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q -Szász Schurer operators

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q -SZÁSZ SCHURER OPERATORS

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Abstract. In this paper, we introduce q -Szász Schurer operators and calculate their moments. The transformation properties, Korovkin type approximation theorem and rate of convergence of the operators are studied. We further obtain global estimates for q -Szász Schurer operators in terms of some Lipschitz classes.

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1. INTRODUCTION

Korovkin type approximation theory has been constructed on linear positive operators and has been an active research field during the last century because of its simple applicability [2]. After the works of A. Lupaş [9] and G.M. Phillips [19], where they introduced different analogues of the q -Bernstein polynomials $B_{n,q}(f; x)$, $n = 1, 2, \dots$, $0 < q < \infty$, intensive research has been conducted on linear positive operators based on q -integers (see [4, 6, 8, 15–18, 20, 21]).

In [4], [3] q -Szász Mirakjan operators were defined and their approximation properties were investigated. For $0 \leq x < \frac{b_n}{(1-q)[n]}$, $f \in C[0, \infty)$, and $\{b_n\}$ is a sequence of positive numbers such that $\lim_{n \rightarrow \infty} b_n = \infty$, in [4], q -Szász Mirakjan operators were defined by

$$S_{n,q}(f)(x) := E_q \left(-[n] \frac{x}{b_n} \right) \sum_{k=0}^{\infty} f \left(\frac{[k]b_n}{[n]} \right) \frac{[n]^k x^k}{[k]! b_n^k},$$

where

$$[k] = \begin{cases} (1-q^k)/(1-q) & \text{if } q \neq 1 \\ k & \text{if } q = 1 \end{cases}, \quad [k]! = \begin{cases} [1][2]\dots[k] & \text{if } k \geq 1 \\ 1 & \text{if } k = 0 \end{cases},$$

$$E_q(x) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}}}{[n]!} x^n = (-(1-q)x; q)_{\infty}; x \in \mathbb{R}, |q| < 1, \quad (1.1)$$

$$(a; q)_{\infty} = \prod_{j=0}^{\infty} (1 - aq^j) \quad (a \in \mathbb{C}).$$

As it was pointed out in [11], it is better to refer to these operators as q -parametric Szász-Mirakjan-Chlodowsky operators because of their structures. In [11], Mahmudov introduced the following q -Szász-Mirakjan operators

$$S_{n,q}^*(f)(x) = \frac{1}{E_q([n]x)} \sum_{k=0}^{\infty} f\left(\frac{[k]}{q^{k-2}[n]}\right) q^{\frac{k(k-1)}{2}} \frac{[n]^k x^k}{[k]!},$$

where $x \in [0, \infty)$, $0 < q < 1$, $f \in C[0, \infty)$ and studied convergence properties of these operators. We note that these operators do not preserve linear functions. In [11], Mahmudov obtained inequalities for the weighted approximation error, and also for the rate of convergence in terms of weighted moduli of continuity, and a Voronoskaja-type formula for q -Szász-Mirakjan operators was derived, too.

Now let $E_q(x)$ be defined by (1.1). In the present paper we introduce q -Szász Schurer operators which is defined for fixed $p \in \mathbb{N}_0$ by

$$S_{n,q}(f; x; p) = \frac{1}{E([n+p]x)} \sum_{k=0}^{\infty} f\left(\frac{[k]}{q^{k-1}[n]}\right) q^{\frac{k(k-1)}{2}} \frac{[n+p]^k x^k}{[k]!} \quad (1.2)$$

where $x \in [0, \infty)$, $0 < q < 1$, $f \in C[0, \infty)$. It is clear from (1.1) that

$$S_{n,q}(1; x; p) = 1, \quad (1.3)$$

for each fixed $p \in \mathbb{N}_0$. Note that in the case $p = 0$, q -Szász Schurer operators reduces to the slightly modified form of the Mahmudov's q -Szász-Mirakjan operators. Furthermore, letting $q \rightarrow 1$ in (1.2), one can get Schurer-Szász-Mirakjan operators (see p.p. 338, [2]).

There are some recent papers related to the q -Szász operators. For instance, in [1] and in [10] different variants of King type q -Szász operators (preserving x^2 and providing better error estimation) has been considered. On the other hand, integral type q -Szász operators were recently investigated in [13], [14], [12] and [7]. These types of modifications of the q -Szász Schurer operators will be the subject of future studies.

In this paper we compute the moments of the q -Szász Schurer operators and investigate their transformation properties. Korovkin type approximation result and rate of convergence of the operators are obtained in Section 3. Section 4 is devoted to global estimates for q -Szász Schurer operators in terms of some Lipschitz classes.

2. AUXILIARY LEMMAS

In this section we calculate the moments of the q -Szász Schurer operators and investigate their transformation properties. We start with the following lemma:

Lemma 1. *For $m \in \mathbb{N}_0$ and fixed $p \in \mathbb{N}_0$, we have*

$$S_{n,q}(t^{m+1}; x; p) = \frac{[n+p]x}{[n]} \sum_{j=0}^m \binom{m}{j} \frac{1}{q^j [n]^{m-j}} S_{n,q}(t^j; x; p). \quad (2.1)$$

Using (1.3) and the above lemma, we get the following results immediately.

Corollary 1. *For fixed $p \in \mathbb{N}_0$, we have*

$$\begin{aligned} S_{n,q}(t; x; p) &= \frac{[n+p]x}{[n]}, \quad S_{n,q}(t^2; x; p) = \frac{1}{q} \left(\frac{[n+p]x}{[n]} \right)^2 + \frac{[n+p]x}{[n]^2}, \\ S_{n,q}(t^3; x; p) &= \frac{1}{q^3} \left(\frac{[n+p]x}{[n]} \right)^3 + \frac{2q+1}{q^2} \frac{[n+p]^2 x^2}{[n]^3} + \frac{[n+p]x}{[n]^3}. \end{aligned}$$

Note that the operators $S_{n,q}(t; x; 0)$ preserve the linear functions while $S_{n,q}^*(f)(x)$ does not preserve them. Now, using linearity of the operators, we have the following:

Corollary 2. *For the second central moment, we have*

$$S_{n,q}((t-x)^2; x; p) = \left[\left(\frac{[n+p]}{q[n]} - 2 \right) \frac{[n+p]}{[n]} + 1 \right] x^2 + \frac{[n+p]}{[n]^2} x.$$

Lemma 2. *For every $x \geq 0$, we have*

$$\begin{aligned} & \frac{1}{E([n+p]x)} \sum_{k=0}^{\infty} \left| \frac{[k]}{q^{k-1}[n]} - x \right| q^{\frac{k(k-1)}{2}} \frac{[n+p]^k x^k}{[k]!} \\ & \leq \left(\left[\left(\frac{[n+p]}{q[n]} - 2 \right) \frac{[n+p]}{[n]} + 1 \right] x^2 + \frac{[n+p]}{[n]^2} x \right)^{1/2}. \end{aligned}$$

From (2.1), it is not difficult to obtain the following result by induction.

Lemma 3. *For the m^{th} moment ($m = 1, 2, \dots$) we have the following:*

$$\begin{aligned} S_{n,q}(t^m; x; p) &= q^{\frac{-m(m-1)}{2}} \left(\frac{[n+p]x}{[n]} \right)^m \\ &+ \frac{b_{m,1}(q)}{[n]} \left(\frac{[n+p]x}{[n]} \right)^{m-1} + \dots + \frac{b_{m,m-1}(q)}{[n]^{m-1}} \frac{[n+p]x}{[n]}, \end{aligned} \quad (2.2)$$

where the coefficients $b_{m,i}(q)$ ($i = 1, 2, \dots, m-1$) are constants depending on m and q . Furthermore, $b_{m,m-1}(q) = 1$ ($m = 1, 2, \dots$).

Now let $C_B[0, \infty)$ denote the space of all real valued continuous bounded functions on $[0, \infty)$ endowed with the norm

$$\|f\| = \sup_{x \in [0, \infty)} |f(x)|, \quad f \in C_B[0, \infty).$$

It is obvious by (1.3) that $S_{n,q}$ maps $C_B[0, \infty)$ into itself.

Let $r \in \mathbb{N}_0 := \{0, 1, \dots\}$ and define the weight function μ_r as follows: $\mu_0(x) := 1$ and $\mu_r(x) := (1 + x^r)^{-1}$ for $x \geq 0$ and $r \in \mathbb{N}_0$. Also for $r \in \mathbb{N}_0$, let $C_r[0, \infty)$ be the space of functions $f \in C[0, \infty)$, such that $\mu_r f$ is uniformly continuous and bounded on $[0, \infty)$, endowed with the norm

$$\|f\|_r := \sup_{x \in [0, \infty)} \mu_r(x) |f(x)| \quad \text{for } f \in C_r[0, \infty).$$

Using (2.2), we easily obtain the following transformation property.

Lemma 4. *For the operators $S_{n,q}$, there exists a constant $K_r(q) > 0$ depending on r and q , such that*

$$\mu_r(x) S_{n,q} \left(\frac{1}{\mu_r}; x; p \right) \leq K_r(q). \quad (2.3)$$

Furthermore, for all $f \in C_r[0, \infty)$, we have

$$\|S_{n,q}(f)\|_r \leq K_r(q) \|f\|_r, \quad (2.4)$$

which guarantees that $S_{n,q}$ maps $C_r[0, \infty)$ into $C_r[0, \infty)$.

3. KOROVKIN TYPE THEOREM

In this section we consider the space

$$E := \left\{ f \in C_2[0, \infty) : \exists \lim_{x \rightarrow \infty} \frac{|f(x)|}{1 + x^2} < \infty \right\},$$

endowed with the norm

$$\|f\|_2 := \sup_{x \in [0, \infty)} \mu_2(x) |f(x)|.$$

Now let $b > 0$. The usual modulus of continuity of f on the closed interval $[0, b]$ is defined by

$$\omega_b(f, \delta) = \sup_{\substack{|t-x| \leq \delta \\ x, t \in [0, b]}} |f(t) - f(x)|. \quad (3.1)$$

It is well known that, for a function $f \in E$, we have $\lim_{\delta \rightarrow \infty} \omega_b(f, \delta) = 0$.

We start by obtaining the rate of convergence of the operators $S_{n,q}(f; x; p)$ to $f(x)$, for all $f \in E$.

Theorem 1. Let $f \in E$ and let $\omega_{b+1}(f, \delta)$ ($b > 0$) be its modulus of continuity on the finite interval $[0, b+1] \subset [0, \infty)$. Then for fixed $q \in (0, 1)$, we have

$$\|S_{n,q}(f; x; p) - f(x)\|_{C[0,b]} \leq 2\omega_{b+1}(f, \delta(q)) + N_f (1+b^2) \left\{ \left[\left(\frac{[n+p]}{q[n]} - 2 \right) \frac{[n+p]}{[n]} + 1 \right] b^2 + \frac{[n+p]}{[n]^2} b \right\}$$

where $\delta = \delta(q) = \left[\left[\left(\frac{[n+p]}{q[n]} - 2 \right) \frac{[n+p]}{[n]} + 1 \right] b^2 + \frac{[n+p]}{[n]^2} b \right]^{1/2}$ and N_f is a positive constant depending on f .

Proof. Let $q \in (0, 1)$ be fixed. For $x \in [0, b]$ and $t \leq b+1$, we can write from (3.1) that

$$|f(t) - f(x)| \leq \omega_{b+1}(f, |t-x|) \leq \left(1 + \frac{|t-x|}{\delta} \right) \omega_{b+1}(f, \delta) \quad (3.2)$$

where $\delta > 0$. On the other hand, for $x \in [0, b]$ and $t > b+1$, using the fact that $t-x > 1$, we have

$$\begin{aligned} |f(t) - f(x)| &\leq M_f (1+x^2+t^2) \leq M_f (2+3x^2+2(t-x)^2) \\ &\leq N_f (1+b^2) (t-x)^2 \end{aligned} \quad (3.3)$$

where $N_f = 6M_f$. Combining (3.2) and (3.3), we get for all $x \in [0, b]$ and $t \geq 0$ that

$$|f(t) - f(x)| \leq N_f (1+b^2) (t-x)^2 + \left(1 + \frac{|t-x|}{\delta} \right) \omega_{b+1}(f, \delta) \text{ and therefore}$$

$$\begin{aligned} |S_{n,q}(f; x; p) - f(x)| &\leq N_f (1+b^2) S_{n,q}((t-x)^2; x; p) \\ &\quad + \left(1 + \frac{S_{n,q}(|t-x|; x; p)}{\delta} \right) \omega_{b+1}(f, \delta). \end{aligned}$$

By Lemma 2 and Corollary 2, we obtain

$$\begin{aligned} &|S_{n,q}(f; x; p) - f(x)| \\ &\leq N_f (1+b^2) S_{n,q}((t-x)^2; x; p) + \left(1 + \frac{[S_{n,q}((t-x)^2; x; p)]^{1/2}}{\delta} \right) \omega_{b+1}(f, \delta) \\ &\leq N_f (1+b^2) \left\{ \left[\left(\frac{[n+p]}{q[n]} - 2 \right) \frac{[n+p]}{[n]} + 1 \right] x^2 + \frac{[n+p]}{[n]^2} x \right\} \\ &\quad + \left(1 + \frac{\left[\left[\left(\frac{[n+p]}{q[n]} - 2 \right) \frac{[n+p]}{[n]} + 1 \right] x^2 + \frac{[n+p]}{[n]^2} x \right]^{1/2}}{\delta} \right) \omega_{b+1}(f, \delta) \\ &\leq N_f (1+b^2) \left\{ \left[\left(\frac{[n+p]}{q[n]} - 2 \right) \frac{[n+p]}{[n]} + 1 \right] b^2 + \frac{[n+p]}{[n]^2} b \right\} + 2\omega_{b+1}(f, \delta(q)), \end{aligned}$$

where $\delta = \delta(q) = \left[\left[\left(\frac{[n+p]}{q[n]} - 2 \right) \frac{[n+p]}{[n]} + 1 \right] b^2 + \frac{[n+p]}{[n]^2} b \right]^{1/2}$. Whence the result follows. \square

Corollary 3. *Let $q := q_n \in (0, 1)$ and fix $p \in \mathbb{N}_0$. Then for all $f \in E$, $\{S_{n,q}(f; x; p)\}$ converges uniformly to f on $[0, b]$ if and only if $\lim_{n \rightarrow \infty} q_n = 1$.*

4. GLOBAL RESULT

In this section we consider the following Lipschitz classes:

$$\begin{aligned} \Delta_h^2 f(x) &: = f(x+2h) - 2f(x+h) + f(x), \\ \omega_r^2(f, \delta) &: = \sup_{h \in (0, \delta]} \|\Delta_h^2 f\|_r, \\ \omega_r^1(f, \delta) &: = \sup \{ \mu_r(x) |f(t) - f(x)| : |t-x| \leq \delta \text{ and } t, x \geq 0 \} \\ L(p_r^2 \alpha) &: = \{ f \in C_r[0, \infty) : \omega_r^2(f; \delta) = O(\delta^\alpha) \text{ as } \delta \rightarrow 0^+ \}, \end{aligned}$$

where $h > 0$ and $0 < \alpha \leq 2$.

Lemma 5. *Fix $r \in \mathbb{N}$, $p \in \mathbb{N}_0$, $q \in (0, 1)$. For the sequence of operators $\{S_{n,q}(f; x; p)\}$, there exists a constant $M_r(q, p)$ such that*

$$\begin{aligned} & \mu_r(x) S_{n,q} \left(\frac{\psi_x^2}{\mu_r}; x; p \right) \\ & \leq M_r(q, p) \left\{ \left[\left(\frac{[n+p]}{q^{2r+1}[n]} - \frac{2}{q^r} \right) \frac{[n+p]}{[n]} + 1 \right] x^2 + \frac{[n+p]}{[n]^2} x \right\} \end{aligned}$$

where $\psi_x^2(t) = (t-x)^2$.

Proof. Now let $r = 1$. Since, for each fixed $p \in \mathbb{N}_0$, $q \in (0, 1)$ and for all $n \in \mathbb{N}$, we have $\frac{[n+p]}{q[n]} - 2 \leq \frac{[n+p]}{q^3[n]} - \frac{2}{q}$, then we can write using Corollary 1 and Corollary 2 that

$$\begin{aligned} & \mu_1(x) S_{n,q} \left(\frac{\psi_x^2}{\mu_1}; x; p \right) \\ & = \mu_1(x) \{ S_{n,q}(\psi_x^2; x; p) + S_{n,q}(t^3; x; p) - 2x S_{n,q}(t^2; x; p) + x^2 S_{n,q}(t; x; p) \} \\ & \leq \mu_1(x) \left\{ \left[\left(\left(\frac{[n+p]}{q^3[n]} - \frac{2}{q} \right) \frac{[n+p]}{[n]} + 1 \right) x^2 \right] \left(1 + \frac{[n+p]}{[n]} x \right) \right. \\ & \quad \left. + \frac{[n+p]}{[n]^2} x \left[\frac{2q+1}{q^2} \frac{[n+p]}{[n]} x - 2x + \frac{1}{[n]} + 1 \right] \right\} \\ & \leq M_r(q, p) \left\{ \left[\left(\left(\frac{[n+p]}{q^3[n]} - \frac{2}{q} \right) \frac{[n+p]}{[n]} + 1 \right) x^2 \right] + \frac{[n+p]}{[n]^2} x \right\}. \end{aligned}$$

Finally, assume that $r \geq 2$. Since, for each fixed $p \in \mathbb{N}_0, q \in (0, 1)$ and for all $n \in \mathbb{N}$, we have $\frac{[n+p]}{q[n]} - 2 \leq \frac{[n+p]}{q^{2r+1}[n]} - \frac{2}{q^r}$, then, we get from Lemma 3 that

$$\begin{aligned} S_{n,q} \left(\frac{\psi_x^2}{\mu_r}; x; p \right) &= S_{n,q} (\psi_x^2; x; p) + S_{n,q}(e_{r+2}; x; p) - 2x S_{n,q}(e_{r+1}; x; p) \\ &\quad + x^2 S_{n,q}(e_r; x; p) = S_{n,q} (\psi_x^2; x; p) \\ &\quad + q^{\frac{-(r-1)(r+2)}{2}} \left(\frac{[n+p]x}{[n]} \right)^{r+2} + \frac{b_{r+2,1}(q)}{[n]} \left(\frac{[n+p]x}{[n]} \right)^{r+1} + \dots + \frac{1}{[n]^{r+1}} \frac{[n+p]x}{[n]} \\ &\quad - 2x \left\{ q^{\frac{-r(r+1)}{2}} \left(\frac{[n+p]x}{[n]} \right)^{r+1} + \frac{b_{r+1,1}(q)}{[n]} \left(\frac{[n+p]x}{[n]} \right)^r + \dots + \frac{1}{[n]^r} \frac{[n+p]x}{[n]} \right\} \\ &\quad + x^2 \left\{ q^{\frac{-r(r-1)}{2}} \left(\frac{[n+p]x}{[n]} \right)^r + \frac{b_{r,1}(q)}{[n]} \left(\frac{[n+p]x}{[n]} \right)^{r-1} + \dots + \frac{1}{[n]^{r-1}} \frac{[n+p]x}{[n]} \right\}, \\ &= \left[\left(\frac{[n+p]}{q[n]} - 2 \right) \frac{[n+p]}{[n]} + 1 \right] x^2 + \frac{[n+p]}{[n]^2} x \\ &\quad + q^{\frac{-r(r-1)}{2}} \left[\left(\frac{[n+p]}{q^{2r+1}[n]} - \frac{2}{q^r} \right) \frac{[n+p]}{[n]} + 1 \right] x^2 \left(\frac{[n+p]x}{[n]} \right)^r \\ &\quad + \frac{[n+p]}{[n]^2} x \left[(b_{r+2}(q) - 2b_{r+1}(q) + b_r(q)) \left(\frac{[n+p]x}{[n]} \right)^r + \dots + \frac{1}{[n]^r} \right] \end{aligned}$$

which implies that

$$\begin{aligned} &\mu_r(x) S_{n,q} \left(\frac{\psi_x^2}{\mu_r}; x; p \right) \\ &\leq M_r(q, p) \left\{ \left[\left(\frac{[n+p]}{q^{2r+1}[n]} - \frac{2}{q^r} \right) \frac{[n+p]}{[n]} + 1 \right] x^2 + \frac{[n+p]}{[n]^2} x \right\}, \end{aligned}$$

whence the result holds. \square

Now, for fixed $p \in \mathbb{N}_0$, consider the space

$$C_p^2[0, \infty) := \{f \in C_p[0, \infty) : f'' \in C_p[0, \infty)\}.$$

Then we have the following result.

Lemma 6. *Let $g \in C_p^2[0, \infty)$. For the operators $S_{n,q}^*(f; x; p) = S_{n,q}(f; x; p) - f\left(\frac{[n+p]x}{[n]}\right) + f(x)$, there exists a positive constant $M_r(q, p)$ such that, for all $x \in [0, \infty)$ and $n \in \mathbb{N}$, we have*

$$\begin{aligned} &\mu_r(x) |S_{n,q}^*(g; x; p) - g(x)| \leq M_r(q, p) \|g''\|_r \\ &\times \left\{ \left[\left(1 + \frac{1}{q^{2r+1}} \right) \frac{[n+p]^2}{[n]^2} - 2 \left(1 + \frac{1}{q^r} \right) \frac{[n+p]}{[n]} + 2 \right] x^2 + \frac{[n+p]x}{[n]^2} \right\}. \end{aligned}$$

Proof. Let $\psi_x(y) = y - x$. Using Taylor formula, we may write that

$$g(y) - g(x) = \psi_x(y)g'(x) + \int_x^y \psi_t(y)g''(t)dt, \quad y \in [0, \infty).$$

Since $S_{n,q}^*(\psi_x(y); x; p) = 0$, we get

$$\begin{aligned} |S_{n,q}^*(g; x; p) - g(x)| &= |S_{n,q}^*(g(y) - g(x); x; p)| \\ &= \left| S_{n,q}^* \left(\int_x^y \psi_t(y)g''(t)dt; x; p \right) \right| \\ &= \left| S_{n,q} \left(\int_x^y \psi_t(y)g''(t)dt; x; p \right) - \int_x^{\frac{[n+p]x}{[n]}} \psi_t \left(\frac{[n+p]x}{[n]} \right) g''(t)dt \right|. \end{aligned}$$

Using the fact that

$$\left| \int_x^y \psi_t(y)g''(t)dt \right| \leq \frac{\|g''\|_r \psi_x^2(y)}{2} \left(\frac{1}{\mu_r(x)} + \frac{1}{\mu_r(y)} \right)$$

and

$$\left| \int_x^{\frac{[n+p]x}{[n]}} \psi_t \left(\frac{[n+p]x}{[n]} \right) g''(t)dt \right| \leq \frac{\|g''\|_r}{2\mu_r(x)} \left(\frac{[n+p]x}{[n]} - x \right)^2,$$

it follows from Lemma 5 that

$$\begin{aligned} \mu_r(x) |S_{n,q}^*(g; x; p) - g(x)| &\leq \frac{\|g''\|_r}{2} \left\{ S_{n,q}(\psi_x^2; x) + \mu_r(x) S_{n,q} \left(\frac{\psi_x^2}{\mu_r}; x \right) \right\} \\ &\quad + \frac{\|g''\|_r}{2} \left(\frac{[n+p]x}{[n]} - x \right)^2 \\ &\leq M_r(q, p) \|g''\|_r \\ &\quad \times \left\{ \left[\left(1 + \frac{1}{q^{2r+1}} \right) \frac{[n+p]^2}{[n]^2} - 2 \left(1 + \frac{1}{q^r} \right) \frac{[n+p]}{[n]} + 2 \right] x^2 + \frac{[n+p]x}{[n]^2} \right\}. \end{aligned}$$

The Lemma is proved. \square

The main result of the section is the following.

Theorem 2. For each fixed $r \in \mathbb{N}$, $p \in \mathbb{N}_0$ and for all $n \in \mathbb{N}$, $f \in C_r[0, \infty)$ and $x \in [0, \infty)$ and fixed $q \in (0, 1)$, there exists an absolute constant $M_r(q, p) > 0$ such that

$$\begin{aligned} \mu_r(x) |S_{n,q}(f; x; p) - f(x)| &\leq \omega_r^1(f; \frac{[n+p]x}{[n]} - x) + M_r(q, p) \\ &\times \omega_r^2 \left(f, \sqrt{\left[\left(1 + \frac{1}{q^{2r+1}}\right) \frac{[n+p]^2}{[n]^2} - 2 \left(1 + \frac{1}{q^r}\right) \frac{[n+p]}{[n]} + 2 \right] x^2 + \frac{[n+p]x}{[n]^2}} \right), \end{aligned}$$

where μ_r is the same as in Section 2. Particularly, if $f \in Lip_r^2 \alpha$ for some $\alpha \in (0, 2]$, then

$$\begin{aligned} \mu_r(x) |S_{n,q}(f; x; p) - f(x)| &\leq \omega_r^1(f; \frac{[n+p]x}{[n]} - x) + M_r(q, p) \\ &\times \left(\left[\left(1 + \frac{1}{q^{2r+1}}\right) \frac{[n+p]^2}{[n]^2} - 2 \left(1 + \frac{1}{q^r}\right) \frac{[n+p]}{[n]} + 2 \right] x^2 + \frac{[n+p]x}{[n]^2} \right)^{\frac{\alpha}{2}} \end{aligned}$$

holds.

We first consider the modified Steklov means (see [5]) of a function $f \in C_r[0, \infty)$ as follows:

$$f_h(y) := \frac{4}{h^2} \int_0^{h/2} \int_0^{h/2} \{2f(y+s+t) - f(y+2(s+t))\} ds dt,$$

where $h > 0$ and $y \geq 0$. In this case, it is clear that

$$f(y) - f_h(y) = \frac{4}{h^2} \int_0^{h/2} \int_0^{h/2} \Delta_{s+t}^2 f(y) ds dt,$$

which guarantees that

$$\|f - f_h\|_r \leq \omega_r^2(f; h). \quad (4.1)$$

Furthermore, we have $f_h''(y) = \frac{1}{h^2} (8\Delta_{h/2}^2 f(y) - \Delta_h^2 f(y))$, which implies

$$\|f_h''\|_r \leq \frac{9}{h^2} \omega_r^2(f; h). \quad (4.2)$$

Then, combining (4.1) with (4.2) we conclude that the Steklov means f_h corresponding to $f \in C_r[0, \infty)$ belongs to $C_r^2[0, \infty)$.

Proof. Let $r \in \mathbb{N}$, $f \in C_r[0, \infty)$ and $x \in [0, \infty)$ be fixed. Assume that, for $h > 0$, f_h denotes the Steklov means of f . For any $n \in \mathbb{N}$, the following inequality holds:

$$|S_{n,q}(f; x; p) - f(x)| \leq S_{n,q}^*(|f(y) - f_h(y)|; x; p) + |f(x) - f_h(x)|$$

$$+ |S_{n,q}^*(f_h; x; p) - f_h(x)| + \left| f\left(\frac{[n+p]x}{[n]}\right) - f(x) \right|.$$

Since $f_h \in C_r^2[0, \infty)$, it follows from Lemma 6 that

$$\begin{aligned} & \mu_r(x) |S_{n,q}(f; x; p) - f(x)| \leq \|f - f_h\|_r \left\{ \mu_r(x) S_{n,q}^*\left(\frac{1}{\mu_r}; x; p\right) + 1 \right\} \\ & + M_r(q, p) \|f_h''\|_p \left\{ \left[\left(1 + \frac{1}{q^{2r+1}}\right) \frac{[n+p]^2}{[n]^2} - 2 \left(1 + \frac{1}{q^r}\right) \frac{[n+p]}{[n]} + 2 \right] x^2 \right. \\ & \left. + \frac{[n+p]x}{[n]^2} \right\} + \mu_r(x) \left| f\left(\frac{[n+p]x}{[n]}\right) - f(x) \right| \\ \leq & \|f - f_h\|_p \left\{ \mu_r(x) S_{n,q}\left(\frac{1}{\mu_r}; x\right) + 3 \right\} \\ & + M_r(q, p) \|f_h''\|_r \left\{ \left[\left(1 + \frac{1}{q^{2r+1}}\right) \frac{[n+p]^2}{[n]^2} - 2 \left(1 + \frac{1}{q^r}\right) \frac{[n+p]}{[n]} + 2 \right] x^2 \right. \\ & \left. + \frac{[n+p]x}{[n]^2} \right\} + \mu_r(x) \left| f\left(\frac{[n+p]x}{[n]}\right) - f(x) \right|. \end{aligned}$$

By (4.1) and (4.2), the last inequality yields that

$$\begin{aligned} & \mu_r(x) |S_{n,q}(f; x; p) - f(x)| \leq M_r(q) \omega_r^2(f; h) \\ & \times \left\{ 1 + \frac{1}{h^2} \left\{ \left[\left(1 + \frac{1}{q^{2r+1}}\right) \frac{[n+p]^2}{[n]^2} - 2 \left(1 + \frac{1}{q^r}\right) \frac{[n+p]}{[n]} + 2 \right] x^2 \right. \right. \\ & \left. \left. + \frac{[n+p]x}{[n]^2} \right\} \right\} + \omega_r^1(f; f\left(\frac{[n+p]x}{[n]}\right) - f(x)). \end{aligned}$$

Thus, choosing $h = \sqrt{\left[\left(1 + \frac{1}{q^{2r+1}}\right) \frac{[n+p]^2}{[n]^2} - 2 \left(1 + \frac{1}{q^r}\right) \frac{[n+p]}{[n]} + 2 \right] x^2 + \frac{[n+p]x}{[n]^2}}$, the first part of the proof is completed. The remaining part of the proof can be easily obtained from the definition of the space $Lip_p^2\alpha$. \square

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