\,\Re(s)\}>0$ *and* $\Re(\nu)>-\frac{1}{2}$ *the Mellin transforms of* $S_{\nu,\,p,q}$ *with respect to* $p,q\ge 0$, *read as follows:*

$$\mathscr{M}\{S_{\nu,p,q}(x)\}(r,s)=\frac{\Gamma(\nu+1)\Gamma(r)\Gamma(s)\Gamma(\nu+s+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\,{}_1\Psi_1\Big[\begin{matrix}(\frac{1}{2}+r,\frac{1}{2})\\(\nu+r+s+1,\frac{1}{2})\end{matrix}\Big|x\Big].$$

Proof. We find from (3.2) *ad definitionem*

$$\begin{aligned}\mathscr{M}\big\{S_{\nu,p,q}(x)\big\}(r,s)&=\int_0^\infty\int_0^\infty p^{r-1}q^{s-1}S_{\nu,p,q}(x)\,\mathrm{d}p\,\mathrm{d}q\\&=\int_0^\infty\int_0^\infty\left(\frac{\Gamma(\nu+1)\,p^{r-1}q^{s-1}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\sum_{n\ge0}\frac{x^n\mathrm{B}\left(\frac{n}{2}+\frac{1}{2},\,\nu+\frac{1}{2};p,q\right)}{n!}\right)\mathrm{d}p\,\mathrm{d}q\\&=\frac{\Gamma(\nu+1)}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\sum_{n\ge0}\frac{x^n}{n!}\int_0^\infty\int_0^\infty p^{r-1}q^{s-1}\mathrm{B}\left(\frac{n}{2}+\frac{1}{2},\,\nu+\frac{1}{2};p,q\right)\mathrm{d}p\,\mathrm{d}q.\end{aligned}$$

By virtue of the formula [7, p.342, Eq. (2.1)]

$$\int_0^\infty \int_0^\infty p^{r-1} q^{s-1} \mathrm{B}(x,\, y\, ;\, p,q)\, \mathrm{d}p\, \mathrm{d}q = \Gamma(r)\Gamma(s)\mathrm{B}(x+r,\, y+s),$$

valid for all $\min\{\Re(r),\ \Re(s)\} > 0$, the double integral becomes

$$\mathscr{M}\left\{S_{\nu,p,q}(x)\right\}(r,s) = \frac{\Gamma(\nu+1)\Gamma(r)\Gamma(s)}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \sum_{n\geq 0} \frac{\mathrm{B}\left(\frac{n}{2}+\frac{1}{2}+r,\, \nu+\frac{1}{2}+s\right) x^n}{n!}.$$

After simplification we infer the asserted formula. □

Remark 2. For $r=1=s$ in Theorem 7 we obtain the integration formula

$$\int_0^\infty \int_0^\infty S_{\nu,p,q}(x)\, \mathrm{d}p\, \mathrm{d}q = \frac{\Gamma(\nu+1)\,(\nu+\frac{1}{2})}{\sqrt{\pi}}\, {}_1\Psi_1\left[\begin{array}{c} (\frac{3}{2},\frac{1}{2}) \\ (\nu+3,\frac{1}{2}) \end{array} \middle| x\right].$$

The following Laguerre polynomial representation for $S_{\nu,p,q}(x)$ has been established by the lines of the same fashion result for $\mathbf{M}_{\nu,p,q}(x)$ in Theorem 3, therefore we omit the proof.

Theorem 8. *For all* $\min\{\Re(p),\ \Re(q)\} > 0$ *and* $\Re(\nu) > -\frac{1}{2}$ *we have*

$$S_{\nu,p,q}(x) = \frac{\Gamma(\nu+1)\,(\nu+\frac{1}{2})_{\alpha+1}}{\sqrt{\pi}\,\mathrm{e}^{p+q}} \sum_{m,n\geq 0} L_m^\alpha(p)\, L_n^\alpha(q)\, (\nu+\alpha+\tfrac{3}{2})_m$$
$$\times\, {}_1\Psi_1\left[\begin{array}{c} (\alpha+n+\frac{3}{2},\frac{1}{2}) \\ (\nu+2\alpha+m+n+3,\frac{1}{2}) \end{array} \middle| x\right],$$

where the Pochhammer symbol notation $(a)_b = \Gamma(a+b)/\Gamma(a)$ *has been employed.*

To end the exposition we obtain a derivation formula for $S_{\nu,\,p,q}(x)$.

Theorem 9. *For all* $\min\{\Re(p),\Re(q)\} > 0$ *and for* $\Re(\nu) > -\frac{1}{2}$ *when* $p=q=0$, *we have*

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 (S_{\nu,p,q}(x)) = S_{\nu,p,q}(x) - \frac{(2\nu+1)}{2(\nu+1)} S_{\nu+1,p,q}(x)\,.$$

Proof. Recall the series representation (3.1) of the Bessel–Struve kernel

$$S_{\nu,p,q}(x) = \frac{\Gamma(\nu+1)}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)} \sum_{n\geq 0} \mathrm{B}\left(\frac{n}{2}+\frac{1}{2},\nu+\frac{1}{2};p,q\right)\frac{x^n}{n!}.$$

With the help of the contiguous relation [7, p. 362, Theorem 3]

$$\mathrm{B}(x+1,y;p,q) + \mathrm{B}(x,y+1;p,q) = \mathrm{B}(x,y;p,q)\,,$$

a direct calculation gives

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 (S_{\nu,p,q}(x)) = \frac{\Gamma(\nu+1)}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)} \sum_{n\geq 2} \left(\mathrm{B}\left(\frac{n}{2}-\frac{1}{2},\nu+\frac{1}{2};p,q\right)\right.$$

$$-\mathrm{B}\left(\frac{n}{2}-\frac{1}{2},\nu+\frac{3}{2};p,q\right)\Bigg)\frac{x^{n-2}}{n!}$$
$$=\frac{\Gamma(\nu+1)}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\sum_{k\geq 0}\mathrm{B}\left(\frac{k}{2}+\frac{1}{2},\nu+\frac{1}{2};p,q\right)\frac{x^k}{k!}$$
$$+\frac{\Gamma(\nu+1)}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\sum_{k\geq 0}\mathrm{B}\left(\frac{k}{2}+\frac{1}{2},\nu+\frac{3}{2};p,q\right)\frac{x^k}{k!}.$$

The rest is obvious. □

4. Log–convexity and Turán type inequalities

In this section, first we prove the log–convexity properties and Turán type inequalities for the normalized variant of modified Struve function of second kind $\mathbf{M}_{\nu,p,q}(x)$. Let us consider the normalized variant $\mathcal{M}_{\nu,p,q}\colon \mathbb{R}_+ \mapsto \mathbb{R}$, defined by

$$\mathcal{M}_{\nu,p,q}(x) = -\Gamma\left(\nu+\tfrac{1}{2}\right)\left(\frac{x}{2}\right)^{-\nu}\mathbf{M}_{\nu,p,q}(x) \tag{4.1}$$
$$= \frac{2}{\sqrt{\pi}}\int_0^1 (1-t^2)^{\nu-\frac{1}{2}}\,\mathrm{e}^{-xt-\frac{p}{t^2}-\frac{q}{1-t^2}}\,\mathrm{d}t. \tag{4.2}$$

Theorem 10. *Let* $\min\{p,q\}\geq 0$. *Then the following assertions are true:*

- *The function* $\nu\mapsto\mathcal{M}_{\nu,p,q}(x)$ *is log–convex on* $(-\frac{1}{2},\infty)$ *for all* $x>0$.
- *The function* $x\mapsto\mathcal{M}_{\nu,p,q}(x)$ *is log–convex on* $(0,\infty)$ *for all* $\nu>-\frac{1}{2}$.
- *The function* $(p,q)\mapsto\mathcal{M}_{\nu,p,q}(x)$ *is log–convex on* $(0,\infty)$ *for all* $x>0$ *and* $\nu>-\frac{1}{2}$.

Moreover, for the same parameter range there holds the Turán inequality

$$\mathcal{M}^2_{\nu,p,q}(x)-\mathcal{M}_{\nu-1,p,q}(x)\mathcal{M}_{\nu+1,p,q}(x)\leq 0, \qquad \nu>-\frac{1}{2}, \tag{4.3}$$

which is equivalent with the Turán–type inequality

$$\mathbf{M}^2_{\nu,p,q}(x)-\mathbf{M}_{\nu-1,p,q}(x)\mathbf{M}_{\nu+1,p,q}(x)\leq\frac{1}{\nu+\frac{1}{2}}\,\mathbf{M}^2_{\nu,p,q}(x), \qquad \nu>-\frac{1}{2}. \tag{4.4}$$

Furthermore, for the same parameter range there holds the Turán inequality

$$\mathcal{M}^2_{\nu,p,q}(x)-\mathcal{M}_{\nu,p-1,q-1}(x)\,\mathcal{M}_{\nu,p+1,q+1}(x)\leq 0, \qquad \nu>-\frac{1}{2}. \tag{4.5}$$

Proof. Using the integral representation (4.2) by aid of classical Hölder–Rogers inequality for integrals, we have

$$\mathcal{M}_{\lambda\nu_1+(1-\lambda)\nu_2,p,q}(x)=\frac{2}{\sqrt{\pi}}\int_0^1(1-t^2)^{\lambda\nu_1+(1-\lambda)\nu_2-\frac{1}{2}}\,\mathrm{e}^{-xt-\frac{p}{t^2}-\frac{q}{1-t^2}}\,\mathrm{d}t$$

$$= \frac{2}{\sqrt{\pi}} \int_0^1 (1-t^2)^{\lambda(\nu_1-\frac{1}{2})+(1-\lambda)(\nu_2-\frac{1}{2})} \mathrm{e}^{-xt-\frac{p}{t^2}-\frac{q}{1-t^2}}\, \mathrm{d}t$$

$$= \frac{2}{\sqrt{\pi}} \int_0^1 \left\{(1-t^2)^{\nu_1-\frac{1}{2}} \mathrm{e}^{-xt-\frac{p}{t^2}-\frac{q}{1-t^2}}\right\}^{\lambda}$$

$$\times \left\{(1-t^2)^{\nu_1-\frac{1}{2}} \mathrm{e}^{-xt-\frac{p}{t^2}-\frac{q}{1-t^2}}\right\}^{1-\lambda} \mathrm{d}t$$

$$\leq \left\{\frac{2}{\sqrt{\pi}} \int_0^1 (1-t^2)^{\nu_1-\frac{1}{2}}\, \mathrm{e}^{-xt-\frac{p}{t^2}-\frac{q}{1-t^2}}\, \mathrm{d}t\right\}^{\lambda}$$

$$\times \left\{\frac{2}{\sqrt{\pi}} \int_0^1 (1-t^2)^{\nu_2-\frac{1}{2}}\, \mathrm{e}^{-xt-\frac{p}{t^2}-\frac{q}{1-t^2}}\, \mathrm{d}t\right\}^{1-\lambda}.$$

This is equivalent to

$$\mathcal{M}_{\lambda\nu_1+(1-\lambda)\nu_2,p,q}(x) \leq \left[\mathcal{M}_{\nu_1,p,q}(x)\right]^{\lambda} \left[\mathcal{M}_{\nu_2,p,q}(x)\right]^{1-\lambda}, \tag{4.6}$$

which proves the first assertion for all $\nu_1, \nu_2 > -\frac{1}{2}$, $\lambda \in [0,1]$ and $x > 0$.

In a similar manner, we can prove the next two assertions that $x \mapsto \mathcal{M}_{\nu,p,q}(x)$ is log–convex for all $\nu > -\frac{1}{2}$, $\lambda \in [0,1]$ and x, $y > 0$, that is

$$\mathcal{M}_{\nu,p,q}(\lambda x + (1-\lambda)y) \leq \left[\mathcal{M}_{\nu,p,q}(x)\right]^{\lambda} \left[\mathcal{M}_{\nu,p,q}(y)\right]^{1-\lambda},$$

while the function $(p,q) \mapsto \mathcal{M}_{\nu,p,q}(x)$ is log–convex for all $p_1, p_2, q_1, q_2 > 0$, $\nu > -\frac{1}{2}$, $\lambda \in [0,1]$ and $x > 0$:

$$\mathcal{M}_{\nu,\lambda p_1+(1-\lambda)p_2,\lambda q_1+(1-\lambda)q_2}(x) \leq \left[\mathcal{M}_{\nu,p_1,q_1}(x)\right]^{\lambda} \left[\mathcal{M}_{\nu,p_2,q_2}(x)\right]^{1-\lambda}. \tag{4.7}$$

Next, choosing $\nu_1 = \nu - 1$, $\nu_2 = \nu + 1$ and $\lambda = \frac{1}{2}$ in (4.6) we conclude the Turán inequality (4.3).

As to the Turán–type inequality (4.4), we replace (4.1) into (4.3) and transform the result by

$$\frac{\Gamma(\nu+\frac{3}{2})\Gamma(\nu-\frac{1}{2})}{\Gamma^2(\nu+\frac{1}{2})} = \frac{2\nu+1}{2\nu-1}, \qquad \nu > \frac{1}{2}.$$

Again, specifying $p_1 = p-1, q_1 = q-1$, $p_2 = p+1, q_2 = q+1$ and $\lambda = \frac{1}{2}$ in (4.7), we deduce the Turán inequality (4.5). □

Also one proves the log–convexity of $S_{\nu,p,q}(x)$ for $x > 0$.

Theorem 11. *Let* $\min\{p,q\} \geq 0$. *Then the following assertions are true:*

- *The function* $x \mapsto S_{\nu,p,q}(x)$ *is log–convex on positive real* x *half–axis for all* $\nu > -\frac{1}{2}$.

- *The function* $(p,q) \mapsto S_{\nu,p,q}(x)$ *is log–convex on* $(0,\infty)$ *for all* $x > 0$ *and* $\nu > -\frac{1}{2}$.

Moreover, for the same parameter range there holds the Turán inequality

$$S^2_{\nu,p,q}(x) - S_{\nu,p-1,q-1}(x) S_{\nu,p+1,q+1}(x) \le 0, \qquad \nu > -\frac{1}{2}, \tag{4.8}$$

Proof. Considering the integral representation (3.2) assuming $x, y > 0$, $\lambda \in [0,1]$ we have

$$S_{\nu,p,q}(\lambda x + (1-\lambda)y) \le \{S_{\nu,p,q}(x)\}^{\lambda} \{S_{\nu,p,q}(y)\}^{1-\lambda}.$$

Indeed, this follows with the aid of the Hölder–Rogers inequality for integrals, *viz.*

$$\int_0^1 \left\{(1-t^2)^{\nu-\frac{1}{2}}\, \mathrm{e}^{xt-\frac{p}{t^2}-\frac{q}{1-t^2}}\right\}^{\lambda} \left\{(1-t^2)^{\nu-\frac{1}{2}}\, \mathrm{e}^{yt-\frac{p}{t^2}-\frac{q}{1-t^2}}\right\}^{1-\lambda} \mathrm{d}t$$

$$\le \left\{ \int_0^1 (1-t^2)^{\nu-\frac{1}{2}}\, \mathrm{e}^{xt-\frac{p}{t^2}-\frac{q}{1-t^2}}\, \mathrm{d}t \right\}^{\lambda} \left\{ \int_0^1 (1-t^2)^{\nu-\frac{1}{2}}\, \mathrm{e}^{yt-\frac{p}{t^2}-\frac{q}{1-t^2}}\, \mathrm{d}t \right\}^{1-\lambda}.$$

completing the proof in the prescribed form.

Letting $p_1, p_2, q_1, q_2 > 0$, $\nu > -\frac{1}{2}$, $\lambda \in [0,1]$ and $x > 0$, we obtain

$$S_{\nu,\lambda p_1+(1-\lambda)p_2,\lambda q_1+(1-\lambda)q_2}(x) \le \left[S_{\nu,p_1,q_1}(x)\right]^{\lambda} \left[S_{\nu,p_2,q_2}(x)\right]^{1-\lambda}.$$

Specifying here $p_1 = p-1, q_1 = q-1$, $p_2 = p+1, q_2 = q+1$ and $\lambda = \frac{1}{2}$ we conclude the Turán inequality (4.8). □

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