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A NOTE ON RADICALS OF ASSOCIATIVE RINGS AND ALTERNATIVE RINGS

S. TUMURBAT AND T. KHULAN AND D. DAYANTSOLMON

To the memory of professor Avirmed Nyamrinchin

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Abstract. Let U be a universal subclass of a universal class V of rings. We investigate connections between radicals in U and V. We define $\mathcal{T}$ and $\mathcal{T}_s$ as follows:

$$\mathcal{T} = \{A \in \mathcal{A}_{ss} \mid \text{every prime homomorphic image of } A \text{ is not a hereditary Amitsur ring}\}$$

$$\mathcal{T}_s = \{A \in \mathcal{A}_{ss} \mid \text{every prime homomorphic image of } A \text{ has no nonzero ideal which is a hereditary Amitsur ring}\}.$$

Let $\gamma \in \{\mathcal{T}, \mathcal{T}_s\}$ and let A be a commutative ring with minimum condition on ideals. We give a sufficient and necessary condition for A to be γ-semisimple.

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1. Introduction

In this note, all rings considered will belong to some arbitrary (but fixed) universal class V of not necessarily associative rings. A universal class of rings is a class of rings that is hereditary (that is, if $A \in V$ and I is an ideal of A, then $I \in V$) and homomorphically closed (that is, if $A \in V$ and I is an ideal of A, then $A/I \in V$). All radicals considered in this paper are in the sense of Kurosh and Amitsur. For the fundamental definitions and properties of radicals, we refer the reader to [1].

For any ring A, let A^0 denote the ring with additive group $(A,+)$ and with multiplication defined by $xy = 0$ for all $x, y \in R$. Such a ring A^0 is called a zero ring. The notation $I \trianglelefteq A$ means that I is an ideal of A. A subring B of a ring A is said to be an accessible subring of A if there exists a finite sequence $C_1, \dots, C_n$ of subrings of A such that $C_i \trianglelefteq C_{i+1}$, for $i = 1, \dots, n-1$, $C_1 = B$ and $C_n = A$. We recall that a radical γ of rings has the ADS property if $\gamma(I) \trianglelefteq A$ for every $I \trianglelefteq A$. Let

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$\mathcal{M}$ be a subclass of the universal class V . If $\mathcal{M}$ determines an upper radical, let $\mathcal{U}_V(\mathcal{M}) = \{A \in V \mid A$ has no nonzero homomorphic image in $\mathcal{M}\}$ that is, the upper radical generated by $\mathcal{M}$ in V. The lower radical in V, induced by the class $\mathcal{M}$, is denoted by $\mathcal{L}_V(\mathcal{M})$. For a radical γ, the semisimple class of γ in V is given by $(S_V)\gamma = \{A \in V \mid \gamma(A) = 0\}$.

2. General rings

In what follows, U denotes a universal subclass of the universal class V.

Proposition 1. *Let γ be a radical class in V. Then $\gamma \cap U$ is a radical class in U.*

Proof. Clear. □

Proposition 2. *Let γ be a radical class in U. Then $\gamma = \mathcal{L}_V(\gamma) \cap U$.*

Proof. It is clear that $\gamma \subseteq \mathcal{L}_V(\gamma) \cap U$.
If the semisimple class $(S_U)\gamma$ of γ is equal to 0, then $\gamma = U$ and $\gamma \supseteq \mathcal{L}_V(\gamma) \cap U$.
Suppose $(S_U)\gamma \neq 0$. Then $\mathcal{U}_V((S_U)\gamma) \supseteq \mathcal{L}_V(\gamma) \supseteq \mathcal{L}_V(\gamma) \cap U$. If there exists a nonzero ring $A \in \mathcal{L}_V(\gamma) \cap U$ with $\gamma(A) = 0$, then $A \in (S_U)\gamma$ and we have $A \in \mathcal{U}_V((S_U)\gamma) \cap (S_U)\gamma = 0$. This leads to a contradiction, so $\gamma \supseteq \mathcal{L}_V(\gamma) \cap U$ □

Now let us consider the following condition in the universal class V.
(ADS) : Every radical γ in V has the ADS property.

Theorem 1. *Let $U \subseteq V$ be universal classes of rings with property (ADS) and let γ be a radical in V. If for every ring $A \in \gamma$, there exists a nonzero accessible subring I_0 with $I_0 \in \gamma \cap U$, then $\mathcal{L}_V(\gamma \cap U) = \gamma$.*

Proof. We put $\gamma' = \mathcal{L}_V(\gamma \cap U)$.
It is clear that $\gamma' \subseteq \gamma$.
Let us suppose that $0 \neq A \in \gamma \setminus \gamma'$, then $\bar{0} \neq \bar{A} = A/\gamma'(A)$ and $\bar{A} \in \gamma$. Hence by assumption there exits a nonzero accessible subring $\overline{I_0} \trianglelefteq \overline{I_1} \trianglelefteq \overline{I_2} \trianglelefteq \cdots \trianglelefteq \overline{I_n} = \bar{A}$ such that $\overline{I_0} \in \gamma \cap U$. Therefore $\gamma'(\overline{I_1}) \neq \bar{0}$ and by ADS property we have $\gamma'(\overline{I_1}) \trianglelefteq \overline{I_2}$. Then by induction $\gamma'(\overline{I_n}) \neq \bar{0}$. Thus $\gamma'(\bar{A}) \neq \bar{0}$. But $\gamma'(A/\gamma'(A)) = \bar{0}$, which leads to a contradiction. Hence $\gamma \subseteq \gamma'$. □

3. Associative rings

In this section, all rings are associative not necessarily with a unity element. In [4], we introduced (hereditary) Amitsur rings and also constructed radicals $\mathcal{T}$ and $\mathcal{T}_s$. A ring A is said to be a (hereditary) Amitsur ring if $\gamma(A[x]) = (\gamma(A[x]) \cap A)[x]$, for all (hereditary) radicals γ. Let us recall the definitions of $\mathcal{T}$ and $\mathcal{T}_s$.
If A_{ss} denotes the class of all associative rings, then

$$\mathcal{T} = \{A \in \mathcal{A}_{ss} \mid \text{every prime homomorphic image of } A \text{ is not a hereditary Amitsur ring}\}.$$

$$\mathcal{T}_s = \{A \in \mathcal{A}_{ss} \mid \text{every prime homomorphic image of } A \text{ has no nonzero ideal which is a hereditary Amitsur ring}\}.$$

Working in the class $\mathcal{A}$ of associative rings, we denote the semisimple class of a radical γ by $S\gamma$.

Proposition 3. *If F is a finite field, then F is not a hereditary Amitsur ring.*

Proof. Let F be a finite field and let us consider the class $\{F\}$. It is clear that $\{F\}$ is a special class of rings. Therefore the upper radical $\gamma = \mathcal{U}(\{F\})$ is a special radical class of rings. Hence γ is a hereditary radical class.

Also it is easy to check that $\gamma(F[x])$ is the ideal of $F[x]$, generated by $x^{p^n} - x$, where p^n is the number of elements in F. So $\gamma(F[x]) \subseteq xF[x]$. Therefore $\gamma(F[x]) \neq (\gamma(F[x]) \cap F)[x]$. Thus F is not a hereditary Amitsur ring. □

Proposition 4. *[4] Let A be a ring without proper prime homomorphic images. If A is an infinite integral domain, then A is a hereditary Amitsur ring.*

Corollary 1. *Let $J = \left\{ \frac{2x}{2y+1} \mid x, y \in \mathbb{Z}, (2x, 2y+1) = 1 \right\}$. Then J is a hereditary Amitsur ring.*

Proposition 5. *Let $\gamma \in \{\mathcal{T}, \mathcal{T}_s\}$ and F be a field. Then*

(1) *$F \in S\gamma$ if and only if F is an infinite field.*
(2) *$F \in \gamma$ if and only if F is a finite field.*

Proof. It follows from Propositions 3 and 4. □

Proposition 6. *For all simple rings A, $\mathcal{T}(A) = \mathcal{T}_s(A)$.*

Proof. If A is a simple ring, then A has no nonzero proper ideals and so it is obvious that $A \in \mathcal{T}$ if and only if $A \in \mathcal{T}_s$ □

Let us denote by $\mathcal{P}$ the class of all prime rings. Let (η, ξ) be a partition of simple rings. Let us consider the following class:

$$s(\xi) = \{A \in \mathcal{P} \mid A \text{ is subdirectly irreducible with heart } \mathrm{H(A)} \in \xi\}$$

We denote by $\mathcal{L}_{sp}\eta$ the lower special radical generated by η.

Theorem 2. *[5] $\mathcal{L}_{sp}\eta \subsetneqq \mathcal{U}(s(\xi))$.*

Let us put:

$$\eta_{\mathcal{T}} = \{A \in \mathcal{T} \mid A \text{ is a simple ring}\}$$
$$\xi_{\mathcal{T}} = \{A \in S\mathcal{T} \mid A \text{ is a simple ring}\}$$
$$\eta_{\mathcal{T}_s} = \{A \in \mathcal{T}_s \mid A \text{ is a simple ring}\}$$
$$\xi_{\mathcal{T}_s} = \{A \in S\mathcal{T}_s \mid A \text{ is a simple ring}\}$$

Corollary 2. $\mathcal{L}_{sp}(\eta_{\mathcal{T}}) = \mathcal{L}_{sp}(\eta_{\mathcal{T}_s}) \subseteq \mathcal{T}_s \subseteq \mathcal{T} \subsetneqq \mathcal{U}(s(\xi_{\mathcal{T}})) = \mathcal{U}(s(\xi_{\mathcal{T}_s}))$.

Proof. By the notations above and Proposition 4 and 6, we have $\mathcal{L}_{sp}(\eta_{\mathcal{T}_s}) \subseteq \mathcal{T}_s \subseteq \mathcal{T} \subseteq \mathcal{U}(s(\xi_{\mathcal{T}})) = \mathcal{U}(s(\xi_{\mathcal{T}_s}))$. Let J be the ring defined on Corollary 1. Then in a similar to the proof of the Theorem 2, we can show that $J \notin \mathcal{T}$ and $J \in \mathcal{U}(s(\xi_{\mathcal{T}_s}))$. Thus $\mathcal{T} \subsetneqq \mathcal{U}(s(\xi_{\mathcal{T}_s}))$. □

Theorem 3. *Let A be a commutative ring, satisfying the minimum condition on ideals. Let $\gamma \in \{\mathcal{T}, \mathcal{T}_s\}$. Then $A \in S\gamma$ if only if $A = F_1 \oplus F_2 \oplus \cdots \oplus F_n$, where F_i is an infinite field for any i, $1 \leq i \leq n$. Moreover, this statement is true for any radical γ such that $\mathcal{L}_{sp}(\eta_{\mathcal{T}_s}) \subseteq \gamma \subseteq \mathcal{U}(s(\xi_{\mathcal{T}_s}))$.*

Proof. Let $A \in S\gamma$. Since A satisfies the minimum condition on ideals, there exists a non-zero minimal ideal $I \trianglelefteq A$. $A \in S\gamma$ implies that A is a semiprime ring. Therefore I is a prime simple ring. Since A is a commutative ring, I is a commutative ring. Hence $aI = I$, for every non-zero $a \in I$. Thus I is a field. It is well known that I is a direct summand of A. So we have $A = I_1 \oplus A_1$, where $I = I_1$ and $A_1 \in S\gamma$. Also, A_1 satisfies all the conditions of the theorem. Hence if we continue this procedure, then we have $A = I_1 \oplus I_2 \oplus \cdots \oplus I_n \oplus \ldots$

Put

$$\begin{aligned} J_1 &= A, \\ J_2 &= I_2 \oplus \cdots \oplus I_n \oplus \ldots. \\ &\vdots \\ J_n &= I_n \oplus \cdots \oplus I_{n+s} \ldots. \end{aligned}$$

Then $J_1 \supsetneqq J_2 \supsetneqq \cdots \supsetneqq J_n \supsetneqq \ldots$ and $J_i \trianglelefteq A$, for each $i = 1, 2, \ldots$.
By assumption there exists $n \in \mathbb{N}$, such that

$$J_n = J_{n+1} = \cdots = .$$

It implies $I_{n+1} = 0$. Since I_i, $1 \leq i \leq n$, is a field, we have

$$A = F_1 \oplus \cdots \oplus F_n, \text{ where } I_i = F_i.$$

Also $A \in S\gamma$ and $F_i \in S\gamma$. Thus by Proposition 5, each F_i is an infinite field.

Let $A = F_1 \oplus \cdots \oplus F_n$, where F_i is an infinite field. Then by Proposition 4 each $F_i \in S\gamma$. Thus $A \in S\gamma$. □

We denote by A_{ss} all associative rings and by C_{ss} all commutative associative rings. It is clear that C_{ss} is a universal subclass of A_{ss}.

Remark 1. It follows from Proposition 1 that $\mathcal{T} \cap C_{ss}$ and $\mathcal{T}_s \cap C_{ss}$ are radicals in C_{ss}.

Theorem 4. *Let $\gamma \in \{\mathcal{T}, \mathcal{T}_s\}$ and A be a subdirectly irreducible semiprime ring. Then A is $\sigma = \mathcal{L}_{A_{ss}}(\gamma \cap C_{ss})$-semisimple if and only if A is not a finite field.*

Proof. It is easy to see that $\sigma \subseteq \gamma$ and A is a prime ring. Suppose that A is not a finite field and not σ-semisimple. Therefore $\sigma(A) \neq 0$ and $\sigma(A) \trianglelefteq A$. It is clear that $\sigma(A)$ is a subdirectly irreducible ring. Let $H(\sigma(A))$ denote the heart of $\sigma(A)$. Since $\sigma(A) \in \sigma$, there exists a nonzero accessible subring I_0 of $\sigma(A)$, which is in $\gamma \cap C_{ss}$. Since A is a prime ring $H(\sigma(A)) \subseteq I_0$. This implies that $H(\sigma(A)$ is a simple commutative ring. Therefore $F = H(\sigma(A)$ is a field. It is easy to see that $F = A$. Thus by Proposition 6, $F \in \mathcal{T}_s \cap C_{ss}$. Then by Proposition 5, A is a finite field, which leads to a contradiction. Hence A is a σ-semisimple ring.

Suppose A is σ-semisimple and $A = F$ is a finite field. Then again by Proposition 5, $A = F \in \gamma \cap C_{ss}$, which leads to a contradiction. □

A class δ of rings is said to be a matrix-extensible class if for all natural numbers n, $A \in \delta$ if and only if $M_n(A) \in \delta$, where $M_n(A)$ is the $n \times n$ matrix ring.

Corollary 3. $\sigma = \mathcal{L}_{A_{ss}}(\gamma \cap C_{ss})$ *is not a matrix-extensible class.*

Proof. Let F be a finite field. Then $F \in \gamma \cap C_{ss}$. Thus, by Theorem 4, $M_n(F)$ is a σ-semisimple class. □

For the next theorem we use the following notations:
$\beta-$ Baer radical, $\mathcal{L}-$ Levitzki radical, $\mathcal{N}-$ Nil radical, $\mathcal{J}-$ Jacobson radical and $\mathcal{G}-$ Brown-McCoy radical. The notation $\alpha \upuparrows r$ means that α and r are not comparable radicals.

Theorem 5.

(i) $\beta = \mathcal{L}_{A_{ss}}(\beta \cap C_{ss})$;
(ii) $\mathcal{L} \neq \mathcal{L}_{A_{ss}}(\mathcal{L} \cap C_{ss}) = \beta$;
(iii) $\mathcal{N} \neq \mathcal{L}_{A_{ss}}(\mathcal{N} \cap C_{ss}) = \beta$;
(iv) $\mathcal{L}_{A_{ss}}(\mathcal{J} \cap C_{ss}) \upuparrows \mathcal{N}$;
(v) $\mathcal{L}_{A_{ss}}(\mathcal{G} \cap C_{ss}) \upuparrows \mathcal{N}$.

Proof.

(i) Let A be a ring in β. Then there exists a non-zero accessible subring I^0 of A, which is a zero ring. Hence $I^0 \in \beta \cap C_{ss}$. Thus by Theorem 1, $\beta = \mathcal{L}_{A_{ss}}(\beta \cap C_{ss})$. Also, we know that $\beta \cap C_{ss} = \mathcal{L} \cap C_{ss} = \mathcal{N} \cap C_{ss}$.

(ii) and (iii) In [6], E.I. Zelmanov constructed a ring A, which is locally nilpotent as well as prime. So $\beta(A) = 0$ while $\mathcal{L}(A) = A$, which implies $\beta \neq \mathcal{L}$. Also, it is easy to see that $\beta \subseteq \mathcal{L}_{A_{ss}}(\mathcal{L} \cap C_{ss}) \subseteq \mathcal{L}_{A_{ss}}(\mathcal{N} \cap C_{ss})$ and $\mathcal{N} \cap C_{ss} \subseteq \beta$. Thus $\beta = \mathcal{L}_{A_{ss}}(\mathcal{L} \cap C_{ss}) = \mathcal{L}_{A_{ss}}(\mathcal{N} \cap C_{ss})$. Also, it is easy to see that $\mathcal{L}_{A_{ss}}(\mathcal{N} \cap C_{ss}) \neq \mathcal{N}$.

(iv) We shall show that $\mathcal{L}_{A_{ss}}(\mathcal{J} \cap C_{ss}) \nsubseteq \mathcal{N}$. Let

$$J = \{\frac{2x}{2y+1} \mid x, y \in \mathbb{Z}, (2x, 2y+1) = 1\}.$$

Then we know that J is a commutative Jacobson radical ring. So $J \in \mathcal{J} \cap C_{ss}$, also $J \in \mathcal{L}_{A_{ss}}(\mathcal{J} \cap C_{ss})$. But J has no nonzero nilpotent elements. Thus $J \notin \mathcal{N}$. In [2] A.Smoktunowicz proved that there exists a simple nil prime ring A. It is clear that $A \notin \mathcal{L}_{A_{ss}}(\mathcal{J} \cap C_{ss})$. Thus $\mathcal{N} \nsubseteq \mathcal{L}_{A_{ss}}(\mathcal{J} \cap C_{ss})$.

(v) It is clear.

□

Corollary 4. *$\mathcal{J} \neq \mathcal{L}_{A_{ss}}(\mathcal{J} \cap C_{ss}) \neq \beta$ and $\mathcal{G} \neq \mathcal{L}_{A_{ss}}(\mathcal{G} \cap C_{ss}) \neq \beta$.*

Corollary 5. *Let $\gamma \in \{\mathcal{T}, \mathcal{T}_s\}$. Then $\beta \subsetneqq \mathcal{L}_{A_{ss}}(\gamma \cap C_{ss})$.*

Proof. All zero rings and all finite fields are in $\gamma \cap C_{ss}$. Thus $\beta \subsetneqq \mathcal{L}_{A_{ss}}(\gamma \cap C_{ss})$. □

4. ALTERNATIVE RINGS

In this section all rings are alternative. An alternative rings is a ring in which multiplication need not be associative, only alternative, that is, $x^2y = x(xy)$ and $yx^2 = (yx)x$, for all $x, y \in A$. We denote the class of all alternative rings by Alt. Let $\mathcal{M}$ be a nonempty class of alternative rings and assume that $\mathcal{M}$ is homomorphically closed. Let us define $\mathcal{M}_1 = \mathcal{M}$. Assuming that $\mathcal{M}_\alpha$ has been defined for every ordinal number α such that $1 \leq \alpha < \beta$, we define $\mathcal{M}_\beta$ to be the class of all alternative rings A such that every nonzero homomorphic image of A contains a nonzero ideal I, which is in $\mathcal{M}_\alpha$ for some $\alpha < \beta$. It is clear that $\mathcal{M}_\alpha \leq \mathcal{M}_\beta$ if $\alpha \leq \beta$ and each class $\mathcal{M}_\alpha$ is homomorphically closed. Let $\mathcal{L}_{Alt}(\mathcal{M}) = \cup_\alpha \mathcal{M}_\alpha$. Then $\mathcal{L}_{Alt}(\mathcal{M})$ determines a radical property and this is the smallest radical class containing $\mathcal{M}$ (see [6]).

Lemma 1 ([3]). *If B is a nonzero accessible subring of an alternative ring A and if B is in $\mathcal{M}$, then $\bar{B}$, the ideal of A generated by B, is in $\mathcal{M}_{q \cdot w_0}$, where q is finite and w_0 is the first infinite ordinal.*

Proposition 7. *An alternative ring A is in $\gamma = \mathcal{L}_{Alt}(\mathcal{M})$ if and only if every nonzero homomorphic image of $\bar{A}$ of A contains a nonzero accessible subring B, such that B is in $\mathcal{M}$.*

Proof. Suppose that A is not in γ. Then $\bar{A} = A/\gamma(A) \neq \bar{0}$. By assumption, $\bar{A}$ contains a nonzero accessible subring B such that B is in $\mathcal{M}$. From Lemma 1, $\bar{B}$ is the ideal of $\bar{A}$ generated by B, which is in $\mathcal{M}_{q \cdot w_0}$. Hence $\bar{B}$ is a γ-radical ideal. Therefore $\bar{0} = \gamma(A/\gamma(A)) = \gamma(\bar{A}) \neq \bar{0}$, which leads to a contradiction. Let $A \in \gamma$. Then every nonzero homomorphic image $\bar{A}$ of A is in γ. By Lemma 1 of [2] and the proof of Theorem 3 of [3], $0 \neq \bar{A}$ contains a nonzero accessible subring $\bar{B}$ such that $\bar{B}$ is in $\mathcal{M}$. □

Theorem 6. *Let γ be a radical in Alt. Then $\mathcal{L}_{Alt}(\gamma \cap A_{ss}) = \gamma$ if and only if every nonzero ring $A \in \gamma$ contains a nonzero accessible subring $B \in \gamma \cap A_{ss}$.*

Proof. It is well known that every radical γ in Alt has the ADS property. Thus the result follows from Theorem 1 and Proposition 7. □

Corollary 6. *Let γ be a hereditary radical in Alt. Then $\mathcal{L}_{Alt}(\gamma \cap A_{ss}) = \gamma$ if and only if every nonzero ring $A \in \gamma$ contains a nonzero associative accessible subring B.*

Proof. Let γ be a radical in Alt such that $\mathcal{L}_{Alt}(\gamma \cap A_{ss}) = \gamma$. By Theorem 6, every nonzero ring $A \in \gamma$ contains a nonzero accessible subring B such that $B \in \gamma \cap A_{ss}$. Thus B is an associative ring.

Let A be a nonzero ring in γ. Then, by the assumption, there exists a nonzero associative accessible subring B such that $B = B_1 \trianglelefteq B_2 \trianglelefteq \cdots \trianglelefteq B_{n-1} \trianglelefteq B_n = A$. Since γ is a hereditary radical, $B_n \in \gamma$. Also $B_{n-1} \in \gamma \ldots B_1 \in \gamma$. Hence $B_1 = B \in \gamma \cap A_{ss}$. Therefore, by Theorem 6, we have $\mathcal{L}_{Alt}(\gamma \cap A_{ss}) = \gamma$. □

Let us denote by $\mathcal{B}$ the Baer radical in Alt. We define a chain of subsets in a ring A by setting $A^{(1)} = A^2, \ldots, A^{(n)} = (A^{(n-1)})^2$. We recall that a ring A is solvable if $A^{(n)} = 0$, for some n. Also it is clear that $A^{(n)} \trianglelefteq A^{(n-1)}$.

Remark 2. From the definition of $\mathcal{B}$ it is easy to prove that $\mathcal{B}$ is generated by all solvable alternative rings. (see [6]).

Corollary 7. $\beta = \mathcal{B} \cap A_{ss}$ *and* $\mathcal{L}_{Alt}(\beta) = \mathcal{B}$.

Proof. By Remark 2 $\beta \subseteq \mathcal{B}$ and $\beta \subseteq A_{ss}$. Hence $\beta \subseteq \mathcal{B} \cap A_{ss}$. It is easy to see that $\mathcal{B} \cap A_{ss} \subseteq \beta$. Hence $\beta = \mathcal{B} \cap A_{ss}$. By Theorem 9 of [6] $\mathcal{B}$ is a hereditary radical. By Proposition 7 and Remark 2, for every ring $A \in \mathcal{B}$ there exists an accessible subring B of A such that B is a solvable ring, that is $B^k = 0$ and $B^{k-1} \neq 0$. Since B is an accessible subring of A, B^{k-1} is an accessible subring of A. But B^{k-1} is a zero ring. Thus $0 \neq B^{k-1} \in A_{ss}$. By Corollary 6, $\mathcal{L}_{Alt}(\beta) = \mathcal{B}$, because $\mathcal{B}$ is a hereditary radical. □

Remark 3. $\mathcal{T} = \mathcal{T}_s$ if and only if $\mathcal{L}_{Alt}(\mathcal{T}) = \mathcal{L}_{Alt}(\mathcal{T}_s)$.

Indeed, by Proposition 2, $\mathcal{T} = \mathcal{L}_{Alt}(\mathcal{T}) \cap A_{ss} = \mathcal{L}_{Alt}(\mathcal{T}_s) \cap A_{ss} = \mathcal{T}_s$.

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REFERENCES

[1] J. W. Gardner and R. Wiegandt, *Radical theory of rings.* New York, NY: Marcel Dekker, 2004, vol. 261, doi: 10.1201/9780203913352.

[2] A. Smoktunowicz, "A simple nil ring exists," *Commun. Algebra*, vol. 30, no. 1, pp. 27–59, 2002, doi: 10.1081/AGB-120006478.

[3] A. Suliński, R. Anderson, and N. Divinsky, "Lower radical properties for associative and alternative rings," *J. Lond. Math. Soc.*, vol. 41, pp. 417–424, 1966, doi: 10.1112/jlms/s1-41.1.417.

[4] S. Tumurbat, "On Amitsur rings," *Quaest. Math.*, vol. 42, no. 5, pp. 665–672, 2019, doi: 10.2989/16073606.2018.1480542.

[5] S. Tumurbat and R. Wiegandt, "A note on special radicals and partitions of simple rings," *Commun. Algebra*, vol. 30, no. 4, pp. 1769–1777, 2002, doi: 10.1081/AGB-120013214.

[6] E. I. Zel'manov, *An example of a finitely generated primitive ring*. Springer US, New York, NY; Pleiades Publishing, New York, NY; MAIK "Nauka/Interperiodica", Moscow, 1979, vol. 20, doi: 10.1007/BF00970042.

Authors' addresses

S. Tumurbat
Department of Mathematics, National University of Mongolia, Ikh Surguuliin Gudamj-1, Ulaanbaatar, Mongolia and, School of Applied Science, Mongolian University of Science and Technology, P.O.Box 75, Ulaanbaatar, Mongolia
E-mail address: stumurbat@hotmail.com

T. Khulan and D. Dayantsolmon
(**Corresponding author**) National University of Mongolia, Department of Mathematics, Ikh Surguuliin Gudamj-1, Ulaanbaatar, Mongolia, P.O.Box-46A/523,210646
E-mail address: hulangaaa@yahoo.com and dayantsolmon@num.edu.mn