Miskolc Mathematical Notes
Vol. 20 (2019), No. 1, pp. 345–352
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2019.2586

A GENERALIZATION OF g-SUPPLEMENTED MODULES

BERNA KOŞAR, CELIL NEBIYEV, AND AYTEN PEKIN

Received 07 April, 2018

Abstract. In this work g-radical supplemented modules are defined which generalize g-supplemented modules. Some properties of g-radical supplemented modules are investigated. It is proved that the finite sum of g-radical supplemented modules is g-radical supplemented. It is also proved that every factor module and every homomorphic image of a g-radical supplemented module is g-radical supplemented. Let R be a ring. Then ${}_RR$ is g-radical supplemented if and only if every finitely generated R-module is g-radical supplemented. In the end of this work, it is given two examples for g-radical supplemented modules separating with g-supplemented modules.

2010 *Mathematics Subject Classification:* 16D10; 16D70

Keywords: small submodules, radical, supplemented modules, radical (generalized) supplemented modules

1. INTRODUCTION

Throughout this paper all rings will be associative with identity and all modules will be unital left modules.

Let R be a ring and M be an R -module. We will denote a submodule N of M by $N \leq M$. Let M be an R -module and $N \leq M$. If $L = M$ for every submodule L of M such that $M = N + L$, then N is called a *small submodule* of M and denoted by $N \ll M$. Let M be an R -module and $N \leq M$. If there exists a submodule K of M such that $M = N + K$ and $N \cap K = 0$, then N is called a *direct summand* of M and it is denoted by $M = N \oplus K$. For any module M, we have $M = M \oplus 0$. $RadM$ indicates the radical of M. A submodule N of an R -module M is called an *essential submodule* of M, denoted by $N \trianglelefteq M$, in case $K \cap N \neq 0$ for every submodule $K \neq 0$. Let M be an R -module and K be a submodule of M. K is called a *generalized small* (briefly, *g-small*) *submodule* of M if for every $T \trianglelefteq M$ with $M = K + T$ implies that $T = M$, this is written by $K \ll_g M$ (in [6], it is called an *e-small submodule* of M and denoted by $K \ll_e M$). It is clear that every small submodule is a generalized small submodule but the converse is not true generally. Let M be an $R-$module. M is called an *hollow module* if every proper submodule of M is small in M. M is called a *local module* if M has the largest submodule, i.e.

a proper submodule which contains all other proper submodules. Let U and V be submodules of M. If $M = U + V$ and V is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V \ll V$, then V is called a *supplement* of U in M. M is called a *supplemented module* if every submodule of M has a supplement in M. Let M be an R-module and $U, V \leq M$. If $M = U + V$ and $M = U + T$ with $T \trianglelefteq V$ implies that $T = V$, or equivalently, $M = U + V$ and $U \cap V \ll_g M$, then V is called a *g-supplement* of U in M. M is called *g-supplemented* if every submodule of M has a g-supplement in M. The intersection of maximal essential submodules of an R-module M is called a *generalized radical* of M and denoted by $Rad_g M$ (in [6], it is denoted by $Rad_e M$). If M have no maximal essential submodules, then we denote $Rad_g M = M$.

Lemma 1 ([2, 4, 6])**.** *Let M be an R -module and $K, L, N, T \leq M$. Then the followings are hold.*

(1) *If $K \leq N$ and N is generalized small submodule of M, then K is a generalized small submodule of M.*

(2) *If K is contained in N and a generalized small submodule of N, then K is a generalized small submodule in submodules of M which contains submodule N.*

(3) *Let S be an R-module and $f : M \to S$ be an R -module homomorphism. If $K \ll_g M$, then $f(K) \ll_g S$.*

(4) *If $K \ll_g L$ and $N \ll_g T$, then $K + N \ll_g L + T$.*

Corollary 1. *Let $M_1, M_2, ..., M_n \leq M$, $K_1 \ll_g M_1$, $K_2 \ll_g M_2$, ..., $K_n \ll_g M_n$. Then $K_1 + K_2 + ... + K_n \ll_g M_1 + M_2 + ... + M_n$.*

Corollary 2. *Let M be an R -module and $K \leq N \leq M$. If $N \ll_g M$, then $N/K \ll_g M/K$.*

Corollary 3. *Let M be an R -module, $K \ll_g M$ and $L \leq M$. Then $(K + L)/L \ll_g M/L$.*

Lemma 2. *Let M be an R-module. Then $Rad_g M = \sum_{L \ll_g M} L$.*

Proof. See [2]. □

Lemma 3. *The following assertions are hold.*

(1) *If M is an $R-$module, then $Rm \ll_g M$ for every $m \in Rad_g M$.*

(2) *If $N \leq M$, then $Rad_g N \leq Rad_g M$.*

(3) *If $K, L \leq M$, then $Rad_g K + Rad_g L \leq Rad_g (K + L)$.*

(4) *If $f : M \longrightarrow N$ is an R-module homomorphism, then $f\left(Rad_g M\right) \leq Rad_g N$.*

(5) *If $K, L \leq M$, then $\frac{Rad_g K + L}{L} \leq Rad_g \frac{K+L}{L}$.*

Proof. Clear from Lemma 1 and Lemma 2. □

Lemma 4. *Let $M = \oplus_{i \in I} M_i$. Then $Rad_g M = \oplus_{i \in I} Rad_g M_i$.*

Proof. Since $M_i \leq M$, then by Lemma 3(2), $Rad_g M_i \leq Rad_g M$ and $\oplus_{i\in I} Rad_g M_i \leq Rad_g M$. Let $x \in Rad_g M$. Then by Lemma 3(1), $Rx \ll_g M$. Since $x \in M = \oplus_{i\in I} M_i$, there exist $i_1, i_2, ..., i_k \in I$ and $x_{i_1} \in M_{i_1}$, $x_{i_2} \in M_{i_2}$, ..., $x_{i_k} \in M_{i_k}$ such that $x = x_{i_1} + x_{i_2} + ... + x_{i_k}$. Since $Rx \ll_g M$, then by Lemma 1(4), under the canonical epimorphism π_{i_t} $(t = 1, 2, ..., k)$ $Rx_{i_t} = \pi_{i_t}(Rx) \ll_g Rx_{i_t}$. Then $x_{i_t} \in Rad_g M_{i_t}$ $(t = 1, 2, ..., k)$ and $x = x_{i_1} + x_{i_2} + ... + x_{i_k} \in \oplus_{i\in I} Rad_g M_i$. Hence $Rad_g M \leq \oplus_{i\in I} Rad_g M_i$ and since $\oplus_{i\in I} Rad_g M_i \leq Rad_g M$, $Rad_g M = \oplus_{i\in I} Rad_g M_i$. □

2. G-RADICAL SUPPLEMENTED MODULES

Definition 1. Let M be an R-module and $U, V \leq M$. If $M = U + V$ and $U \cap V \leq Rad_g V$, then V is called a generalized radical supplement (briefly, g-radical supplement) of U in M. If every submodule of M has a generalized radical supplement in M, then M is called a generalized radical supplemented (briefly, g-radical supplemented) module.

Clearly we see that every g-supplemented module is g-radical supplemented. But the converse is not true in general. (See Example 1 and 2.)

Lemma 5. *Let M be an R-module and $U, V \leq M$. Then V is a g-radical supplement of U in M if and only if $M = U + V$ and $Rm \ll_g V$ for every $m \in U \cap V$.*

Proof. $(\Rightarrow)$ Since V is a g-radical supplement of U in M, $M = U + V$ and $U \cap V \leq Rad_g V$. Let $m \in U \cap V$. Since $U \cap V \leq Rad_g V$, $m \in Rad_g V$. Hence by Lemma 3(1), $Rm \ll_g V$.

$(\Leftarrow)$ Since $Rm \ll_g V$ for every $m \in U \cap V$, then by Lemma 2, $U \cap V \leq Rad_g V$ and hence V is a g-radical supplement of U in M. □

Lemma 6. *Let M be an R-module, $M_1, U, X \leq M$ and $Y \leq M_1$. If X is a g-radical supplement of $M_1 + U$ in M and Y is a g-radical supplement of $(U + X) \cap M_1$ in M_1, then $X + Y$ is a g-radical supplement of U in M.*

Proof. Since X is a g-radical supplement of $M_1 + U$ in M, $M = M_1 + U + X$ and $(M_1 + U) \cap X \leq Rad_g X$. Since Y is a g-radical supplement of $(U + X) \cap M_1$ in M_1, $M_1 = (U + X) \cap M_1 + Y$ and $(U + X) \cap Y = (U + X) \cap M_1 \cap Y \leq Rad_g Y$. Then $M = M_1 + U + X = (U + X) \cap M_1 + Y + U + X = U + X + Y$ and, by Lemma 3(3), $U \cap (X + Y) \leq (U + X) \cap Y + (U + Y) \cap X \leq Rad_g Y + (M_1 + U) \cap X \leq Rad_g Y + Rad_g X \leq Rad_g (X + Y)$. Hence $X + Y$ is a g-radical supplement of U in M. □

Lemma 7. *Let $M = M_1 + M_2$. If M_1 and M_2 are g-radical supplemented, then M is also g-radical supplemented.*

Proof. Let $U \leq M$. Then 0 is a g-radical supplement of $M_1 + M_2 + U$ in M. Since M_1 is g-radical supplemented, there exists a g-radical supplement X of

$(M_2+U)\cap M_1=(M_2+U+0)\cap M_1$ in M_1. Then by Lemma 6, $X+0=X$ is a g-radical supplement of M_2+U in M. Since M_2 is g-radical supplemented, there exists a g-radical supplement Y of $(U+X)\cap M_2$ in M_2. Then by Lemma 6, $X+Y$ is a g-radical supplement of U in M. □

Corollary 4. *Let $M=M_1+M_2+...+M_k$. If M_i is g-radical supplemented for every $i=1,2,...,k$, then M is also g-radical supplemented.*

Proof. Clear from Lemma 7. □

Lemma 8. *Let M be an $R-$module, $U,V\leq M$ and $K\leq U$. If V is a g-radical supplement of U in M, then $(V+K)/K$ is a g-radical supplement of U/K in M/K.*

Proof. Since V is a g-radical supplement of U in M, $M=U+V$ and $U\cap V\leq Rad_gV$. Then $M/K=U/K+(V+K)/K$ and by Lemma 3(5), $(U/K)\cap((V+K)/K)=(U\cap V+K)/K\leq\left(Rad_gV+K\right)/K\leq Rad_g[(V+K)/K]$. Hence $(V+K)/K$ is a g-radical supplement of U/K in M/K. □

Lemma 9. *Every factor module of a g-radical supplemented module is g-radical supplemented.*

Proof. Clear from Lemma 8. □

Corollary 5. *The homomorphic image of a g-radical supplemented module is g-radical supplemented.*

Proof. Clear from Lemma 9. □

Lemma 10. *Let M be a g-radical supplemented module. Then every finitely $M-$generated module is g-radical supplemented.*

Proof. Clear from Corollary 4 and Corollary 5. □

Corollary 6. *Let R be a ring. Then ${}_RR$ is g-radical supplemented if and only if every finitely generated $R-$module is g-radical supplemented.*

Proof. Clear from Lemma 10. □

Theorem 1. *Let M be an $R-$module. If M is g-radical supplemented, then M/Rad_gM is semisimple.*

Proof. Let $U/Rad_gM\leq M/Rad_gM$. Since M is g-radical supplemented, there exists a g-radical supplement V of U in M. Then $M=U+V$ and $U\cap V\leq Rad_gV$. Thus $M/Rad_gM=U/Rad_gM+\left(V+Rad_gM\right)/Rad_gM$ and

$$\begin{aligned}\left(U/Rad_gM\right)\cap\left(\left(V+Rad_gM\right)/Rad_gM\right)&=\left(U\cap V+Rad_gM\right)/Rad_gM\\&\leq\left(Rad_gV+Rad_gM\right)/Rad_gM\\&=Rad_gM/Rad_gM=0.\end{aligned}$$

Hence $M/Rad_gM=U/Rad_gM\oplus\left(V+Rad_gM\right)/Rad_gM$ and U/Rad_gM is a direct summand of M. □

Lemma 11. *Let M be a g-radical supplemented module and $L \leq M$ with $L \cap Rad_g M = 0$. Then L is semisimple. In particular, a g-radical supplemented module M with $Rad_g M = 0$ is semisimple.*

Proof. Let $X \leq L$. Since M is g-radical supplemented, there exists a g-radical supplement T of X in M. Hence $M = X + T$ and $X \cap T \leq Rad_g T \leq Rad_g M$. Since $M = X + T$ and $X \leq L$, by Modular Law, $L = L \cap M = L \cap (X + T) = X + L \cap T$. Since $X \cap T \leq Rad_g M$ and $L \cap Rad_g M = 0$, $X \cap L \cap T = L \cap X \cap T \leq L \cap Rad_g M = 0$. Hence $L = X \oplus L \cap T$ and X is a direct summand of L. □

Proposition 1. *Let M be a g-radical supplemented module. Then $M = K \oplus L$ for some semisimple module K and some module L with essential generalized radical.*

Proof. Let K be a complement of $Rad_g M$ in M. Then by [5, 17.6], $K \oplus Rad_g M \trianglelefteq M$. Since $K \cap Rad_g M = 0$, then by Lemma 11, K is semisimple. Since M is g-radical supplemented, there exists a g-radical supplement L of K in M. Hence $M = K + L$ and $K \cap L \leq Rad_g L \leq Rad_g M$. Then by $K \cap Rad_g M = 0$, $K \cap L = 0$. Hence $M = K \oplus L$. Since $M = K \oplus L$, then by Lemma 4, $Rad_g M = Rad_g K \oplus Rad_g L$. Hence $K \oplus Rad_g M = K \oplus Rad_g L$. Since $K \oplus Rad_g L = K \oplus Rad_g M \trianglelefteq M = K \oplus L$, then by [1, Proposition 5.20], $Rad_g L \trianglelefteq L$. □

Proposition 2. *Let M be an $R-$module and $U \leq M$. The following statements are equivalent.*

(1) *There is a decomposition $M = X \oplus Y$ with $X \leq U$ and $U \cap Y \leq Rad_g Y$.*

(2) *There exists an idempotent $e \in End(M)$ with $e(M) \leq U$ and $(1-e)(U) \leq Rad_g (1-e)(M)$.*

(3) *There exists a direct summand X of M with $X \leq U$ and $U/X \leq Rad_g (M/X)$.*

(4) *U has a g-radical supplement Y such that $U \cap Y$ is a direct summand of U.*

Proof. (1) ⇒ (2) For a decomposition $M = X \oplus Y$, there exists an idempotent $e \in End(M)$ with $X = e(M)$ and $Y = (1-e)(M)$. Since $e(M) = X \leq U$, we easily see that $(1-e)(U) = U \cap (1-e)(M)$. Then by $Y = (1-e)(M)$ and $U \cap Y \leq Rad_g Y$, $(1-e)(U) = U \cap (1-e)(M) = U \cap Y \leq Rad_g Y = Rad_g (1-e)(M)$.

(2) ⇒ (3) Let $X = e(M)$ and $Y = (1-e)(M)$. Since $e \in End(M)$ is idempotent, we easily see that $M = X \oplus Y$. Then $M = U + Y$. Since $e(M) = X \leq U$, we easily see that $(1-e)(U) = U \cap (1-e)(M)$. Since $M = U + Y$ and $U \cap Y = U \cap (1-e)(M) = (1-e)(U) \leq Rad_g (1-e)(M) = Rad_g Y$, Y is a g-radical supplement of U in M. Then by Lemma 8, $M/X = (Y + X)/X$ is a g-radical supplement of U/X in M/X. Hence $U/X = (U/X) \cap (M/X) \leq Rad_g (M/X)$.

(3) ⇒ (4) Let $M = X \oplus Y$. Since $X \leq U$, $M = U + Y$. Let $t \in U \cap Y$ and $Rt + T = Y$ for an essential submodule T of Y. Let $((T + X)/X) \cap (L/X) = 0$ for a submodule L/X of M/X. Then $(L \cap T + X)/X = ((T + X)/X) \cap (L/X) = 0$ and $L \cap T + X = X$. Hence $L \cap T \leq X$ and since $X \cap Y = 0$, $L \cap T \cap Y \leq X \cap Y = 0$. Since $L \cap Y \cap T = L \cap T \cap Y = 0$ and $T \trianglelefteq Y$, $L \cap Y = 0$. Since $X \leq L$ and

$M = X + Y$, by Modular Law, $L = L \cap M = L \cap (X + Y) = X + L \cap Y = X + 0 = X$. Hence $L/X = 0$ and $(T + X)/X \trianglelefteq M/X$. Since $Rt + T = Y$, $R(t + X) + (T + X)/X = (Rt + X)/X + (T + X)/X = (Rt + T + X)/X = (Y + X)/X = M/X$. Since $t \in U$, $t + X \in U/X \leq Rad_g(M/X)$ and hence $R(t + X) \ll_g M/X$. Then by $R(t + X) + (T + X)/X = M/X$ and $(T + X)/X \trianglelefteq M/X$, $(T + X)/X = M/X$ and then $X + T = M$. Since $X + T = M$ and $T \leq Y$, by Modular Law, $Y = Y \cap M = Y \cap (X + T) = X \cap Y + T = 0 + T = T$. Hence $Rt \ll_g Y$ and by Lemma 5, Y is a g-radical supplement of U in M. Since $M = X \oplus Y$ and $X \leq U$, by Modular Law, $U = U \cap M = U \cap (X \oplus Y) = X \oplus U \cap Y$. Hence $U \cap Y$ is a direct summand of U.

(4) $\Rightarrow$ (1) Let $U = X \oplus U \cap Y$ for a submodule X of U. Since Y is a g-radical supplement of U in M, $M = U + Y$ and $U \cap Y \ll_g Y$. Hence $M = U + Y = (X \oplus U \cap Y) + Y = X \oplus Y$. □

Lemma 12. *Let V be a g-radical supplement of U in M. If U is a generalized maximal submodule of M, then $U \cap V$ is a unique generalized maximal submodule of V.*

Proof. Since U is a generalized maximal submodule of M and $V/(U \cap V) \simeq (V + U)/U = M/U$, $U \cap V$ is a generalized maximal submodule of V. Hence $Rad_g V \leq U \cap V$ and since $U \cap V \leq Rad_g V$, $Rad_g V = U \cap V$. Thus $U \cap V$ is a unique generalized maximal submodule of V. □

Definition 2. Let M be an $R-$module. If every proper essential submodule of M is generalized small in M or M has no proper essential submodules, then M is called a generalized hollow module.

Clearly we see that every hollow module is generalized hollow.

Definition 3. Let M be an $R-$module. If M has a large proper essential submodule which contain all essential submodules of M or M has no proper essential submodules, then M is called a generalized local module.

Clearly we see that every local module is generalized local.

Proposition 3. *Let M be an $R-$module and $Rad_g M \neq M$. Then M is generalized hollow if and only if M is generalized local.*

Proof. ($\Longrightarrow$) Let M be generalized hollow and let L be a proper essential submodule of M. Then $L \ll_g M$ and by Lemma 2, $L \leq Rad_g M$. Thus $Rad_g M$ is a proper essential submodule of M which contain all proper essential submodules of M.

($\Longleftarrow$) Let M be a generalized local module, T be the largest proper essential submodule of M and L be a proper essential submodule of M. Let $L + S = M$ with $S \trianglelefteq M$. If $S \neq M$, then $L + S \leq T \neq M$. Thus $S = M$ and $L \ll_g M$. □

Definition 4. Let M be an R-module and $U, V \leq M$. If $M = U + V$ and $U \cap V \ll_g M$, then V is called a weak g-supplement of U in M. If every submodule of M has a weak g-supplement in M, then M is called a weakly g-supplemented module. (See [3]).

Clearly we can see that if M is a weakly g-supplemented module, then M is g-semilocal ($M/Rad_g M$ is semisimple, see [3]).

Proposition 4. *Generalized hollow and generalized local modules are weakly g-supplemented, so are g-semilocal.*

Proof. Clear from definitions. □

Proposition 5. *Let M be a g-radical supplemented module with $Rad_g M \ll_g M$. Then M is weakly g-supplemented.*

Proof. Clear from definitions. □

Example 1. Consider the $\mathbb{Z}-$module $\mathbb{Q}$. Since $Rad_g \mathbb{Q} = Rad \mathbb{Q} = \mathbb{Q}$, ${}_{\mathbb{Z}}\mathbb{Q}$ is g-radical supplemented. But, since ${}_{\mathbb{Z}}\mathbb{Q}$ is not supplemented and every nonzero submodule of ${}_{\mathbb{Z}}\mathbb{Q}$ is essential in ${}_{\mathbb{Z}}\mathbb{Q}$, ${}_{\mathbb{Z}}\mathbb{Q}$ is not g-supplemented.

Example 2. Consider the $\mathbb{Z}-$module $\mathbb{Q} \oplus \mathbb{Z}_{p^2}$ for a prime p. It is easy to check that $Rad_g \mathbb{Z}_{p^2} \neq \mathbb{Z}_{p^2}$. By Lemma 4, $Rad_g \left(\mathbb{Q} \oplus \mathbb{Z}_{p^2}\right) = Rad_g \mathbb{Q} \oplus Rad_g \mathbb{Z}_{p^2} \neq \mathbb{Q} \oplus \mathbb{Z}_{p^2}$. Since $\mathbb{Q}$ and $\mathbb{Z}_{p^2}$ are g-radical supplemented, by Lemma 7, $\mathbb{Q} \oplus \mathbb{Z}_{p^2}$ is g-radical supplemented. But $\mathbb{Q} \oplus \mathbb{Z}_{p^2}$ is not g-supplemented.

REFERENCES

[1] F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules (Graduate Texts in Mathematics)*. New York: Springer, 1998.

[2] B. Koşar, C. Nebiyev, and N. Sökmez, "G-supplemented modules," *Ukrainian Mathematical Journal*, vol. 67, no. 6, pp. 861–864, 2015, doi: 10.1007/s11253-015-1127-8.

[3] C. Nebiyev and H. H. Ökten, "Weakly g-supplemented modules," *European Journal of Pure and Applied Mathematics*, vol. 10, no. 3, pp. 521–528, 2017.

[4] N. Sökmez, B. Koşar, and C. Nebiyev, "Genelleştirilmiş küçük alt modüller," in *XIII. Ulusal Matematik Sempozyumu*. Kayseri: Erciyes Üniversitesi, 2010.

[5] R. Wisbauer, *Foundations of Module and Ring Theory*. Philadelphia: Gordon and Breach, 1991.

[6] D. X. Zhou and X. R. Zhang, "Small-essential submodules and morita duality," *Southeast Asian Bulletin of Mathematics*, vol. 35, pp. 1051–1062, 2011.

Authors' addresses

Berna Koşar

Department of Mathematics, Ondokuz Mayıs University, 55270, Kurupelit-Atakum, Samsun, Turkey

E-mail address: bernak@omu.edu.tr

Celil Nebiyev
Department of Mathematics, Ondokuz Mayıs University, 55270, Kurupelit-Atakum, Samsun, Turkey
E-mail address: cnebiyev@omu.edu.tr

Ayten Pekin
Department of Mathematics, İstanbul University, İstanbul, Turkey
E-mail address: aypekin@istanbul.edu.tr