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# REGULARITY AND ENTROPY SOLUTIONS OF SOME ELLIPTIC EQUATIONS

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*Abstract.* Here, we would study regularity of solutions and existence at least one entropy solution for $L^1$-data and duality solution. This result would improve some results of Laplacian differential equations.

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## 1. INTRODUCTION

First, we summarize some result notions of Schwratz space and tempered distributions. The schwartz space $\mathcal{S}(\mathbb{R}^n)$ is a topological vector space of all $f : \mathbb{R}^n \to \mathbb{C}$ such that $f \in C^\infty(\mathbb{R}^n)$ and $x^\alpha \partial^\beta f(x)$ is bounded. For every pair of multi-induces $\alpha, \beta \in \mathbb{N}^n$, we set

$$|||f|||_{\alpha,\beta} := \sup_x |x^\alpha \partial^\beta f|$$

which induces a family of semi-norms on $\mathcal{S}(\mathbb{R}^n)$. A tempered distribution is a continuous linear functional $T : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ and $\mathcal{S}'(\mathbb{R}^n)$ is the space of all tempered distributions. $\mathcal{D}(\mathbb{R}^n)$ is the space $C_c^\infty(\mathbb{R}^n)$ endowed with the topology in which $f_n \to 0$ means that, there is a compact set $K$; such that $Supp f_n \subseteq K$ $(n = 1, 2, ...)$ and for each $\alpha \in \mathbb{N}^n$, $D^\alpha f_n \to 0$ uniformly.
The fourier transform of a function $f \in \mathcal{S}(\mathbb{R}^n)$ is the function $\widehat{f} : \mathbb{R}^n \to \mathbb{C}$ defined by

$$\widehat{f}(k) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(x) e^{-ikx} dx.$$

It is well known that
1)$\widehat{\ }: \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous one to one.
2) $\widehat{\partial^\alpha f}(k) = (ik)^\alpha \widehat{f}(k)$.
3)$\widehat{(-ix)^\beta f}(k) = \partial^\beta \widehat{f}(k)$.
For regularity of Laplacian, Ma and Thompson [8], Ma [7] proved regularity, where

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$f \in C[0,1]$. Moreover, Lee and Sim [6] proved it but for $f \in L^1(0,1)$. Recently, interior regularity have studied by many mathematician:

M. Cozzi [4] studied regularity theory of weak solutions for the second order linear elliptic differential equations $-div(A(.)\nabla u) = f$ in $\Omega$, where $\Omega$ is an open bounded subset of $\mathbb{R}^n$ and $A = [a_{ij}]$ is $n \times n$ matrix uniformly elliptic, $a_{ij} \in C^{0,1}_{loc}(\Omega)$ and $f \in L^2(\Omega)$. In fact, it is proved that for any $\Omega' \subset\subset \Omega$, $||u||_{H^2(\Omega)} \le C(||u||_{L^2(\Omega)} + ||f||_{L^2(\Omega)})$. Moreover, it was shown the interior $H^{2s-\epsilon}$ regularity for weak solutions of some linear elliptic differential equations.

J. Siljander, J. M. Urbano [11] studied the Serrin-type interior regulatity result.

$$u \in L^{2+\epsilon}_{loc}(\Omega_T) \Rightarrow \text{regularity}$$

for a weak solution in the energy space $L^\infty_t L^2_x$ satisfying in appropriate vorticity estimates for

$$\partial_t u + (u.\nabla)u + \nabla p = 0 \text{ and } \mathrm{div} u = 0.$$

S. Gustafson and co authors [5] gave an interior regularity criteria for suitable weak solutions of the 3D Navier-Stokes equations. In fact they considered the regularity problem for a suitable weak solution $(u,p) : \Omega \times I \to \mathbb{R}^3 \times \mathbb{R}$ of three-dimensional incompressible Navier-Stokes equations (NS)

$$\begin{cases} u_t - \Delta u + (u.\nabla)u + \nabla p = f & \text{in } \Omega \\ \mathrm{div} u = 0 & \text{in } \Omega \times I \end{cases}$$

and proved $u \in L^\infty(Q_{z,r})$ for some $B_{x,r} \times (t - r^2, t) = Q_{z,r} \subseteq \Omega \times I$, $r > 0$.

These brand of problems have potential applications to the modeling of combustion, thermal explosions, nonlinear heat generation, gravitational equilibrium of polytropic stars, glaciology, non-Newtonian fluids, and the flow through porous media.

In this paper, we would study the regular property of

$$-\Delta_\alpha u + \lambda u = f, \tag{1.1}$$

in $\mathcal{D}'(\Omega)$, where we define $\Delta_\alpha := \alpha_1 \frac{\partial^2}{\partial x_1^2} + \ldots + \alpha_n \frac{\partial^2}{\partial x_n^2}$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$, ($\alpha_i > 0, \forall i$), our aim is to show $u \in C^\infty(\mathbb{R}^n)$.

## 2. Results

Here, we use of $\approx$ for equivalent norms.

**Lemma 1.** *Suppose that $m \in \mathbb{Z}$, $\lambda > 0$ and let $u, f \in \mathcal{S}'(\mathbb{R}^n)$ satisfy (1.1) as distributions. If $f \in W^{m,2}(\mathbb{R}^n)$, then $u \in W^{m+2,2}(\mathbb{R}^n)$ and there exists a constant $C$ such that $\|u\|_{W^{m+2,2}} \le C\|f\|_{W^{m,2}}$.*

*Proof.* Taking the Fourier transform of (1.1), we have

$$\mathcal{F}(-\alpha_1 u_{x_1x_1} - \ldots - \alpha_n u_{x_nx_n} + \lambda u) = \mathcal{F}(f),$$

$$(\alpha_1\xi_1^2+...+\alpha_n\xi_n^2+\lambda)\mathcal{F}(u)=\mathcal{F}(f).$$

From theorem 5.2.3 of [3] for any $m \in \mathbb{Z}$ and $a,b>0$ we note to the following equivalent norms:

$$W^{m,2}(\mathbb{R}^n):=\{u\in \mathcal{S}'(\mathbb{R}^n);\mathcal{F}^{-1}[(a+b|\xi|^2)^{\frac{m}{2}}\mathcal{F}(u)]\in L^2(\mathbb{R}^n)\}$$

$$\|u\|_{W^{m,2}}\approx \|\mathcal{F}^{-1}[(a+b|\xi|^2)^{\frac{m}{2}}\mathcal{F}(u)]\|_{L^2}, \qquad u\in W^{m,2}(\mathbb{R}^n).$$

By using Parseval theorem

$$\|\mathcal{F}^{-1}[(a+b|\xi|^2)^{\frac{m}{2}}\mathcal{F}(u)]\|_{L^2}=\|(a+b|\xi|^2)^{\frac{m}{2}}\mathcal{F}(u)\|_{L^2}.$$

Thus,

$$\|(a+b|\xi|^2)^{\frac{m}{2}}\mathcal{F}(u)\|_{L^2}\approx \|(\alpha_1\xi_1^2+...+\alpha_n\xi_n^2+\lambda)^{\frac{m}{2}}\mathcal{F}(u)\|_{L^2} \tag{2.1}$$

and

$$(\alpha_1\xi_1^2+...+\alpha_n\xi_n^2+\lambda)^{\frac{m+2}{2}}\mathcal{F}(u)=(\alpha_1\xi_1^2+...+\alpha_n\xi_n^2+\lambda)^{\frac{m}{2}}\mathcal{F}(f)$$

so the result follows from (2.1).

□

We now consider the case of a general domain $\Omega$

**Theorem 1.** *Suppose that $\lambda\in\mathbb{R}$ and $u,f\in\mathcal{D}'(\Omega)$ satisfy the equation (1.1) in $\mathcal{D}'(\Omega)$.*
*(i) If $f\in W^{m,2}_{loc}(\Omega)$ and $u\in W^{n,2}_{loc}(\Omega)$ for some $m\geq 0$ and $n\in\mathbb{Z}$, then $u\in W^{m+2,2}_{loc}(\Omega)$ and for every $\Omega_2\subset\subset\Omega_1\subset\subset\Omega$, there exists a constant $C$ (depending only on $m$, $\Omega_2$ and $\Omega_1$) such that $\|u\|_{W^{m+2,2}(\Omega_2)}\leq C(\|f\|_{W^{m,2}(\Omega_1)}+\|u\|_{W^{n,2}(\Omega_1)})$.*
*(ii) If $f\in C^\infty(\Omega)$ and $u\in W^{n,2}_{loc}(\Omega)$ for some $n\in\mathbb{Z}$ then $u\in C^\infty(\Omega)$.*

*Proof.* We proceed in two steps.
**Step 1:** Consider $M''\subset\subset M'\subset\subset\Omega$ and $k\in\mathbb{Z}$. If $u\in W^{k,2}(M')$ and $f\in W^{k-1,2}(M')$ solve the equation (1.1) in $\mathcal{D}'(\Omega)$, thus, $u\in W^{k+1,2}(M'')$ and there exists $C$ such that $\|u\|_{W^{k+1,2}(M'')}\leq C(\|f\|_{W^{k-1,2}(M')}+\|u\|_{W^{k,2}(M')})$. To show this, consider $\rho\in C_c^\infty(\mathbb{R}^n)$ such that $\rho\equiv 1$ on $M''$ and supp $\rho\subset M'$ and define $v\in\mathcal{D}'(\mathbb{R}^n)$ by $v=\rho u$, i.e.

$$(v,\varphi)_{D'(\mathbb{R}^n),D(\mathbb{R}^n)}=(u,\rho\varphi)_{D'(M'),D(M')}.$$

Clearly $v\in W^{k,2}(\mathbb{R}^n)$ and $\|v\|_{W^{k,2}(\mathbb{R}^n)}\leq C\|u\|_{W^{k,2}(M')}$.
$v$ solves the equation

$$-\Delta_\alpha v+v=T_1+T_2+T_3 \tag{2.2}$$

in $\mathcal{D}'(\mathbb{R}^n)$, where the distributions $T_1, T_2$, and $T_3$ are defined by

$$(T_1,\varphi)_{\mathcal{D}'(\mathbb{R}^n),\mathcal{D}(\mathbb{R}^n)}=(f+(1-\lambda)u,\rho\varphi)_{\mathcal{D}'(M'),\mathcal{D}(M')},$$

$$(T_2,\varphi)_{D'(\mathbb{R}^n),\mathcal{D}(\mathbb{R}^n)}=-(u,\Delta_\alpha\rho.\varphi)_{\mathcal{D}'(M'),\mathcal{D}(M')},$$

$$(T_3,\varphi)_{\mathcal{D}'(\mathbb{R}^n),\mathcal{D}(\mathbb{R}^n)} = -2(u,(\alpha.\nabla\rho)\varphi)_{\mathcal{D}'(M'),\mathcal{D}(M')},$$

for every $\varphi \in C_c^{\infty}(\mathbb{R}^n)$, since

$$\begin{aligned}(-\Delta_\alpha v+v,\varphi)_{D'(\mathbb{R}^n),D(\mathbb{R}^n)} &= (-\alpha_1 v_{x_1x_1}-...-\alpha_n v_{x_nx_n}+v,\varphi)_{D'(\mathbb{R}^n),D(\mathbb{R}^n)} \\ = -(\alpha_1 \big[(u,\rho_{x_1x_1}\varphi)_{D'(M'),D(M')} &+ 2(u,\rho_{x_1}\varphi)_{D'(M'),D(M')} + (u_{x_1x_1},\rho\varphi)_{D'(M'),D(M')}\big] \\ +...+\alpha_n\big[(u,\rho_{x_nx_n}\varphi)_{D'(M'),D(M')} &+ 2(u,\rho_{x_n}\varphi)_{D'(M'),D(M')} + (u_{x_nx_n},\rho\varphi)_{D'(M'),D(M')}\big]) \\ +(u,\rho\varphi)_{D'(M'),D(M')} &= (-\alpha_1 u_{x_1x_1}-...-\alpha_n u_{x_nx_n}+\lambda u,\rho\varphi)_{D'(M'),D(M')} \\ -(u,(\alpha_1\rho_{x_1x_1}+...+\alpha_n\rho_{x_nx_n})\varphi)_{D'(M'),D(M')} &- 2(u,(\alpha_1\rho_{x_1}+...+\alpha_n\rho_{x_n})\varphi)_{D'(M'),D(M')}.\end{aligned}$$

Thus $T_j \in W^{k-1,2}(\mathbb{R}^n)$ and

$$\|T_j\|_{W^{k-1,2}(\mathbb{R}^n)} \le C(\|f\|_{W^{k-1,2}(M')} + \|u\|_{W^{k,2}(M')}),$$

for $j=1,2,3$. Applying (2.1) and lemma 1, we deduce $v \in W^{k+1,2}(\mathbb{R}^n)$ and $\|v\|_{W^{k+1,2}(\mathbb{R}^n)} \le C(\|f\|_{W^{k-1,2}(M')} + \|u\|_{W^{k,2}(M')})$.

**Step 2:** (Conclusion) Without loss of generality, we may assume $n=-\ell \le 0$. Let $\Omega_2 \subset\subset \Omega_1 \subset\subset \Omega$. Consider a family $(M_j)_{0\le j\le m+\ell+1}$ of open subsets of $\Omega$, such that

$$\Omega_2 = M_{m+\ell+1} \subset\subset ... \subset\subset M_0 \subset\subset \Omega_1$$

(one constructs easily such a family). It follows from Step 1 that $u \in W^{-\ell+1,2}(M_0)$ and can

$$\begin{aligned}\|u\|_{W^{-\ell+1,2}(M_0)} &\le C(\|f\|_{W^{-\ell-1,2}(\Omega_1)} + \|u\|_{W^{-\ell,2}(\Omega_1)}) \\ &\le C(\|f\|_{W^{m,2}(\Omega_1)} + \|u\|_{W^{n,2}(\Omega_1)}).\end{aligned} \tag{2.3}$$

(2.3) and lemma 1 imply that $u \in W^{-\ell+2,2}(M_1)$ and

$$\begin{aligned}\|u\|_{W^{-\ell+2,2}(M_1)} &\le C(\|f\|_{W^{-\ell,2}(M_0)} + \|u\|_{W^{-\ell+1,2}(M_0)}) \\ &\le C(\|f\|_{W^{m,2}(\Omega_1)} + \|u\|_{W^{n,2}(\Omega_1)}).\end{aligned} \tag{2.4}$$

Iterating the above argument, $u \in W^{m+2,2}(M_{m+\ell+1}) = W^{m+2,2}(\Omega_2)$ and that there exists $C$ in which

$$\|u\|_{W^{m+2,2}(\Omega_2)} \le C(\|f\|_{W^{m,2}(\Omega_1)} + \|u\|_{W^{n,2}(\Omega_1)}).$$

Hence, property (i) satisfies since $\Omega_1$ and $\Omega_2$ are arbitrary. Property (ii) follows from Property (i) and $C^{\infty}(\Omega) = \bigcap_{m\ge 0} W^{m,2}_{loc}(\Omega)$. □

Before paying to entropy solutions, we remember some notions:

Let $f, g$ and $q$ be functions in $L^{\infty}(\Omega)$, $u$ and $v$ be the solutions of

$$\begin{cases} -div(A(x)\nabla u) + q(x)u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.5}$$

and

$$\begin{cases} -div(A^*(x)\nabla v) + q(x)v = g & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{2.6}$$

respectively, where $A^*$ is the transposed matrix of $A$ and

$$A(x)y.y \geq \alpha|y|^2, \ |A(x)| \leq \beta \tag{2.7}$$

for every $y \in \mathbb{R}^n$, $0 < \alpha \leq \beta$. It is well known that (2.5) has a weak unique solution (Theorem 1.6.1 [1]). Since both $u$ and $v$ belong to $H_0^1(\Omega)$, $u$ can be chosen as test function in the formulation of weak solution for $v$ and vice versa. One obtains

$$\int fv = \int A(x)\nabla u.\nabla v + \int q(x)u.v = \int A^*(x)\nabla v.\nabla u + \int q(x)v.u = \int ug$$

for every $f, g \in L^\infty(\Omega)$, where $u$ and $v$ solve the corresponding problems with data $f$ and $g$ respectively. $u, v \in L^\infty(\Omega)$ (Theorem 2.3 [9]), but we remark that the two integrals are well-defined also if $f \in L^1(\Omega)$ and $u \in L^1(\Omega)$ (always maintaining the assumption that $g$ and so $v$ is a bounded function). This fact inspired to Guido Stampacchia the following definition of solution for (2.5) if the datum is in $L^1(\Omega)$.

**Definition 1.** Suppose that $f \in L^1(\Omega)$. A function $u \in L^1(\Omega)$ is called a *duality solution* with datum $f$ if one has $\int ug = \int fv$, for every $g \in L^\infty(\Omega)$, where $v$ is the solution of

$$\begin{cases} -div(A^*(x)\nabla v) + q(x)v = g & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

**Theorem 2** (Stampacchia, theorem 3.3 of [9]). *For $f \in L^1(\Omega)$ there exists a unique duality solution with datum $f$. Furthermore, $u \in L^q(\Omega)$ for every $q < \frac{N}{N-2}$.*

*Remark* 1. In special case if

$$A = \begin{bmatrix} \alpha_1 & 0 & 0 & . & . & . & 0 \\ 0 & \alpha_2 & 0 & . & . & . & 0 \\ . & & & & & & . \\ . & & & & & & . \\ . & & & & & & . \\ 0 & 0 & . & . & . & 0 & \alpha_n \end{bmatrix}$$

and $q(x) = \lambda$, problems (2.5) and (2.6) change to

$$\begin{cases} -\Delta_\alpha u + \lambda u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad \text{and} \quad \begin{cases} -\Delta_\alpha v + \lambda v = g & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

Respectively.

**Definition 2.** For $k > 0$, set

$$T_k(s) := \max\{-k, \min\{s, k\}\}$$

and

$$\tau_0^{1,2} = \{u : \Omega \longrightarrow \mathbb{R}\, measurable : T_k(u) \in H_0^1(\Omega), \ \forall k > 0\},$$

It is well known that $\nabla T_k(u) = \nabla u \chi_{\{|u| \leq k\}}$ .

**Lemma 2.** *Suppose that $u \in \tau_0^{1,2}(\Omega)$, $q \in L^\infty(\Omega)$ and (2.7) valid. Then there is $\alpha > 0$ in which*

$$\alpha \int_\Omega |\nabla T_k(u)|^2 \leq \int_\Omega A(x)\nabla u.\nabla T_k(u) + \int_\Omega q(x)uT_k(u).$$

*Proof.* By (2.7)

$$\int_\Omega A(x)\nabla u.\nabla T_k(u) + \int_\Omega q(x)uT_k(u) \geq \gamma \int_\Omega |\nabla u|^2 + \int_\Omega q(x)uT_k(u)$$
$$\geq \gamma \int_\Omega |\nabla u|^2 - \int_\Omega |q(x)||u||T_k(u)|,$$

since $|T_k(u)| \leq |u|$, $|q(x)| \leq b$ (almost every where) for a suitable $b > 0$ and from Poincaré inequality

$$\gamma \int_\Omega |\nabla u|^2 - \int_\Omega |q(x)||u||T_k(u)| \geq \gamma \int_\Omega |\nabla u|^2 - \int_\Omega |q(x)||u|^2$$
$$\geq \gamma \int_\Omega |\nabla u|^2 - bC_q \int_\Omega |\nabla u|^2 = \alpha \int_\Omega |\nabla T_k(u)|^2,$$

for $\alpha := \gamma - bC_q$, where $C_q$ is multiplier in Poincaré inequality. □

**Definition 3.** Suppose that $f \in L^1(\Omega)$. A function $u \in \tau_0^{1,2}(\Omega)$ is called an enteropy solution of (2.5) if

$$\int_\Omega A(x)\nabla u.\nabla T_k(u-\varphi) + \int_\Omega q(x)uT_k(u-\varphi) \leq \int_\Omega f T_k(u-\varphi), \tag{2.8}$$

for every $k > 0$ and for every $\varphi$ in $H_0^1(\Omega) \cap L^\infty(\Omega)$.

**Theorem 3.** *Suppose that $f \in L^1(\Omega)$. Then there exists an entropy solution $u$ for* (2.5).

*Proof.* We do by approximation; Suppose that $f_n = T_n(f)$ and by the Lax-Miligram theorem, there exists a weak solution $u_n$ for

$$\begin{cases} -div(A(x)\nabla u_n) + q(x)u_n = f_n & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Let $k > 0$. Taking $T_k(u_n)$ as test function and using of lemma 2,

$$\alpha \int_\Omega |\nabla T_k(u_n)|^2 \leq \int_\Omega A(x)\nabla u_n.\nabla T_k(u_n) + \int_\Omega q(x)u_nT_k(u_n)$$
$$= \int_\Omega f_nT_k(u_n) \leq k||f||_{L^1(\Omega)}.$$

Therefore, $(T_k(u_n))_n$ is bounded in $H_0^1(\Omega)$ for a fixed $k$. This implies that there exists a function $v_k \in H_0^1(\Omega)$ such that, up to subsequences $T_k(u_n)$ converges to $v_k$

weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$. From lemma 2, one can deduce that

$$\int_\Omega |\nabla(u_n - u_m)|^q \leq C_q \| f_n - f_m\|^q_{L^1(\Omega)}$$

and since $(f_n)_n$ is a Cauchy sequence in $L^1(\Omega)$, so $(u_n)_n$ is a Cauchy sequence in $W_0^{1,q}(\Omega)$ and then $u_n$ converges strongly to a suitable $u \in W_0^{1,q}(\Omega)$. For every $q < \frac{N}{N-1}$, $\nabla u_n$ converges to $\nabla u$ almost everywhere in $\Omega$. Thus, $T_k(u_n)$ converges strongly to $T_k(u)$ in $L^2(\Omega)$, and so $v_k = T_k(u)$. Therefore, by Fatou lemma,

$$\alpha \int_\Omega |\nabla T_k(u_n)|^2 \leq \liminf_{n\to+\infty} \alpha \int_\Omega |\nabla T_k(u_n)|^2 \leq k||f||_{L^1(\Omega)},$$

which implies that $u$ belongs to $\tau_0^{1,2}$. Fix $k > 0$, $\varphi$ in $H_0^1(\Omega) \cap L^\infty(\Omega)$, and $v := T_k(u_n - \varphi)$ as test function in the weak formulation of (2.5). Then

$$\int_\Omega A(x)\nabla u_n.\nabla T_k(u_n-\varphi) + \int_\Omega q(x)u_n T_k(u_n-\varphi) = \int_\Omega f_n T_k(u_n-\varphi).$$

For the right hand side we have $T_n \longrightarrow I$ as $n \longrightarrow \infty$ and $f_n = T_n(f) \longrightarrow f$. Thus, $f_n \longrightarrow f$ point wise in $L^1(\Omega)$ and $|f_n T_k(u_n-\varphi)| \leq 2k|f|$. Lebesgue theorem implies that

$$\lim_{n\to+\infty} \int_\Omega f_n T_k(u_n-\varphi) = \int_\Omega f\, T_k(u-\varphi),$$

while the left hand side can be rewritten as

$$\int_\Omega A(x)\nabla T_k(u_n-\varphi).\nabla T_k(u_n-\varphi) + \int_\Omega A(x)\nabla\varphi.T_k(u_n-\varphi) + \int_\Omega q(x)u_n T_k(u_n-\varphi).$$

The first term is non-negative, thus, the almost everywhere convergence of $\nabla u_n$ to $\nabla u$ follows by Fatou lemma,

$$\int_\Omega A(x)\nabla T_k(u-\varphi).\nabla T_k(u-\varphi) \leq \liminf_{n\to+\infty} (\int_\Omega A(x)\nabla T_k(u_n-\varphi).\nabla T_k(u_n-\varphi).$$

For the second, since $u_n$ converges to $u$ in $H_0^1(\Omega)$ so $u_n - \varphi$ to $u - \varphi$ in $H_0^1(\Omega)$, then $T_k(u_n-\varphi)$ to $T_k(u-\varphi)$ in $H_0^1(\Omega)$ ) and since $-\nabla(A(x)\nabla\varphi) \in H_0^1$

$$< -\nabla(A(x)\nabla\varphi), T_k(u_n-\varphi) > \longrightarrow < -\nabla(A(x)\nabla\varphi), T_k(u-\varphi) >$$

i.e.

$$\int_\Omega A(x)\nabla\varphi.T_k(u-\varphi) = \lim_{n\to+\infty} \int_\Omega A(x)\nabla\varphi.T_k(u_n-\varphi).$$

For third term since

$$|q(x)u_n T_k(u_n-\varphi)| \leq 2k\|q\|_\infty |u|$$

by Lebesgue dominated convergence theorem

$$\int_{\Omega} q(x)uT_k(u-\varphi) = \lim_{n\to+\infty}\int_{\Omega} q(x)u_nT_k(u_n-\varphi).$$

Then by cancelling equal terms:

$$\int_{\Omega} A(x)\nabla u.\nabla T_k(u-\varphi) + \int_{\Omega} q(x)uT_k(u-\varphi) \leq \int_{\Omega} fT_k(u-\varphi),$$

so $u$ is an entropy solution of (2.5). □

**Theorem 4.** *Let $f \in L^1(\Omega)$ and $u$ be an entropy solution of (2.5) with datum f. Then u belongs to $W_0^{1,q}(\Omega)$ for every $q < \frac{N}{N-1}$ and it is a distributional solution for (2.5).*

*Proof.* Taking $\varphi = 0$ in (2.8)

$$\alpha\int_{\Omega} |\nabla T_k(u)|^2 \leq \int_{\Omega} A(x)\nabla u.\nabla T_k(u) + \int_{\Omega} q(x)uT_k(u) = \int_{\Omega} fT_k(u) \leq k||f||_{L^1(\Omega)}.$$

Proof of Theorem 4.1 in [9] shows that $u \in W_0^{1,q}(\Omega)$ for every $q < \frac{N}{N-1}$. We now fix $h > 0$ and choose $\varphi = T_h(u)$ as test function in (2.8). Then

$$\int_{\Omega} A(x)\nabla u.\nabla T_k(u-T_h(u)) + \int_{\Omega} q(x)uT_k(u-T_h(u)) \leq \int_{\Omega} fT_k(u-T_h(u)).$$

Moreover,

$$T_k(u-T_h(u)) = \begin{cases} u-T_h(u) & -k \leq u-T_h(u) \leq k, \\ k & k \leq u-T_h(u), \\ -k & u-T_h(u) \leq -k, \end{cases}$$

where

$$u-T_h(u) = \begin{cases} 0 & -h \leq u \leq h, \\ u-h & h \leq u, \\ u+h & u \leq -h. \end{cases}$$

Therefore, if $|u| \leq h$, then $T_k(u-T_h(u)) = 0$. Moreover, if $h-k \leq |u| \leq h+k$, then $T_k(u-T_h(u)) = u-T_h(u)$. Thus,

$$\int_{\{h-k\leq|u|\leq h+k\}} A(x)\nabla u.\nabla u + \int_{\{|u|\geq h\}} q(x)uT_k(u-T_h(u))$$
$$= \int_{\{|u|\geq h\}} fT_k(u-T_h(u)) \leq k\int_{\{|u|\geq h\}} |f|.$$

Defining $A_h = \{|u| \geq h\}$, $m(A_h) \longrightarrow 0$ as $h \longrightarrow \infty$ (since $u \in W_0^{1,1}(\Omega)$, thus, in $L^1(\Omega)$). From $f \in L^1(\Omega)$,

$$\lim_{h\to+\infty}\int_{\{|u|\geq h\}} |f| = 0,$$

hence by recalling (2.7)

$$\lim_{h\to+\infty}\int_{\{h-k\le|u|\le h+k\}}|\nabla u|^2=0. \tag{2.9}$$

For $h>0$, $\eta$ in $C_0^1(\Omega)$ and $\varphi=T_h(u)-\eta$ as test function in the entropy formulation (2.8), where $k=\|\eta\|_{L^\infty(\Omega)}$ then

$$\int_\Omega A(x)\nabla u.\nabla T_k(u-T_h(u)+\eta)+\int_\Omega q(x)uT_k(u-T_h(u)+\eta)\le\int_\Omega fT_k(u-T_h(u)+\eta).$$

By Lebesgue dominated theorem and choice of $k$

$$\lim_{h\to+\infty}\int_\Omega fT_k(u-T_h(u)+\eta)=\int_\Omega fT_k(\eta)=\int_\Omega f\eta.$$

For the left hand side, using again the choice of $k$

$$\int_{\{|u|\le h\}}(A(x)\nabla u.\nabla T_k(u-T_h(u)+\eta)+\int_{\{|u|\le h\}}q(x)uT_k(u-T_h(u)+\eta))$$
$$+\int_{\{|u|\ge h\}}(A(x)\nabla u.\nabla T_k(u-T_h(u)+\eta)+\int_{\{|u|\ge h\}}q(x)uT_k(u-T_h(u)+\eta)).$$

Since $A$ is bounded, so $u\in W_0^{1,1}(\Omega)$ and $\eta\in C_0^1(\Omega)$. For $\{|u|\le h\}$ we have $T_k(u-T_h(u)+\eta)=T_k(\eta)$. Thus, by Lebesgue dominated theorem

$$\lim_{h\to+\infty}\int_{\{|u|\le h\}}A(x)\nabla u.\nabla T_k(\eta)=\lim_{h\to+\infty}\int_{\{|u|\le h\}}A(x)\nabla u.\nabla\eta=\int_\Omega A(x)\nabla u.\nabla\eta.$$

Similarly

$$\int_{\{|u|\le h\}}q(x)uT_k(u-T_h(u)+\eta)=\int_{\{|u|\le h\}}q(x)uT_k(\eta)=\int_{\{|u|\le h\}}q(x)u\eta.$$

Then

$$\lim_{h\to+\infty}\int_{\{|u|\le h\}}q(x)u\eta=\int_\Omega q(x)u\eta.$$

Since

$$\{|u-T_h(u)+\eta|\le k,|u|\ge h\}\subseteq\{h-2k\le|u|\le h+2k\}$$

by (2.7) and choice of $k$

$$\int_{\{|u|\ge h\}}A(x)\nabla u.\nabla T_k(u-T_h(u)+\eta)\le|\int_{\{|u|\ge h\}}A(x)\nabla u.\nabla T_k(u-T_h(u)+\eta)|$$
$$\le\int_{\{|u|\ge h\}}A(x)|\nabla u|.|\nabla T_k(u-T_h(u)+\eta)|\le\beta\int_{\{h-2k\le|u|\le h+2k\}}|\nabla u|(|\nabla u|+|\nabla\eta|).$$

Thus by (2.9) and Hölder Inequality

$$\beta \int_{\{h-2k\le|u|\le h+2k\}} |\nabla u|(|\nabla u| + |\nabla \eta|) \le \beta \int_{\{h-2k\le|u|\le h+2k\}} (|\nabla u|^2 + |\nabla u|.|\nabla \eta|)$$
$$\le \beta[\int_{\{h-2k\le|u|\le h+2k\}} |\nabla u|^2]^{\frac{1}{2}}.[\int_{\{h-2k\le|u|\le h+2k\}} |\nabla \eta|^2]^{\frac{1}{2}} = 0.$$

Therefore,

$$\lim_{h\to+\infty} \int_{\{|u|\ge h\}} (A(x)\nabla u.\nabla T_k(u - T_h(u) + \eta) = 0$$

and

$$|q(x)uT_k(u - T_h(u) + \eta))| \le k\|q\|_\infty |u|$$

so $q(x)uT_k(u - T_h(u) + \eta)) \in L^1(\Omega)$. Thus, $\lim_{h\to+\infty} \int_{\{|u|\ge h\}} k\|q\|_\infty |u| = 0$ and

$$\int_{\{|u|\ge h\}} q(x)uT_k(u - T_h(u) + \eta)) = 0$$

Putting together the results,

$$\int_\Omega (A(x)\nabla u.\nabla \eta + q(x)u\eta) \le \int_\Omega f\eta,$$

for any $\eta \in C_0^1(\Omega)$. Exchanging $\eta$ with $-\eta$ we obtain the reverse inequality so that $u$ is a distributional solution of (2.5). □

Finally, we would show uniqueness of entropy solution.

**Theorem 5.** *Let $f \in L^1(\Omega)$. Then the entropy solution of (2.5) is unique.*

*Proof.* We proceed in three steps.

**Step 1** (An entropy solution is a duality solution): Consider $g$ is in $L^\infty(\Omega)$ and $v$ is a weak solution of

$$\begin{cases} -div(A^*(x)\nabla v) + q(x)v = g & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

According to the Stampacchia's theorem [2], $v \in L^\infty(\Omega)$. We repeat the proof of Theorem 4. By choosing $\varphi = T_h(u) - v$ in the entropy formulation, for $h > 0$ and $k = \|v\|_{L^\infty(\Omega)}$:

$$\int_\Omega A(x)\nabla u.\nabla T_k(u - T_h(u) + v) + \int_\Omega q(x)uT_k(u - T_h(u) + v) \le \int_\Omega fT_k(u - T_h(u) + v).$$

Similar to theorem 4, from Lebesgue dominated theorem and choose of $k$,

$$\lim_{h\to+\infty} \int_\Omega fT_k(u - T_h(u) + v) = \int_\Omega fv,$$

Moreover, the left hand side can be rewritten as

$$\int_{\{|u|\le h\}}(A(x)\nabla u.\nabla v+q(x)uv)+\int_{\{|u|\ge h\}}A(x)\nabla u.\nabla T_k(u-T_h(u)+v)$$
$$+\int_{\{|u|\ge h\}}q(x)uT_k(u-T_h(u)+v).$$

For the second and the third term, similar to the proof of Theorem 4 (using (2.9))

$$\lim_{h\to+\infty}\int_{\{|u|\ge h\}}A(x)\nabla u.\nabla T_k(u-T_h(u)+v)+\int_{\{|u|\ge h\}}q(x)uT_k(u-T_h(u)+v)=0,$$

and the first term can be rewritten as

$$\int_{\{|u|\le h\}}(A(x)\nabla u.\nabla v+q(x)uv)=\int_{\Omega}(A(x)\nabla T_h(u).\nabla v+q(x)T_h(u)v)$$
$$=\int_{\Omega}(A^*(x)\nabla v.\nabla T_h(u)+q(x)vT_h(u))=\int_{\Omega}gT_h(u),$$

since $T_h(u)\in H_0^1(\Omega)$ can be chosen as test function in the problem solved by $v$. Then, by Lebesgue dominated theorem,

$$\int_{\{|u|\le h\}}(A(x)\nabla u.\nabla v+q(x)uv)=\lim_{h\to+\infty}\int_{\{|u|\le h\}}(A(x)\nabla u.\nabla v+q(x)uv)=\int_{\Omega}gu.$$

Putting together the results, we obtain

$$\int_{\Omega}gu\le\int_{\Omega}fv.$$

Exchanging $g$ with $-g$ (and so $v$ with $-v$, by linearity), we obtain the reverse inequality, therefore, $u$ is a duality solution of (2.5).

**Step 2** (An entropy solution is a solution obtained by approximation (See [10])): Suppose that $f_n\in L^\infty(\Omega)$ be a sequence of functions that converges to $f\in L^1(\Omega)$, and suppose that $u_n$ be the solution of

$$\begin{cases}-div(A^*(x)\nabla u_n)+q(x)u_n=f_n & \text{in }\Omega\\ u_n=0 & \text{on }\partial\Omega\end{cases}$$

$u_n\in H_0^1(\Omega)\cap L^\infty(\Omega)$, so that $\varphi=u_n$ is an admissible choice in the entropy formulation for $u$. Thus,

$$\int_{\Omega}A(x)\nabla u.\nabla T_k(u-u_n)+\int_{\Omega}q(x)uT_k(u-u_n)\le\int_{\Omega}fT_k(u-u_n).$$

On the other hand, $T_k(u-u_n)$ belongs to $H_0^1(\Omega)$ and so it can be chosen as test function in the weak formulation for $u_n$. Then

$$\int_{\Omega}A(x)\nabla u_n.\nabla T_k(u-u_n)+\int_{\Omega}q(x)u_nT_k(u-u_n)=\int_{\Omega}f_nT_k(u-u_n).$$

Thus,

$$\int_{\Omega} A(x)\nabla(u-u_n).\nabla T_k(u-u_n)+\int_{\Omega} q(x)(u-u_n)T_k(u-u_n)\le\int_{\Omega}(f-f_n)T_k(u-u_n).$$

By (2.7) and lemma 2

$$\alpha\int_{\Omega}|\nabla T_k(u-u_n)|^2\le\int_{\Omega} A(x)\nabla(u-u_n).\nabla T_k(u-u_n)+\int_{\Omega} q(x)(u-u_n)T_k(u-u_n)$$
$$\le k\|(f-f_n)\|_{L^1(\Omega)}.$$

approaching $n\longrightarrow\infty$ , $T_k(u-u_n)\longrightarrow 0$ in $H_0^1(\Omega)$ and this implies that $u_n$ converges to the entropy solution $u$. From solutions obtained by approximation are unique, hence, the entropy solution $u$ is unique.

**Step 3** (There exists at most an entropy solution): Here, we follow [2]. Suppose that $u$ and $v$ be two entropy solutions of (2.5), with the same datum $f$, and let $h>k>0$. Then $\varphi=T_h(v)$ is admissible in the entropy formulation for $u$ and $\varphi=T_h(u)$ is admissible in the entropy formulation for $v$. Thus,

$$\int_{\Omega} A(x)\nabla u.\nabla T_k(u-T_h(v))+\int_{\Omega} q(x)uT_k(u-T_h(v))\le\int_{\Omega} fT_k(u-T_h(v)),$$

and

$$\int_{\Omega} A(x)\nabla v.\nabla T_k(v-T_h(u))+\int_{\Omega} q(x)vT_k(v-T_h(u))\le\int_{\Omega} fT_k(v-T_h(u)).$$

Summing these two inequalities,

$$\int_{\Omega} A(x)\nabla u.\nabla T_k(u-T_h(v))+\int_{\Omega} q(x)uT_k(u-T_h(v))$$
$$+\int_{\Omega} A(x)\nabla v.\nabla T_k(v-T_h(u))+\int_{\Omega} q(x)vT_k(v-T_h(u)),$$

in the left hand side is less than or equal to

$$\int_{\Omega} f(T_k(u-T_h(v))+T_k(v-T_h(u)))$$

in the right hand side. From oddness of $T_k(s)$ and Lebesgue dominated theorem,

$$\lim_{h\to+\infty}\int_{\Omega} f(T_k(u-T_h(v))+T_k(v-T_h(u)))=0.$$

Hence,

$$\limsup_{h\to+\infty}\int_{\Omega} A(x)\nabla u.\nabla T_k(u-T_h(v))+\int_{\Omega} q(x)uT_k(u-T_h(v))$$
$$+\int_{\Omega} A(x)\nabla v.\nabla T_k(v-T_h(u))+\int_{\Omega} q(x)uT_k(v-T_h(u))\le 0.$$

For the sake of simplicity we will suppose from now on that $u \geq 0$ and $v \geq 0$, since the proof turns out to be considerably simplified. We refer to [2] for the proof in the general case of changing sign solutions. We set

$$\Omega = \{u \leq h, v \leq h\} \cup \{u > h, v \leq h\} \cup \{v > h\} = E_0^h \cup F_1^h \cup F_2^h,$$

and

$$\Omega = \{v \leq h, u \leq h\} \cup \{v > h, u \leq h\} \cup \{u > h\} = E_0^h \cup F_3^h \cup F_4^h.$$

Then

$$\int_{E_0^h} A(x)\nabla u.\nabla T_k(u - T_h(v)) + \int_{E_0^h} q(x)uT_k(u - T_h(v))$$
$$= \int_{E_0^h} A(x)\nabla u.\nabla T_k(u - v) + \int_{E_0^h} q(x)uT_k(u - v).$$

Similarly,

$$\int_{E_0^h} A(x)\nabla v.\nabla T_k(v - T_h(u)) + \int_{E_0^h} q(x)vT_k(v - T_h(u))$$
$$= \int_{E_0^h} A(x)\nabla v.\nabla T_k(v - u) + \int_{E_0^h} q(x)vT_k(v - u),$$

On $F_1^h$,

$$\int_{F_1^h} A(x)\nabla u.\nabla T_k(u - T_h(v)) = \int_{\{u>h, v\leq h, 0\leq u-v\leq k\}} A(x)\nabla u.\nabla(u - v).$$

On $\{u > h, v \leq h, 0 \leq u - v \leq k\}$ it valid $h < u \leq h + k$ and $h - k < v \leq h$, so

$$\left|\int_{F_1^h} A(x)\nabla u.\nabla T_k(u - T_h(v))\right| \leq \beta \int_{\{h<u\leq h+k, h-k<v\leq h\}} |\nabla u||\nabla v|.$$

By (2.9)

$$\lim_{h\to+\infty} \int_{\{h<u\leq h+k\}} |\nabla u|^2 = 0,$$

and

$$\lim_{h\to+\infty} \int_{\{h-k<v\leq h\}} |\nabla v|^2 = 0,$$

Hence, by Hölder inequality

$$\lim_{h\to+\infty} \left|\int_{F_1^h} A(x)\nabla u.\nabla T_k(u - T_h(v))\right| = 0;$$

and $|q(x)uT_k(u - T_h(v))| \leq k\|q\|_\infty |u|$, so $q(x)uT_k(u - T_h(v)) \in L^1(\Omega)$. Thus,

$$\lim_{h\to+\infty} \int_{F_1^h} k\|q\|_\infty |u| = 0,$$

hence,

$$\int_{F_1^h} q(x)uT_k(u-T_h(v))=0.$$

Repeating the same for $F_3^h$ we have

$$\lim_{h\to+\infty}\left|\int_{F_3^h} A(x)\nabla v.\nabla T_k(v-T_h(u))\right|=0.$$

and

$$\int_{F_3^h} q(x)vT_k(v-T_h(u))=0$$

Moreover on $F_2^h$,

$$\int_{F_2^h}(A(x)\nabla u.\nabla T_k(u-T_h(v))+q(x)uT_k(u-T_h(v)))$$
$$=\int_{\{v>h,0\le u<h+k\}}(A(x)\nabla u.\nabla u)+q(x)uu)\ge 0,$$

and similarly on $F_4^h$,

$$\int_{F_4^h}(A(x)\nabla v.\nabla T_k(v-T_h(u))+q(x)vT_k(v-T_h(u)))$$
$$=\int_{\{u>h,0\le v<h+k\}}(A(x)\nabla v.\nabla v)+q(x)vv)\ge 0,$$

Putting the results together,

$$\limsup_{h\to+\infty}\int_{E_0^h}(A(x)\nabla(u-v).\nabla T_k(u-v)+q(x)(u-v)T_k(u-v))\le 0,$$

which, by Fatou lemma, implies, from $E_0^h$ "fills" $\Omega$ as $h\longrightarrow+\infty$,

$$0\le\int_{\Omega}(A(x)\nabla(u-v).\nabla T_k(u-v)+q(x)(u-v)T_k(u-v))\le 0,$$

Using (2.7) and lemma 2 we have $\nabla T_k(u-v)\equiv 0$, thus $u=v$. □

*Remark* 2. In special case if

$$A=\begin{bmatrix}\alpha_1 & 0 & 0 & . & . & . & 0\\ 0 & \alpha_2 & 0 & . & . & . & 0\\ . & & & & & & .\\ . & & & & & & .\\ . & & & & & & .\\ 0 & 0 & . & . & . & 0 & \alpha_n\end{bmatrix}$$

and $q(x)=\lambda$, problem (2.5) can be rewritten

$$\begin{cases} -div(A(x)\nabla u) + q(x)u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \Longrightarrow \begin{cases} -\Delta_\alpha u + \lambda u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Then above theorems are satisfied for this problem.

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