IDENTITIES AND CONGRUENCES INVOLVING THE GEOMETRIC POLYNOMIALS

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Received 13 January, 2018

Abstract. In this paper, we investigate the umbral representation of the geometric polynomials \( w_n(x) \) to derive some properties involving these polynomials. Furthermore, for any prime number \( p \) and any polynomial \( f \) with integer coefficients, we show \( (f(w_n))^p \equiv f(w_1) \pmod{p} \) and we give other curious congruences.

2010 Mathematics Subject Classification: 05A18; 05A40; 11A07

Keywords: geometric umbra, geometric polynomials, identities, congruences

1. INTRODUCTION

The geometric numbers are quantities arising from enumerative combinatorics and have nice number-theoretic properties. In combinatorics, the \( n \)-th geometric number (named also the \( n \)-th ordered Bell number) counts the number of ways to partition the set \([n] := \{1, \ldots, n\}\) into ordered subsets \([2, 3, 6]\). The geometric polynomials are defined by

\[
w_n(x) = \sum_{k=0}^{n} \binom{n}{k} k! x^k
\]

and satisfy the recurrence relation

\[
(x+1)w_n(x) = x \sum_{j=0}^{n} \binom{n}{j} w_j(x), \quad n \geq 1,
\]

where \( \binom{n}{k} \) is the \( (n,k) \)-th Stirling number of the second kind \([2, 26]\). These polynomials have attracted attention from many researchers, see for instance \([9, 10, 15–17]\). For \( x = 1 \) we obtain the geometric numbers \( w_n := w_n(1) = \sum_{k=0}^{n} \binom{n}{k} k! \), for more information about these numbers, see \([6–8, 11, 12, 14, 28, 29]\). More generally, let \( w_n(x; r, s) \) be the \( n \)-th \( (r,s) \)-geometric polynomial defined by

\[
w_n(x; r, s) = \sum_{k=0}^{n} \binom{n}{k} \binom{k+r}{k+s} (k+s)! x^k.
\]

This polynomial generalizes the geometric polynomial \( w_n(x) = w_n(x; 0, 0) \) and the polynomial \( w_n(x; r, r) \) introduced by Mezô \([18]\). Here, \( \binom{n}{k}_r \) denotes the \( (n,k) \)-th \( r \)-Stirling number of the second kind \([4]\). One can see easily that

\[
w_0(x; r, s) = s!,
\]

\[
w_1(x; r, s) = s!(r + (s+1)x),
\]

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We note that this generalization can be viewed as a particular case of that defined by Kargin et al. [16]. As it shown below, these polynomials are also linked to the absolute \( r \)-Stirling numbers of first kind denoted by \( \left[ n \atop k \right]_r \).

Recall that the \( r \)-Stirling numbers can be defined by [4, 26]

\[
(x)_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n+r}{k+r} (x+r)^k \quad \text{and} \quad (x+r)^n = \sum_{k=0}^{n} \binom{n+r}{k+r} (x)_k,
\]

where \( (\alpha)_n = \alpha \cdots (\alpha - n + 1) \) if \( n \geq 1 \), \( (\alpha)_0 = 1 \).

This work is motivated by application of the umbral calculus method to determine identities and congruences involving Bell numbers and polynomials in the works of Gessel [13], Sun et al. [27], Mező et al. [19] and Benyattou et al. [1]. In this paper, we will talk about identities and congruences involving the \((r,s)\)-geometric polynomials based on the geometric umbra defined by \( W_n^x := w_n(x) \). For more information about umbral calculus, see [5, 13, 22–25].

2. IDENTITIES INVOLVING THE \((r,s)\)-GEOMETRIC POLYNOMIALS

The above recurrence relation is equivalent to \((x+1)W_n^x = x(W_n+1)^n, n \geq 1\). Furthermore, we have

**Proposition 1.** Let \( f \) be a polynomial and \( r, s \) be non-negative integers. Then

\[
(x+1) f(W_n + r) = xf(W_n + r + 1) + f(r),
\]

\[
(W_n + r)_{n+r} = (n+r)!x^n(x+1)^r,
\]

\[
(W_n + r-s)^n(W_n)_s = x^n w_n(x;r,s),
\]

\[
(W_n + r)^n(W_n + r)_s = (x+1)^sw_n(x;r,s).
\]

**Proof.** It suffices to show the first identity for \( f(x) = x^n \). For \( r = 0 \) we have

\[(x+1)W_n^x - x(W_n+1)^n = \delta_{(n=0)} \quad \text{Assume it is true for } r-1, \text{ then if we set} \]

\[
h_n(r) := (x+1)(W_n + r)^n - x(W_n + r + 1)^n
\]

we obtain \( h_n(r) = \sum_{j=0}^{n} \binom{n}{j} h_j(r-1) = \sum_{j=0}^{n} \binom{n}{j} (r-1)^j = r^n \), which concludes the induction step. For the other identities, since \((x)_n = \sum_{k=0}^{n} (-1)^{n-k} \left[ n \atop k \right]_r x^k \) and \((x)_n \) is a sequence of binomial type \([20, 23]\), we obtain

\[
(W_n + r)_{n+r} = \sum_{j=0}^{n+r} \binom{n+r}{j} (r)_j (W_n)_{n+r-j} = (n+r)!x^n(x+1)^r.
\]

So, the polynomials \( x^sw_n(x;r,s) \) and \((x+1)^sw_n(x,r,s) \) must be, respectively,

\[
\sum_{j=0}^{n} \binom{n+r}{j+r} (W_n)_{j+s} = \sum_{j=0}^{n} \binom{n+r}{j+r} (W_n - s)_j (W_n)_s = (W_n + r - s)^n(W_n)_s,
\]
\[ \sum_{j=0}^{n} \binom{n+r}{j+r} (w_x + s)^{j+r} = \sum_{j=0}^{n} \binom{n+r}{j+r} (w_x)^{j+r} (w_x + s) = (w_x + r)^n (w_x + s)_n. \]

The last two identities of Proposition 1 lead to:

**Corollary 1.** Let \( r; s \) be non-negative integers and \( f \) be a polynomial. Then
\[ (x + 1)^s f(w_x + r - s)(w_x)_r = x^s f(w_x + r)(w_x + s)_r. \]

**Proposition 2.** Let \( P_n \) and \( T_n \) be the polynomials
\[ P_n(x; r) = \sum_{j=0}^{n} (-1)^j \binom{n+r}{j+r} x^{n-j} \quad \text{and} \quad T_n(x; r) = \sum_{j=0}^{n} \binom{n+r}{j+r} x^j. \]

Then \( (w_x - r - 1)_n = n! P_n(x; r) \) and \( (w_x + n + r)_n = n! T_n(x; r) \).

**Proof.** It suffices to observe that
\[
(w_x - r - 1)_n = \sum_{j=0}^{n} \binom{n}{j} (-r - 1)_j (w_x)^{n-j} = n! \sum_{j=0}^{n} (-1)^j \binom{n+r}{j+r} x^{n-j},
\]
\[
(w_x + n + r)_n = \sum_{j=0}^{n} \binom{n}{j} (n + r)_j (w_x)^{n-j} = n! \sum_{j=0}^{n} \binom{n+r}{j+r} x^j.
\]

The following theorem can be served to derive several identities and congruences for the \((r, s)\)-geometric polynomials.

**Theorem 1.** Let \( m, s \) be non-negative integers and \( f \) be a polynomial. Then
\[ (x + 1)^m f(w_x) - x^m f(w_x + m) = \sum_{k=0}^{m-1} f(k)(x + 1)^{m-1-k} x^k, \quad m \geq 1. \]

**Proof.** Set \( f(x) = \sum_{k=0}^{n} a_k x^k \) and use Proposition 1 to obtain
\[
(x + 1) f(w_x) - x f(w_x + 1) = f(0) + \sum_{k=0}^{n} a_k \left( (x + 1) w_x^k - x (w_x + 1)^k \right) = f(0).
\]

So, the identity is true for \( m = 1 \). Assume it is true for \( m \). Then
\[
(x + 1)^{m+1} f(w_x) = (x + 1) \left( \sum_{k=0}^{m-1} (x + 1)^{m-1-k} x^k f(k) + x^m f(w_x + m) \right)
\]
\[
= \sum_{k=0}^{m-1} (x + 1)^{m-k} x^k f(k) + x^m (x + 1) f(w_x + m)
\]

\[ \square \]
and since \((x + 1) f(w_x + m) - x f(w_x + m + 1) = f(m)\), we can write

\[
(x + 1)^{m+1} f(w_x) = \sum_{k=0}^{m-1} (x + 1)^{m-k} x^k f(k) + x^m \left( x f(w_x + m + 1) + f(m) \right)
\]

\[
\sum_{k=0}^{m-1} (x + 1)^{m-k} x^k f(k) + x^m f(m) + x^{m+1} f(w_x + m + 1)
\]

\[
\sum_{k=0}^{m} (x + 1)^{m-k} x^k f(k) + x^{m+1} f(w_x + m + 1)
\]

which concludes the induction step. □

We note that for \(f(x) = x^n\) and \(x = 1\) in Theorem 1 we obtain Proposition 3.3 given in [8].

**Corollary 2.** For any polynomial \(f\) there holds

\[
f(w_x) = \frac{1}{1+x} \sum_{k \geq 0} f(k) \left( \frac{x}{1+x} \right)^k, \quad x > -\frac{1}{2}.
\]

**Proof.** For \(m = 1\) in Theorem 1, when we replace \(f(x)\) by \(f(x + r)\) we get the identity \(f(r) = (x + 1) f(w_x + r) - x f(w_x + r + 1)\). Then

\[
\text{RHS} = \lim_{n \to \infty} \frac{1}{1+x} \sum_{k=0}^{n} \left( \frac{x}{1+x} \right)^k \left( (x + 1) f(w_x + k) - x f(w_x + k + 1) \right)
\]

\[
= \lim_{n \to \infty} \left( f(w_x) - \left( \frac{x}{1+x} \right)^{n+1} f(w_x + n + 1) \right) = f(w_x)
\]

which completes the proof. □

**Corollary 3.** Let \(n, r, s\) be non-negative integers. For \(f(x) = (x + r)^n (x + s)^s\) or \((x + r - s)^n (x)^s\) in Corollary 2 we obtain

\[
w_n(x; r, s) = \frac{s!}{(1+x)^{s+1}} \sum_{k \geq 0} \binom{k+s}{s} \left( \frac{x}{1+x} \right)^k, \quad x > -\frac{1}{2}.
\]

**Corollary 4.** For any integers \(r \geq 0, s \geq 0\) and \(n \geq 1\) the polynomial \(w_n(x, r, s + r)\) has only real non-positive zeros.

**Proof.** From Corollary 3 we may state

\[
x^r (x + 1)^s w_{n+1}(x; r, s + r) = x \frac{d}{dx} \left( x^r (x + 1)^{s+1} w_n(x; r, s + r) \right)
\]
and using the recurrence relation of r-Stirling numbers we conclude that this identity remains true for all real number \( x \). So, by induction on \( n \), it follows that \( w_n(x;r,s + r), n \geq 1 \), has only real non-positive zeros.

**Lemma 1.** For any non-negative integers \( n \geq 2 \) there holds

\[
(1 + x)w_{n-1}(x) = \sum_{k=1}^{n} \binom{n}{k} (k-1)! x^k.
\]

**Proof.** From the definition of geometric polynomials, we have

\[
(1 + x)w_{n-1}(x) = \sum_{k=1}^{n-1} \binom{n-1}{k} k! x^k + \sum_{k=1}^{n-1} \binom{n-1}{k} k! x^{k+1} \\
= \sum_{k=1}^{n} \left( \binom{n-1}{k} + \binom{n-1}{k-1} \right) (k-1)! x^k \\
= \sum_{k=1}^{n} \binom{n}{k} (k-1)! x^k.
\]

For more explicit formulae for geometric polynomials, see for example [15].

**Proposition 3.** Let \( n, r, s \) be non-negative integers. Then

\[
\log \left( 1 + \sum_{n \geq 1} \frac{w_n(x;r,s) t^n}{s! n!} \right) = (r + (s + 1)x)t + (s + 1)(x + 1) \sum_{n \geq 2} w_{n-1}(x) \frac{t^n}{n!}.
\]

In particular, for \( r = s = 0 \) we get

\[
\log \left( 1 + \sum_{n \geq 1} \frac{w_n(x) t^n}{s! n!} \right) = xt + (x + 1) \sum_{n \geq 2} w_{n-1}(x) \frac{t^n}{n!}.
\]

**Proof.** One can verify easily that the exponential generating function of the polynomials \( w_n(x;r,s) \) is to be \( s! \exp(rt)(1-x(\exp(t) - 1))^{-s-1} \). Then, upon using this generating function and the last Lemma, we can write

\[
LHS = rt - (s + 1) \ln(1-x(\exp(t) - 1)) \\
= rt + (s + 1) \sum_{k \geq 1} \frac{x^k}{k} (\exp(t) - 1)^k \\
= rt + (s + 1) \sum_{k \geq 1} (k-1)! x^k \sum_{n \geq k} \frac{n!}{k!} \frac{t^n}{n!}.
\]
3. Congruences Involving the \((r,s)\)-Geometric Polynomials

In this section, we give some congruences involving the \((r,s)\)-geometric polynomials. Let \(\mathbb{Z}_p\) be the ring of \(p\)-adic integers and for two polynomials \(f(x), g(x) \in \mathbb{Z}_p[x]\), the congruence \(f(x) \equiv g(x) \pmod{p\mathbb{Z}_p[x]}\) means that the corresponding coefficients of \(f(x)\) and \(g(x)\) are congruent modulo \(p\). This congruence will be used later as \(f(x) \equiv g(x)\) and we will use \(a \equiv b\) instead of \(a \equiv b \pmod{p}\).

**Proposition 4.** Let \(n; r; s\) be non-negative integers and \(p\) be a prime number. Then, for any polynomial \(f\) with integer coefficients there holds

\[
\sum_{k=0}^{n-1} f(k)(x+1)^{n-k-1} x^k \equiv f(w_n).
\]

In particular, for \(f(x) = (x + r - s)^n(x)^s\) or \((x + r)^n (x + s)^s\) we get, respectively,

\[
\sum_{k=0}^{n-1} (r - k)^n (k)^s (x + 1)^{n-k-1} x^k \equiv x^s w_n(x; r, s),
\]

\[
\sum_{k=0}^{n-1} (r + k)^n (s + k)^s (x + 1)^{n-k-1} x^k \equiv (x + 1)^s w_n(x; r, s).
\]

**Proof.** For \(m = p\) be a prime number, Theorem 1 implies

\[
LHS = (x + 1)^P f(w_x) - x^P f(w_x + p) \equiv (x^P + 1) f(w_x) - x^P f(w_x) = f(w_x).
\]

For the particular cases, use Proposition 1. \(\square\)

**Corollary 5.** Let \(n; r; s; m; q\) be non-negative integers and \(p\) be a prime number. Then, for any polynomials \(f\) and \(g\) with integer coefficients there holds

\[
(f(w_x))^p g(w_x) \equiv f(w_x) g(w_x).
\]

In particular, we have \(w_{mp+q}(x; r, s) \equiv w_{m+q}(x; r, s)\).

**Proof.** By Fermat’s little theorem and by twice application of Proposition 4 we may state

\[
LHS \equiv \sum_{k=0}^{n-1} (f(k))^p g(k)(x+1)^{n-k-1} x^k \equiv \sum_{k=0}^{n-1} f(k) g(k)(x + 1)^{n-k-1} x^k = RHS.
\]
We note that, for \( f(x) = x^m \), \( g(x) = x^q \) and \( x = 1 \), Corollary 5 may be seen as a particular case of Theorem 3.1 given in [8].

**Corollary 6.** For any non-negative integers \( m \geq 1 \), \( n, r, s \) and any prime number \( p \), there hold

\[
(x + 1)^{s+1} (w_{m(p-1)}(x; r, s) - s!) \equiv -(s - r')_s (x + 1)^{r'} x^{p-r'}, \quad r' \neq 0, \\
(x + 1)^{s+1} (w_{m(p-1)}(x; r, s) - s!) \equiv -s!(x^p + 1), \quad r' = 0.
\]

where \( r' \equiv r \) and \( r' \in \{0, 1, \ldots, p - 1\} \).

**Proof.** Set \( n = m(p-1) \) in Proposition 4. If \( r' \neq 0 \) we get

\[
(x + 1)^s w_{m(p-1)}(x; r, s) \equiv \sum_{k=0}^{p-1} (r' + k)^{s+k} (x + 1)^{p-1-k} x^k
\]

\[
= \sum_{k=0}^{p-1} \sum_{r'+k \neq p} (s + k)_s (x + 1)^{p-1-k} x^k
\]

\[
= \sum_{k=0}^{p-1} (s + k)_s (x + 1)^{p-1-k} x^k
\]

\[
- (s - r' + p)_s (x + 1)^{r'-1} x^{p-r'}
\]

\[
\equiv (x + 1)^s w_0(x; 0, s) - (s - r')_s (x + 1)^{r'-1} x^{p-r'}
\]

\[
\equiv s!(x + 1)^s - (s - r')_s (x + 1)^{r'-1} x^{p-r'}
\]

and if \( r' = 0 \) we get

\[
(x + 1)^{s+1} w_{m(p-1)}(x; r, s) \equiv \sum_{k=1}^{p-1} (s + k)_s (x + 1)^{p-k} x^k
\]

\[
= \sum_{k=0}^{p-1} (s + k)_s (x + 1)^{p-k} x^k - s!(x + 1)^p
\]

\[
= (x + 1)^{s+1} w_0(x; 0, s) - s!(x + 1)^p
\]

\[
= s!(x + 1)^{s+1} - s!(x^p + 1).
\]

which complete the proof. \( \square \)

**Remark 1.** For \( r = s = m - 1 = 0 \) in Corollary 6 or \( n = p \) in Lemma 1 we obtain \( (x + 1)w_{p-1}(x) \equiv x - x^p \) which gives for \( x = 1 \) the known congruence \( w_{p-1} \equiv 0 \), see [8].
Now, we give some curious congruences on \((r,s)\)-geometric polynomials and on \((r_1,\ldots,r_q)\)-geometric polynomials defined below.

**Theorem 2.** For any integers \(n,m,r,s \geq 0\) and any prime number \(p \nmid m\), there holds

\[
\sum_{k=1}^{p-1} \frac{w_{n+k}(x;r,s)}{(-m)_k} \equiv (-m)^n (w_{p-1}(x;r,m,s) - s!).
\]

**Proof.** Upon using the identity \(x^s w_n(x;r,s) = (w_x + r - s)^n (w_x)_s\) and the known congruence \((-m)^{-k} \equiv (p^{-1}) m^{-1-k}\) we obtain

\[
x^s LHS \equiv \sum_{k=0}^{p-1} \binom{p-1}{k} m^{-1-k} (w_x + r - s)^{n+k} (w_x)_s
\]

\[
= (w_x + r - s)^n (w_x + r + m - s)^{p-1} (w_x)_s
\]

\[
= \sum_{j=0}^{n} \binom{n}{j} (-m)^{n-j} (w_x + r + m - s)^{j+p-1} (w_x)_s
\]

\[
= (-m)^n (w_x + r + m - s)^{p-1} (w_x)_s
\]

\[
+ \delta(n \geq 1) \sum_{j=1}^{n} \binom{n}{j} (-m)^{n-j} (w_x + r + m - s)^{j+p-1} (w_x)_s
\]

\[
= x^n (-m)^n w_{p-1}(x;r+m,s)
\]

\[
+ \delta(n \geq 1) x^n \sum_{j=1}^{n} \binom{n}{j} (-m)^{n-j} w_{p+j-1}(x;r+m,s)
\]

\[
\equiv x^n (-m)^n w_{p-1}(x;r+m,s)
\]

\[
+ \delta(n \geq 1) x^n \sum_{j=1}^{n} \binom{n}{j} (-m)^{n-j} w_j(x;r+m,s)
\]

\[
= x^n (-m)^n w_{p-1}(x;r+m,s) + \delta(n \geq 1) x^n (w_n(x;r,s) - (-m)^n s!)
\]

\[
= x^n [(-m)^n w_{p-1}(x;r+m,s) + w_n(x;r,s) - (-m)^n s!],
\]

where \(\delta\) is the Kronecker’s symbol, i.e. \(\delta(n \geq 1) = 1\) if \(n \geq 1\) and 0 otherwise. \(\square\)

Let \(r_q = (r_1,\ldots,r_q)\) be a vector of non-negative integers and let

\[
w_n(x;r_q) = \sum_{j=0}^{n+|r_q|-1} \binom{n+|r_q|}{j+r_q} j! (j+r_q)! x^j, \quad 0 \leq r_1 \leq \cdots \leq r_q.
\]
where $\binom{n+q}{j+q}$ are the $(r_1, \ldots, r_q)$-Stirling numbers defined by Mihoubi et al. [21]. This polynomial is a generalization of the $r$-geometric polynomials $w_n(x;r) := w_n(x;r,r)$.

**Proposition 5.** For any non-negative integers $n,m$ and any prime $p \nmid m$, there holds

$$x^r q \sum_{k=1}^{p-1} \frac{w_{n+k}(x;r_q)}{(-m)^k} \equiv (-m)^n (-m)_{r_1} \cdots (-m)_{r_q} (w_{p-1}(x;m,0) - 1).$$

In particular, for $q = 1$ and $r_q = r$ we obtain

$$x^r \sum_{k=1}^{p-1} \frac{w_{n+k}(x;r,r)}{(-m)^k} \equiv (-m)^n (-m)_r (w_{p-1}(x;m,0) - 1).$$

**Proof.** By the identity $(w_n)_n = n!x^n$ and by [21, Th. 10] we have

$$x^r q w_n(x;r_q) = \sum_{j=0}^{n+|r_q|-1} \binom{n+|r_q|}{j+|r_q|} (w_x)^j r_q \binom{n+|r_q|}{j+|r_q|} (w_x-r_q)^j r_q = w_n^r (w_x)_{r_1} \cdots (w_x)_{r_q} \sum_{k=0}^{|r_q|} a_k (r_q) w_{n+k} \sum_{j=0}^{|r_q|} a_j (r_q) w_{n+j}(x),$$

where $\sum_{k=0}^{|r_q|} a_k (r_q) u^k = (u)_{r_1} \cdots (u)_{r_q}$. So, by application of Theorem 2 we get

$$x^r q \sum_{k=1}^{p-1} \frac{w_{n+k}(x;r_q)}{(-m)^k} = \sum_{j=0}^{|r_q|} a_j (r_q) \sum_{k=1}^{p-1} \frac{w_{n+j+k}(x;0,0)}{(-m)^k} \equiv \sum_{j=0}^{|r_q|} a_j (r_q) (-m)^{n+j} (w_{p-1}(x;m,0) - 1) \equiv (-m)^n (-m)_{r_1} \cdots (-m)_{r_q} (w_{p-1}(x;m,0) - 1).$$

\[\square\]
Remark 2. Since \( x^q \, w_n(x; r_q) = w^d_n(w_x)_{r_1} \cdots (w_x)_{r_q} \), then, for \( g(x) = x^q(x)_{r_1} \cdots (x)_{r_q} \) and \( f(x) = x^{m} \) in Corollary 5 we obtain
\[
 w_{mp+q}(x; r_q) \equiv w_{m+q}(x; r_q) .
\]
\[
 w_{m(p-1)}(x; r_q) \equiv w_{0}(x; r_q) , \quad r_1 \cdots r_q \neq 0 , \quad m \geq 0 .
\]

Corollary 7. Let \( a_0(x), \ldots, a_t(x) \) be polynomials with integer coefficients,
\[
 R_{n,t}(x; r, s) = \sum_{i=0}^{t} a_i(x) w_{n+i}(x; r, s) \quad \text{and} \quad L_t(x, y) = \sum_{i=0}^{t} a_i(x) y^i .
\]
Then, for any non-negative integers \( n, m, r, s \) and any prime \( p \mid m \), there hold
\[
 \sum_{k=1}^{p-1} \frac{R_{n+k,t}(x; r, s)}{(-m)^k} = (-m)^n L_t(x, -m)(w_{p-1}(x; r + m, s) - s!) .
\]

Proof. Theorem 2 implies
\[
 \sum_{k=1}^{p-1} \frac{R_{n+k,t}(x; r, s)}{(-m)^k} = \sum_{j=0}^{t} a_j(x) \sum_{k=1}^{p-1} \frac{w_{n+k+j}(x; r, s)}{(-m)^k} 
\]
\[
 = \sum_{j=0}^{t} a_j(x)(-m)^{n+j}(w_{p-1}(x; r + m, s) - s!) 
\]
\[
 = (-m)^n L_t(x, -m)(w_{p-1}(x; r + m, s) - s!) .
\]

4. Congruences involving \( w_n(x; r, s) \), \( P_n(x, r) \) and \( T_n(x, r) \)

The following theorem gives connection in congruences between the polynomials \( w_n \) and \( P_n \).

Theorem 3. Let \( n, r \) be non-negative integers and \( p \) be a prime number. Then, for \( m \in \{0, \ldots, p-1\} \) there holds
\[
 \sum_{k=m}^{p-1} (-x)^k \frac{w_n(x; r + k, k)}{(k-m)!} \equiv (-1)^m m! (r + m)^n P_{p-1}(x, m). 
\]
In particular, for \( m = 0 \), we get
\[
 \sum_{k=0}^{p-1} \frac{(-x)^k w_n(x; r + k, k)}{k!} \equiv P^n(1 + x + \cdots + x^{p-1}) .
\]
hence, it follows $LHS \equiv -1 \cdot m! \sum_{k=0}^{p-1} (m+1)_{p-1-k} (w(x+r+k,k)$

$$LHS \equiv -1 \cdot m! \sum_{k=0}^{p-1} (m-p+1)_{p-1-k} (w(x+r)^n(w_k)_k$$

$$LHS \equiv -1 \cdot m! \sum_{k=0}^{p-1} \binom{p-1}{k} (m-p+1)_{p-1-k} (w(x+r)^n(-w_k)_k$$

$$LHS \equiv -1 \cdot m! (m-p+1-w(x))_{p-1}(w(x+r)^n$$

$$LHS \equiv -1 \cdot m! (w(x-m+p-1)_{p-1}(w(x+r)^n$$

$$LHS \equiv -1 \cdot m! (w(x-m+p-r+m)^n(w(x-m+p-1)p-1$$

But for $j \geq 1$ we have

$$(w(x)-m)_{j}(w(x-m+p-1)_{p-1} = (w(x-m-p-1)_{p-1}$$

$$LHS \equiv -1 \cdot m! (r+m)^n \mathcal{T}_{p-1}(x,m).$$

hence, it follows $LHS \equiv -1 \cdot m! (r+m)^n \mathcal{T}_{p-1}(x,m).$ $\square$

A connection in congruences between the polynomials $w_n$ and $\mathcal{T}_n$ is to be:

**Theorem 4.** For any integers $n, m, r \geq 0$ and any prime $p$, there holds

$$\sum_{k=0}^{p-1} (-m)_{p-1-k}(x+1)^k w_n(x; r + m, k) \equiv -r^n \mathcal{T}_{p-1}(x;m).$$

**Proof.** Upon using the identity $(x+1)^w w_n(x;r,s) = (w(x)+r)^n(w(x)+s)_s$ and the known congruence $(m)_{p-1-k} \equiv \binom{p-1}{k}(-m)_{p-1-k}$ we obtain

$$LHS \equiv \sum_{k=0}^{p-1} \binom{p-1}{k} (m)_{p-1-k}(w(x+r+m)^n(w(x+k)_k$$
\[
\begin{align*}
&= \sum_{k=0}^{p-1} \binom{p-1}{k} (m)_{p-1-k} (w_x + r + m)^n (w_x + 1)_k \\
&= (w_x + r + m)^n (w_x + m + 1)_{p-1} \\
&= (w_x + m + r)^n (w_x + m + p - 1)_{p-1} \\
&= \sum_{j=0}^{n} \binom{n + r}{j + r} (w_x + m + p - 1)_{j+p-1} \\
&= \sum_{j=0}^{n} \binom{n + r}{j + r} (j + p - 1)! \mathcal{T}_{j+p-1}(x; m-j) \\
&= (p-1)! \mathcal{T}_{p-1}(x; m) + \sum_{j=1}^{n} \binom{n + r}{j + r} (j + p - 1)! \mathcal{T}_{j+p-1}(x; m-j) \\
&= -r^n \mathcal{T}_{p-1}(x; m).
\end{align*}
\]

\[\square\]

**Corollary 8.** Let \( \mathcal{R}_{n,t}(x; r, s) \) be as in Corollary 7. Then, for any non-negative integers \( n, m, r, s \) and any prime \( p \nmid m \), there holds

\[
\sum_{k=0}^{p-1} (-1)^k \binom{k}{m} \mathcal{R}_{n,t}(x; r + k, k) \equiv (-1)^m (r + m)^n \mathcal{L}_t(x, r + m) \mathcal{P}_{p-1}(x, m).
\]

**Proof.** Theorem 3 implies

\[
LHS = \sum_{j=0}^{t} a_j(x) \sum_{k=m}^{p-1} (-1)^k \binom{k}{m} \frac{w_{n+j}(x; r + k, k)}{k!}
\equiv \sum_{j=0}^{t} a_j(x) (-1)^m (r + m)^{n+j} \mathcal{P}_{p-1}(x, m)
\equiv (-1)^m (r + m)^n \mathcal{L}_t(x, r + m) \mathcal{P}_{p-1}(x, m).
\]

\[\square\]

**References**


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