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NEW GENERALIZED MIDPOINT TYPE INEQUALITIES FOR FRACTIONAL INTEGRAL

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Abstract. Here, our first aim to establish a new identity for differentiable function involving Riemann-Liouville fractional integrals. Then, we obtain same generalized midpoint type inequalities utilizing convex and concave function.

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1. INTRODUCTION

In recent years, the Hermite-Hadamard inequality, which is the first fundamental result for convex mappings with a natural geometrical interpretation and many applications, has drawn attention much interest in elementary mathematics.

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are considerable significant in the literature (see, e.g.,[17, p.137], [7]). These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if f is concave.

In [12], U. S. Kırmacı give the following identity and using this identiy, obtain some bounds for the left hand side of the inequality (1.1)

Lemma 1. *Let $f : I^* \to \mathbb{R}$ be differentiable function on I^*, $a, b \in I^*$ (I^* is interior of I) with $a < b$. If $f' \in L[a,b]$, then we have*

$$\frac{1}{b-a}\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) \tag{1.2}$$

$$=(b-a)\left[\int_0^{\frac{1}{2}} tf'(ta+(1-t)b)dt+\int_{\frac{1}{2}}^{1}(1-t)\,f'(ta+(1-t)b)dt\right].$$

Over the last twenty years, the numerous studies have focused on to obtain new bound for left hand side and right and side of the inequality (1.1). For some examples, please refer to ([2, 4, 6–8, 14, 15, 18, 19, 24])

In the following we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper.

Definition 1. Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J^{\alpha}_{a+}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt,\ \ x>a$$

and

$$J^{\alpha}_{b-}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt,\ \ x<b$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J^{0}_{a+}f(x)=J^{0}_{b-}f(x)=f(x)$.

It is remarkable that Sarikaya et al.[22] first give the following interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a positive function with $0\leq a<b$ and $f\in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right)\leq\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a+}f(b)+J^{\alpha}_{b-}f(a)\right]\leq\frac{f(a)+f(b)}{2} \tag{1.3}$$

with $\alpha>0$.

Sarıkaya and Yıldırım also give the following Hermite-Hadamard type inequality for the Riemann-Lioville fractional integrals in [23].

Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be a positive function with $a<b$ and $f\in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right)\leq\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right]\leq\frac{f(a)+f(b)}{2}. \tag{1.4}$$

Whereupon Sarikaya et al. obtain the Hermite-Hadamard inequality for Riemann-Lioville fractional integrals, many authors have studied to generalize this inequality and establish Hermite-Hadamard inequality other fractional integrals such as k-fractional integral, Hadamard fractional integrals, Katugampola frtactional integrals, Conformable fractional integrals, etc. For some of them, please see ([1, 3, 5, 9, 11, 13, 16, 20, 21, 25–27]). For more information about fraction calculus please refer to [10].

In the following section, we establish some new generalized midpoint type inequalities for Riemann-Liouville fractional integrals.

2. Generalized midpoint inequalities for Riemann-Liouville fractional integral operators

First, we give the following lemma which will be used frequently later.

Lemma 2. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $f'\in L[a,b]$, then for all $x\in[a,b]$ the following equality for fractional integrals holds:*

$$\frac{\Gamma(\alpha+1)}{b-a}\left[(x-a)^{1-\alpha}J^{\alpha}_{b-}f(a+b-x)+(b-x)^{1-\alpha}J^{\alpha}_{a+}f(a+b-x)\right] \tag{2.1}$$
$$-f(a+b-x)$$

$$=\frac{(x-a)^2}{b-a}\int_0^1\left(1-t^{\alpha}\right)f'(tb+(1-t)(a+b-x))\,dt$$

$$+\frac{(b-x)^2}{b-a}\int_0^1\left(t^{\alpha}-1\right)f'(ta+(1-t)(a+b-x))\,dt.$$

Proof. Integrating the by parts, we have

$$\int_0^1\left(1-t^{\alpha}\right)f'(tb+(1-t)(a+b-x))\,dt \tag{2.2}$$
$$=\left(1-t^{\alpha}\right)\frac{f'(tb+(1-t)(a+b-x))}{x-a}\Bigg|_0^1$$
$$+\frac{\alpha}{x-a}\int_0^1 t^{\alpha-1}f'(tb+(1-t)(a+b-x))\,dt$$

$$=-\frac{f(a+b-x)}{x-a}+\frac{\Gamma(1+\alpha)}{(x-a)^{\alpha+1}}\int_{a+b-x}^{b}(u-(a+b-x))^{\alpha-1}f(u)du$$
$$=-\frac{f(a+b-x)}{x-a}+\frac{\Gamma(1+\alpha)}{(x-a)^{\alpha+1}}J_{b-}^{\alpha}f(a+b-x).$$

Similarly, we get

$$\int_0^1\left(t^{\alpha}-1\right)f'(ta+(1-t)(a+b-x))\,dt \tag{2.3}$$
$$=-\frac{f(a+b-x)}{b-x}+\frac{\Gamma(1+\alpha)}{(b-x)^{\alpha+1}}J_{a+}^{\alpha}f(a+b-x).$$

By the identities (2.2) and (2.3), we obtain the required result (2.1). □

Theorem 3. *$f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $0\le a<b$. If $|f'|^q$ is convex on $[a,b]$ for some fixed $q>1$, then for all $x\in[a,b]$ following inequality for fractional integrals holds:*

$$\left|\frac{\Gamma(\alpha+1)}{b-a}\left[(x-a)^{1-\alpha}J_{b-}^{\alpha}f(a+b-x)+(b-x)^{1-\alpha}J_{a+}^{\alpha}f(a+b-x)\right]\right. \tag{2.4}$$
$$\left.-f(a+b-x)\right|$$
$$\le\frac{1}{b-a}\left(\frac{\alpha p}{\alpha p+1}\right)^{\frac{1}{p}}\left[(x-a)^2\left[\frac{|f'(b)|^q+|f'(a+b-x)|^q}{2}\right]^{\frac{1}{q}}\right.$$
$$\left.+(b-x)^2\left[\frac{|f'(a)|^q+|f'(a+b-x)|^q}{2}\right]^{\frac{1}{q}}\right]$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By the Lemma 2, we have

$$\left|\frac{\Gamma(\alpha+1)}{b-a}\left[(x-a)^{1-\alpha}J_{b-}^{\alpha}f(a+b-x)+(b-x)^{1-\alpha}J_{a+}^{\alpha}f(a+b-x)\right]\right. \tag{2.5}$$
$$\left.-f(a+b-x)\right|$$
$$\le\frac{(x-a)^2}{b-a}\int_0^1\left|1-t^{\alpha}\right|\left|f'(tb+(1-t)(a+b-x))\right|dt$$

$$+\frac{(b-x)^2}{b-a}\int_0^1 |t^\alpha - 1|\left|f'(ta+(1-t)(a+b-x))\right|dt.$$

Using the Hölder's inequality and convexity of $|f'|^q$, we obtain

$$\int_0^1 |1-t^\alpha|\left|f'(tb+(1-t)(a+b-x))\right|dt \tag{2.6}$$
$$\leq \left(\int_0^1 |1-t^\alpha|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'(tb+(1-t)(a+b-x))\right|^q dt\right)^{\frac{1}{q}}$$
$$\leq \left(\int_0^1 (1-t^{p\alpha})\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \left[t\left|f'(b)\right|^q+(1-t)\left|f'(a+b-x)\right|^q\right]dt\right)^{\frac{1}{q}}$$
$$=\left(\frac{\alpha p}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{|f'(b)|^q+|f'(a+b-x)|^q}{2}\right)^{\frac{1}{q}}.$$

Here we use

$$(A-B)^q \leq A^q - B^q,$$

for any $A > B \geq 0$ and $q \geq 1$.

Similarly we have

$$\int_0^1 |t^\alpha - 1|\left|f'(ta+(1-t)(a+b-x))\right|dt \tag{2.7}$$
$$\leq\left(\frac{\alpha p}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q+|f'(a+b-x)|^q}{2}\right)^{\frac{1}{q}}.$$

If we substitute the inequalities (2.6) and (2.7) in (2.5), then we obtain the desired result. □

Corollary 1. *Under assumption of Theorem 3 with $x=\frac{a+b}{2}$, we have the following inequality*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right| \tag{2.8}$$
$$\leq\frac{b-a}{4}\left(\frac{\alpha p}{\alpha p+1}\right)^{\frac{1}{p}}\left[\left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|f'(b)|^q+|f'(a)|^q}{4}\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{b-a}{4}\left(\frac{4\alpha p}{\alpha p+1}\right)^{\frac{1}{p}}\left[\left|f'(a)\right|+\left|f'(b)\right|\right].$$

Proof. The proof of the first inequality in (2.8) is obvious from the convexity of $|f'|^q$. For the proof of second inequality, let $a_1 = 3|f'(a)|^q$, $b_1 = |f'(b)|^q$, $a_2 = |f'(a)|^q$ and $b_2 = 3|f'(b)|^q$. Using the fact that,

$$\sum_{k=1}^{n}(a_k+b_k)^s \leq \sum_{k=1}^{n} a_k^s + \sum_{k=1}^{n} b_k^s,\ 0 \leq s < 1$$

the desired result can be obtained straightforwardly. □

Theorem 4. *$f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $0 \leq a < b$. If $|f'|^q$ is convex on $[a,b]$ for some fixed $q \geq 1$, then for all $x \in [a,b]$ following inequality for fractional integrals holds:*

$$\left|\frac{\Gamma(\alpha+1)}{b-a}\left[(x-a)^{1-\alpha} J_{b-}^{\alpha} f(a+b-x) + (b-x)^{1-\alpha} J_{a+}^{\alpha} f(a+b-x)\right]\right.$$
$$\left. - f(a+b-x)\right|$$

$$\leq \frac{1}{b-a}\left(\frac{\alpha}{\alpha+1}\right)^{1-\frac{1}{q}}$$
$$\times\left[(x-a)^2\left(\frac{\alpha}{2(\alpha+2)}\left|f'(b)\right|^q + \left[\frac{(\alpha+1)(\alpha+2)-2}{2(\alpha+1)(\alpha+2)}\right]\left|f'(a+b-x)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left. + (b-x)^2\left(\frac{\alpha}{2(\alpha+2)}\left|f'(a)\right|^q + \left[\frac{(\alpha+1)(\alpha+2)-2}{2(\alpha+1)(\alpha+2)}\right]\left|f'(a+b-x)\right|^q\right)^{\frac{1}{q}}\right].$$

Proof. By the Lemma 2 and the power mean inequality, we have

$$\left|\frac{\Gamma(\alpha+1)}{b-a}\left[(x-a)^{1-\alpha} J_{b-}^{\alpha} f(a+b-x) + (b-x)^{1-\alpha} J_{a+}^{\alpha} f(a+b-x)\right]\right. \tag{2.9}$$
$$\left. - f(a+b-x)\right|$$
$$\leq \frac{(x-a)^2}{b-a}\int_0^1 |1-t^{\alpha}|\left|f'(tb+(1-t)(a+b-x))\right| dt$$
$$+ \frac{(b-x)^2}{b-a}\int_0^1 |t^{\alpha}-1|\left|f'(ta+(1-t)(a+b-x))\right| dt.$$

$$\leq \frac{(x-a)^2}{b-a}\left(\int_0^1 |1-t^\alpha|\,dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 |1-t^\alpha|\left|f'(tb+(1-t)(a+b-x))\right|^q dt\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)^2}{b-a}\left(\int_0^1 |t^\alpha-1|\,dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 |t^\alpha-1|\left|f'(ta+(1-t)(a+b-x))\right|^q dt\right)^{\frac{1}{q}}.$$

Using the convexity of $|f'|^q$, we obtain

$$\int_0^1 |1-t^\alpha|\left|f'(tb+(1-t)(a+b-x))\right|^q dt$$

$$\leq \int_0^1 (1-t^\alpha)\left[t\left|f'(b)\right|^q+(1-t)\left|f'(a+b-x)\right|^q\right]dt$$

$$=\frac{\alpha}{2(\alpha+2)}\left|f'(b)\right|^q+\left[\frac{(\alpha+1)(\alpha+2)-2}{2(\alpha+1)(\alpha+2)}\right]\left|f'(a+b-x)\right|^q$$

and similarly we have

$$\int_0^1 |t^\alpha-1|\left|f'(ta+(1-t)(a+b-x))\right|^q dt$$

$$\leq \frac{\alpha}{2(\alpha+2)}\left|f'(a)\right|^q+\left[\frac{(\alpha+1)(\alpha+2)-2}{2(\alpha+1)(\alpha+2)}\right]\left|f'(a+b-x)\right|^q$$

which completes the proof. □

Corollary 2. *Under assumption of Theorem 4 with $x=\frac{a+b}{2}$, we have the following inequality*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J_{b-}^\alpha f\left(\frac{a+b}{2}\right)+J_{a+}^\alpha f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq \frac{b-a}{4}\left(\frac{\alpha}{\alpha+1}\right)^{1-\frac{1}{q}}$$

$$\times\left[\left(\frac{\alpha(3\alpha+5)}{4(\alpha+1)(\alpha+2)}\left|f'(b)\right|^{q}+\left[\frac{(\alpha+1)(\alpha+2)-2}{4(\alpha+1)(\alpha+2)}\right]\left|f'(a)\right|^{q}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{\alpha(3\alpha+5)}{4(\alpha+1)(\alpha+2)}\left|f'(a)\right|^{q}+\left[\frac{(\alpha+1)(\alpha+2)-2}{4(\alpha+1)(\alpha+2)}\right]\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\right].$$

Theorem 5. *$f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $0\leq a<b$. If $|f'|^{q}$ is concave on $[a,b]$ for some fixed $q>1$, then for all $x\in[a,b]$ following inequality for fractional integrals holds:*

$$\left|\frac{\Gamma(\alpha+1)}{b-a}\left[(x-a)^{1-\alpha}J_{b-}^{\alpha}f(a+b-x)+(b-x)^{1-\alpha}J_{a+}^{\alpha}f(a+b-x)\right]\right.$$
$$-f(a+b-x)|$$

$$\leq\frac{1}{b-a}\left(\frac{\alpha p}{\alpha p+1}\right)^{\frac{1}{p}}\left[(x-a)^{2}\left|f'\left(\frac{a+2b-x}{2}\right)\right|+(b-x)^{2}\left|f'\left(\frac{2a+b-x}{2}\right)\right|\right]$$

where $\frac{1}{q}+\frac{1}{p}=1$.

Proof. By the Lemma 2 and the Hölder inequality, we have

$$\left|\frac{\Gamma(\alpha+1)}{b-a}\left[(x-a)^{1-\alpha}J_{b-}^{\alpha}f(a+b-x)+(b-x)^{1-\alpha}J_{a+}^{\alpha}f(a+b-x)\right]\right. \tag{2.10}$$
$$-f(a+b-x)|$$

$$\leq\frac{(x-a)^{2}}{b-a}\int_{0}^{1}|1-t^{\alpha}|\left|f'(tb+(1-t)(a+b-x))\right|dt$$

$$+\frac{(b-x)^{2}}{b-a}\int_{0}^{1}|t^{\alpha}-1|\left|f'(ta+(1-t)(a+b-x))\right|dt$$

$$\leq\frac{(x-a)^{2}}{b-a}\left(\int_{0}^{1}(1-t^{\alpha})^{p}dt\right)^{\frac{1}{p}}\left(\int_{0}^{1}\left|f'(tb+(1-t)(a+b-x))\right|^{q}dt\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)^2}{b-a}\left(\int_0^1 (1-t^\alpha)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'(ta+(1-t)(a+b-x))\right|^q dt\right)^{\frac{1}{q}}.$$

Since $|f'|^q$ is concave on $[a,b]$, by using Jensen integral inequality, we obtain

$$\int_0^1 \left|f'(tb+(1-t)(a+b-x))\right|^q dt \tag{2.11}$$
$$=\int_0^1 t^0\left|f'(tb+(1-t)(a+b-x))\right|^q dt$$
$$\le\left(\int_0^1 t^0 dt\right)\left|f'\left(\frac{1}{\int_0^1 t^0 dt}\int_0^1 t^0(tb+(1-t)(a+b-x))\,dt\right)\right|^q$$
$$=\left|f'\left(\frac{a+2b-x}{2}\right)\right|^q$$

and similarly,

$$\int_0^1 \left|f'(ta+(1-t)(a+b-x))\right|^q dt \le \left|f'\left(\frac{2a+b-x}{2}\right)\right|^q. \tag{2.12}$$

By substituting the inequalities (2.11) and (2.12) in (2.10) and using the fact that

$$\int_0^1 (1-t^\alpha)^p\,dt \le \int_0^1 (1-t^{p\alpha})\,dt = \frac{\alpha p}{\alpha p+1},$$

we obtain the desired result. □

Corollary 3. *Under assumptions of Theorem 5, if we choose $x=\frac{a+b}{2}$, then we have the inequality*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J_{b-}^\alpha f\left(\frac{a+b}{2}\right)+J_{a+}^\alpha f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right|$$

$$\le\left(\frac{\alpha p}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{b-a}{4}\right)\left[\left|f'\left(\frac{a+3b}{4}\right)\right|+\left|f'\left(\frac{3a+b}{4}\right)\right|\right].$$

Theorem 6. *$f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $0\le a<b$. If $|f'|^q$ is concave on $[a,b]$ for some fixed $q\ge 1$, then for all $x\in[a,b]$ following inequality for fractional integrals holds:*

$$\left|\frac{\Gamma(\alpha+1)}{b-a}\left[(x-a)^{1-\alpha}J^{\alpha}_{b-}f(a+b-x)+(b-x)^{1-\alpha}J^{\alpha}_{a+}f(a+b-x)\right]\right.$$
$$\left.-f(a+b-x)\right|$$

$$\le\frac{1}{b-a}\left(\frac{\alpha}{\alpha+1}\right)^{2-\frac{1}{q}}\left[(x-a)^2\left|f'\left(\frac{2\alpha(\alpha+2)b+[(\alpha+1)(\alpha+2)-2](a-x)}{2(\alpha+2)}\right)\right|^q\right.$$
$$\left.+(b-x)^2\left|f'\left(\frac{2\alpha(\alpha+2)a+[(\alpha+1)(\alpha+2)-2](b-x)}{2(\alpha+2)}\right)\right|^q\right].$$

Proof. From the ineqauality (2.9) we have

$$\left|\frac{\Gamma(\alpha+1)}{b-a}\left[(x-a)^{1-\alpha}J^{\alpha}_{b-}f(a+b-x)+(b-x)^{1-\alpha}J^{\alpha}_{a+}f(a+b-x)\right]\right. \tag{2.13}$$
$$\left.-f(a+b-x)\right|$$

$$\le\frac{(x-a)^2}{b-a}\left(\int_0^1|1-t^\alpha|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1|1-t^\alpha|\left|f'(tb+(1-t)(a+b-x))\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^2}{b-a}\left(\int_0^1|t^\alpha-1|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1|t^\alpha-1|\left|f'(ta+(1-t)(a+b-x))\right|^q dt\right)^{\frac{1}{q}}.$$

Since $|f'|^q$ is concave on $[a,b]$, by using Jensen integral inequality, we obtain

$$\int_0^1|1-t^\alpha|\left|f'(tb+(1-t)(a+b-x))\right|^q dt$$
$$\le\left(\int_0^1(1-t^\alpha)\,dt\right)f'\left(\frac{1}{\int_0^1(1-t^\alpha)\,dt}\int_0^1(1-t^\alpha)(tb+(1-t)(a+b-x))\,dt\right)$$
$$=\frac{\alpha}{\alpha+1}\left|f'\left(\frac{2\alpha(\alpha+2)b+[(\alpha+1)(\alpha+2)-2](a-x)}{2(\alpha+2)}\right)\right|^q$$

and similarly,

$$\int_0^1|t^\alpha-1|\left|f'(ta+(1-t)(a+b-x))\right|^q dt$$

$$\leq \frac{\alpha}{\alpha+1}\left|f'\left(\frac{2\alpha(\alpha+2)a+[(\alpha+1)(\alpha+2)-2](b-x)}{2(\alpha+2)}\right)\right|^q .$$

This completes the proof. □

Corollary 4. *Under assumptions of Theorem 5 with $x=\frac{a+b}{2}$, then we have the inequality*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right|$$
$$\leq\left(\frac{\alpha}{\alpha+1}\right)^{2-\frac{1}{q}}\left(\frac{b-a}{4}\right)\left[\left|f'\left(\frac{\alpha(3\alpha+5)b+[(\alpha+1)(\alpha+2)-2]a}{4(\alpha+2)}\right)\right|^q\right.$$
$$\left.+\left|f'\left(\frac{\alpha(3\alpha+5)a+[(\alpha+1)(\alpha+2)-2]b}{4(\alpha+2)}\right)\right|^q\right].$$

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